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Local Existence and Estimations for a Semilinear Wave Equation in Two Dimension Space.

AMEL ATALLAH BARAKET

Sunto. – *In questo articolo dimostriamo un teorema di esistenza locale per un problema di Cauchy associato ad un'equazione delle onde semilineare in dimensione due. In questo problema la prima condizione iniziale è identicamente nulla, la seconda appartiene a $L^2(\mathbb{R}^2)$, è a simmetria radiale e a supporto compatto. Per dimostrare questo teorema stabiliamo prima una disuguaglianza di tipo Moser–Trudinger per il problema lineare associato e concludiamo grazie ad un'applicazione di un metodo di punto fisso.*

Summary. – *In this paper we prove a local existence theorem for a Cauchy problem associated to a semi linear wave equation with an exponential nonlinearity in two dimension space. In this problem, the first Cauchy data is equal to zero, the second is in $L^2(\mathbb{R}^2)$, radially symmetric and compactly supported. To prove this theorem, we first show a Moser-Trudinger type inequality for the linear problem and then we use a fixed point method to achieve the proof of the result.*

1. – Introduction.

In this work we study the local existence in time for a Cauchy problem associated to a semilinear wave equation:

$$\text{(1)} \qquad \begin{cases} \Box u + u e^{\alpha_0 u^2} = 0 \\ \quad u_{|t=0} \quad = 0 \\ \quad \partial_t u_{|t=0} = f, \end{cases}$$

where $\Box u(t, x) = \partial_t^2 u(t, x) - \Delta_x u(t, x)$, $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Here f is in $L^2(\mathbb{R}^2)$, radially symmetric, compactly supported and α_0 is a positive real. The aim of this paper is to prove a local existence theorem for the problem (1).

THEOREM 1.1. – *For every α_0 in $[0, 4\pi[$, for every f in $L^2(\mathbb{R}^2)$ radially symmetric with compact support satisfying $\|f\|_{L^2} \leq 1$, there exists a positive real T_0 such that the problem* (1) *has a solution u in $\mathcal{C}^0([0, T_0], H^1(\mathbb{R}^2)) \cap \mathcal{C}^1([0, T_0], L^2(\mathbb{R}^2))$.*

First we remark that in dimension two the problem is different from higher dimensions. In dimension $n \geqslant 3$, the analogue of (1) is the following problem:

$$(2) \qquad \begin{cases} \Box u + |u|^{p-1} u = 0 & \text{on } \mathbb{R} \times \mathbb{R}^n, \\ u_{|t=0} \qquad = \varphi & \text{on } \mathbb{R}^n \\ \partial_t u|_{t=0} \quad = \psi & \text{on } \mathbb{R}^n. \end{cases}$$

It has been widely investigated and we have mainly 3 cases.

If the exponent p is subcritical, $\left(\text{which means } p < p_c \text{ where } p_c = \dfrac{n+2}{n-2}\right)$ after many works essentially due to Lions [L], Strauss [Str], Struwe [St]..., Ginibre and Velo [GV1] showed the existence of a unique global solution $u \in \mathcal{C}^0(\mathbb{R}, H^1(\mathbb{R}^n)) \cap \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^n))$, for initial data (φ, ψ) in $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

For the critical case, $p = p_c$, Ginibre, Soffer and Velo [GSV] for radial data, then Shatah and Struwe [SS] in the general case showed the existence of a unique global solution belonging also to $L^{p_c}_{loc}(\mathbb{R}, L^{2p_c}(\mathbb{R}^n))$. The most important ingredient of their proof is the Strichartz inequality (see for example [GV2]). Indeed, we have for v the unique solution in $\mathcal{C}^0(\mathbb{R}, H^1(\mathbb{R}^n)) \cap \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^n))$ of the linear Cauchy problem

$$(2') \qquad \begin{cases} \Box v \qquad = F & \text{on } \mathbb{R} \times \mathbb{R}^n, \\ v_{|t=0} \quad = \varphi & \text{on } \mathbb{R}^n \\ \partial_t v|_{t=0} = \psi & \text{on } \mathbb{R}^n, \end{cases}$$

where $F \in L^1(\mathbb{R}, L^2(\mathbb{R}^n))$, $(\varphi, \psi) \in \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and for $T > 0$

$$(3) \qquad \|v\|_{L^q([0,T], L^r(\mathbb{R}^n))} \leqslant C_q (\|F\|_{L^1([0,T], L^2(\mathbb{R}^n))} + \|\varphi\|_{H^1(\mathbb{R}^n)} + \|\psi\|_{L^2(\mathbb{R}^n)})$$

with $\dfrac{1}{q} + \dfrac{n}{r} = \dfrac{n}{2} - 1$, $q \geqslant \dfrac{n+1}{n-1}$ if $n \geqslant 4$ and $q > 2$ if $n = 3$.

The nonlinear term $|u|^{p_c - 1} u$ in (2) is then considered as the second member $F(u)$. However in the case $n = 2$, the estimates (3) are not available; they will be replaced by another type of inequalities.

Finally in the case $p > p_c$, there are only some partial results in special cases.

We now return to the dimension two. Instead of the Sobolev injection $H^1(\mathbb{R}^n) \hookrightarrow L^{p_c+1}(\mathbb{R}^n)$ used in dimension $n \geqslant 3$, we will use here the injection of $H^1_0(\Omega)$ (Ω bounded) in the Orlicz space, this will be developed in the next section.

On the other hand, multiplying formally the first equation in (1) by $\partial_t u$ and integrating on $\mathbb{R}^2$, we obtain

$$\frac{1}{2}\|\nabla_x u(t,\cdot)\|^2_{L^2} + \frac{1}{2}\|\partial_t u(t,\cdot)\|^2 L^2 + \frac{1}{2\alpha_0}\int_{\mathbb{R}^2} e^{\alpha_0 u^2}\,dx = \text{constant}\,,$$

so if we want to use an iterative schema to solve the problem (1), we have to control the last integral term in the left hand side of the above equality; this will be achieved by the use of a Moser-Trudinger type inequality.

Let us note that this inequality is the most important ingredient of the proof of theorem 1.1 and in order to obtain it, we need very precise estimates on the solution of the linear Cauchy problem associated to (1). So, for this reason, we take the radial assumption on f. Moreover, if we want to prove a global existence result, we need to show an analogue of theorem 1.1, but for u satisfying (1) with $u_{|t=0} = g$. However, we restrict ourselves here to the case g equal to zero in order to obtain the Moser-Trudinger type inequality mentioned above.

We also mention that in a recent paper, Nakamura and Ozawa [NO] showed the existence of a unique global solution of the associated Cauchy problem for the semilinear wave equation with non linearity of exponential growth. In their work the initial data (φ, ψ) belong to $\dot{H}^{\frac{n}{2}}(\mathbb{R}^n)\times\dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$ but they are also supposed to be sufficently small.

This paper is organized as follows. In section two we first recall an inequality proved by Trudinger [Tr] and sharpened by Moser [M], then we show an improved inequality for the solution of the linear problem associated to (1). Finally in section three we complete the proof of Theorem 1.1.

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2. – Inequalities of Moser-Trudinger type.

2.1. *The Moser-Trudinger's inequality.*

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geqslant 2$ and $W_0^{1,q}(\Omega)$ be the Banach space obtained from $C_0^\infty(\Omega)$ by completion with the norm

$$\|u\|_{1,q} = \|\nabla_x u\|_{L^q} = \left(\int_\Omega |\nabla_x u|^q\,dx\right)^{1/q},$$

where $\nabla_x u$ is the gradient of u. It is well known that if $1 < q < n$, $W_0^{1,q}(\Omega)$ is continuously embedded in $L^p(\Omega)$ with $\dfrac{1}{p} = \dfrac{1}{q} - \dfrac{1}{n}$, and if $q > n$ this space is continuously embedded in the Hölder space $C^{0,\nu}(\Omega)$ with Hölder exponent $\nu = 1 - \frac{n}{q}$. For the case $q = n$, it is easy to find examples of unbounded functions in $W_0^{1,n}(\Omega)$. However, Trudinger showed in [T] that $W_0^{1,n}(\Omega)$ is continuously embedded in the Orlicz space L_ϕ with $\phi(t) = e^{\alpha|t|^{n/n-1}}$; this means that there exists positive constants C and α such that

$$\int_\Omega e^{\alpha u^{p^*}} dx \leqslant C$$

for all u in the unit ball of $W_0^{1,n}(\Omega)$. Here $p^* = \frac{n}{n-1}$. Later on, Moser sharpened this result in proving the following one:

THEOREM 2.1 ([M]). – *Let* $u \in W_0^{1,n}(\Omega)$, $n \geqslant 2$ *and*

$$\int_\Omega |\nabla_x u|^n dx \leqslant 1 .$$

Then there exists a constant C *which depends only on* n *such that*

$$\int_\Omega e^{\alpha u^{p^*}} dx \leqslant C|\Omega| ,$$

where $p^* = \frac{n}{n-1}$, $\alpha \leqslant \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$, $|\Omega| = \int_\Omega dx$ *and* ω_{n-1} *is the* $(n-1)$ *dimensional volume of the unit sphere* S^{n-1}. *The integral on the left is finite for any positive* α *but if* $\alpha > \alpha_n$ *it can be made arbitrary large by an appropriate choice of* u.

On the other hand Lions [L] proved that this embedding is not compact. Indeed if Ω is the unit ball, he considered $(u_k)_k$ defined by $u_k(x) = f_k(-n\log|x|)$ where

$$f_k(t) = \begin{cases} \left(\dfrac{k}{\alpha_n}\right)^{\frac{n-1}{n}} \dfrac{t}{k} & \text{if } t \leqslant k \\ \left(\dfrac{k}{\alpha_n}\right)^{\frac{n-1}{n}} & \text{if } t \geqslant k. \end{cases}$$

Clearly $\|\nabla_x u_k\|_{L^n} = 1$ and u_k is weakly convergent to 0 in $W_0^{1,n}(\Omega)$ but $\|\exp(\alpha_n |u_k|^{\frac{n}{n-1}})\|_{L^1} \geqslant \frac{\omega_{n-1}}{n}$.

In the case $n = 2$, the result of Moser can be stated as follows.

There exists a positive constant C such that for every u in $H_0^1(\Omega)$ satisfying $\|\nabla u\|_{L^2} \leqslant 1$, we have

$$\int_\Omega e^{\alpha u^2}\, dx \leqslant C|\Omega| \,,$$

for every $\alpha \leqslant 4\pi$.

The aim of the next paragraph is to show an improved estimate for solutions of the linear wave equation and for some α larger than 4π.

2.2. *An inequality of Moser-Trudinger type for the linear problem.*

We consider the linear problem for the wave equation associated to (1)

$$\begin{cases} \Box v &= 0 \\ v_{|t=0} &= 0 \\ \partial_t v_{|t=0} &= f\,, \end{cases} \tag{4}$$

with f radially symmetric, supp $f \subset B(0, R)$ and $\|f\|_{L^2} \leqslant 1$. The principal result of this paragraph is the following

PROPOSITION 2.2. – *The Cauchy problem (4) has a unique global solution v, which is radially symmetric and $v \in \mathcal{C}^0(\mathbb{R}, H^1(\mathbb{R}^2)) \cap \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^2))$. Moreover, there exists an absolute constant C_1 such that for every $0 \leqslant \alpha < 8\pi$, and every positive t*

$$\int_{B(0,\, t+R)} e^{\alpha v^2}\, dx \leqslant C_1 \left(\frac{t^2}{8\pi - \alpha} + (t+R)^2 \right). \tag{5}$$

REMARK 1. – The hypothesis f radially symmetric, $v_{|t=0} = 0$ in problem (4) and $u_{|t=0} = 0$ in (1) are only technical. It seems that (5) should hold for f with compact support and $v_{|t=0} = g$.

REMARK 2. – If we don't suppose $\|f\|_{L^2} \leqslant 1$, we have the same result as in Proposition 2.2 but for $\alpha < \dfrac{8\pi}{\|f\|_{L^2}^2}$ and in the right hand side of (5) α is replaced by $\alpha \|f\|_{L^2}^2$.

PROOF OF PROPOSITION 2.2. – Using the Hadamard representation, the solution of problem (4) can be written as follows. For any $t > 0$ and x in $\mathbb{R}^2$, we have

$$v(t, x) = \frac{1}{2\pi} \int_{|x-y|<t} (t^2 - |x-y|^2)^{-1/2} f(y)\, dy \,, \tag{6}$$

and it is clear that v is radially symmetric. On the other hand by finite propagation speed principle and since $\operatorname{supp} f \subset B(0, R)$, the support of $v(t, \cdot)$ is contained in $B(0, t+R)$ for every positive t. For every x in $B(0, t+R)$, let $\varrho = |x|$; then we have the following lemma.

LEMMA 2.3. – *For every positive t and every x in $B(0, t+R)$ such that $\varrho \geqslant t$ we have*

$$|v(t, x)| \leqslant \sqrt{\frac{t}{2\pi\varrho}}. \tag{7}$$

PROOF. – Since v is radially symmetric we can assume without loss of generality that $x = (\varrho, 0)$; using polar coordinates $y = (r\cos\theta, r\sin\theta)$ we obtain by (6)

$$v(t, x) = \frac{1}{\pi\sqrt{2\varrho}} \int_{\varrho - t}^{\varrho + t} r^{1/2} f(r) \left(\int_0^\alpha (\cos\theta - \cos\alpha)^{-1/2} \, d\theta \right) dr, \tag{8}$$

where $\cos\alpha = \frac{1}{2r\varrho}(r^2 + \varrho^2 - t^2)$.

Let us compute the integral

$$I = \int_0^\alpha (\cos\theta - \cos\alpha)^{-1/2} \, d\theta .$$

If we introduce a new variable s such that

$$\cos\theta = \frac{1 + \cos\alpha}{2} + s\left(\frac{1 - \cos\alpha}{2}\right),$$

we obtain

$$I = \frac{\sqrt{2}}{\sqrt{3 + \cos\alpha}} \int_{-1}^{1} \frac{ds}{\sqrt{1 - s^2}\sqrt{1 + s\gamma}}$$

where $\gamma = \frac{1 - \cos\alpha}{3 + \cos\alpha}$. Since $r \in [\varrho - t, \varrho + t]$, we have $\cos\alpha \in [0, 1]$, $\gamma \in \left[0, \frac{1}{3}\right]$, $\sqrt{1 + s\gamma} \geqslant \sqrt{\frac{2}{3}}$ and $\left(\frac{3 + \cos\alpha}{2}\right)^{1/2} \geqslant \left(\frac{3}{2}\right)^{1/2}$, which lead to

$$I \leqslant \pi . \tag{9}$$

Coming back to (8), we have by the Cauchy-Schwartz inequality, $\|f\|_{L^2} \leqslant 1$ and (9)

$$
\begin{aligned}
|v(t, x)| &\leqslant \frac{1}{\sqrt{2\varrho}} \left(\int_{\varrho - t}^{\varrho + t} r f^2(r)\, dr \right)^{1/2} \left(\int_{\varrho - t}^{\varrho + t} dr \right)^{1/2} \\
&\leqslant \frac{1}{2\sqrt{\pi\varrho}} \|f\|_{L^2} \sqrt{2t} \\
&\leqslant \sqrt{\frac{t}{2\pi\varrho}} \,.
\end{aligned}
$$

The proof of Lemma 2.3 is complete. ■

Now, in the next lemma, we will estimate the solution v for $\varrho < t$.

LEMMA 2.4. – *For every positive t, every x in $\mathbb{R}^2$ satisfying $0 < |x| = \varrho < t$, there exists an absolute constant M such that*

$$
|v(t, x)| \leqslant \left(M + \frac{1}{4\pi} \log \frac{t}{\varrho} \right)^{1/2} . \tag{10}
$$

PROOF. – According to (6), for $0 < |x| = \varrho < t$, we obtain after using polar coordinates

$$
v(t, x) = \frac{1}{2\pi\sqrt{2\varrho}} \int_0^{t+\varrho} r^{1/2} f(r) \left(\int_0^{2\pi} \left(\frac{1}{\cos\theta + \lambda} \right)_+^{1/2} d\theta \right) dr \tag{11}
$$

where $\lambda = \lambda(r) = \dfrac{t^2 - r^2 - \varrho^2}{2r\varrho}$ and $\varphi_+ = \max(\varphi, 0)$ for every real function φ.

Applying the Cauchy-Schwartz inequality to (11), we get

$$
\begin{aligned}
|v(t, x)| &\leqslant \frac{1}{2\pi} \left(\int_0^{t+\varrho} r f^2(r)\, dr \right)^{1/2} \left[\int_0^{t+\varrho} \frac{1}{2\varrho} \left(\int_0^{2\pi} \left(\frac{1}{\cos\theta + \lambda} \right)_+^{1/2} d\theta \right)^2 dr \right]^{1/2} \\
&\leqslant \frac{1}{(2\pi)^{3/2}} \left(\int_0^{t+\varrho} \left(\frac{1}{2\varrho} \left(\frac{1}{\cos\theta + \lambda} \right)_+^{1/2} d\theta \right)^2 dr \right)^{1/2}
\end{aligned} \tag{12}
$$

Let

$$
F(t, \varrho) = \int_0^{t+\varrho} \frac{1}{2\varrho} \left(\int_0^{2\pi} \left(\frac{1}{\cos\theta + \lambda} \right)_+^{1/2} d\theta \right)^2 dr = \int_0^{t+\varrho} G(t, r, \varrho)\, dr \,.
$$

In order to obtain the inequality (10), we will estimate $F(t, \varrho)$ in the following Lemma.

LEMMA 2.5. – *For every positive t and every $0 < \varrho < t$ we have*

$$F(t, \varrho) \leqslant M + (2\pi^2)\log\frac{t}{\varrho}, \tag{13}$$

where M is the constant appearing in Lemma 2.4.

PROOF OF LEMMA 2.5. – First we remark that λ is a strictly decreasing function of r. Let $r_0 = -2\varrho + \sqrt{3\varrho^2 + t^2}$, $r_1 = t - \varrho$, $r_2 = \sqrt{t^2 - \varrho^2}$ then $0 < r_0 < r_1 < r_2 < t + \varrho$ and $\lambda(r_0) = 2$, $\lambda(r_1) = 1$, $\lambda(r_2) = 0$, $\lambda(t + \varrho) = -1$. Thus

$$F(t, \varrho) = \int_0^{r_0} G(t, \varrho, r)\, dr + \int_{r_0}^{t-\varrho} G(t, \varrho, r)\, dr + \int_{t-\varrho}^{\sqrt{t^2-\varrho^2}} G(t, \varrho, r)\, dr$$

$$+ \int_{\sqrt{t^2-\varrho^2}}^{t+\varrho} G(t, \varrho, r)\, dr = (a) + (b) + (c) + (d).$$

In what follows we will set by

$$J = \int_0^{2\pi} \left(\frac{1}{\cos\theta + \lambda} \right)_+^{1/2} d\theta$$

and we estimate the terms (a), (b), (c) and (d).

REMARK. – All the constants C_i appearing in the proof are absolute constants.

Estimation of (a). Since $r \in (0, r_0)$, we have $\lambda(r) > 2$ and:

$$J = 2\int_0^{\pi} (\cos\theta + \lambda)^{-1/2}\, d\theta \leqslant \frac{2\pi}{\sqrt{\lambda - 1}}$$

so

$$(a) \leqslant 4\pi^2 \int_0^{r_0} \frac{r}{t^2 - (r + \varrho)^2}\, dr\,.$$

Computing this last integral and using the definition of r_0 we obtain

$$4\pi^2 \int_0^{r_0} \frac{r}{t^2-(r+\varrho)^2}\,dr = 4\pi^2 \int_{\varrho}^{r_0+\varrho} \frac{u-\varrho}{t^2-u^2}\,du$$

$$= 2\pi^2 \left[\log(t^2-\varrho^2) - \log(t^2-(r_0+\varrho)^2) - \frac{\varrho}{t}\left(\log\frac{t+\varrho}{t-\varrho}\right) \right.$$

$$\left. + \frac{\varrho}{t}\log\left(\frac{t+r_0+\varrho}{t-r_0-\varrho}\right)\right]$$

$$= 2\pi^2 \left[\left(1+\frac{\varrho}{t}\right)\log(t+\varrho) + \left(1-\frac{\varrho}{t}\right)\log(t-\varrho)\right.$$

$$\left. + \left(\frac{\varrho}{t}-1\right)\log(t-r_0-\varrho) - \left(1+\frac{\varrho}{t}\right)\log(r_0+\varrho+t)\right]$$

$$= 2\pi^2 \left[\left(\frac{\varrho}{t}-1\right)\log 2 - \frac{2\varrho}{t}\log(t+\varrho+\sqrt{3\varrho^2+t^2})\right.$$

$$\left. + \left(\frac{\varrho}{t}-1\right)\log\varrho + \left(1+\frac{\varrho}{t}\right)\log(2\varrho+\sqrt{3\varrho^2+t^2})\right].$$

Since $\varrho < t$ we then obtain

$$(a) \quad \leqslant (2\pi^2)\left[\left(\frac{\varrho}{t}-1\right)\log 2 + \left(\frac{\varrho}{t}-1\right)\log\varrho + \left(1+\frac{\varrho}{t}\right)\log 4t - \frac{2\varrho}{t}\log 2t\right]$$

$$\leqslant 2\pi^2\left[\left(1+\frac{\varrho}{t}\right)\log 2 + \left(\frac{\varrho}{t}-1\right)\log\frac{\varrho}{t}\right]$$

then

$$(a) \quad \leqslant 2\pi^2\left(2\log 2 + \log\frac{t}{\varrho}\right). \tag{14}$$

Now we will show that all the other terms of $F(t,\varrho)$ are bounded from above.

Estimation of (b). Making the change of variable $u=\cos\theta$ in J we obtain

$$J = 2\int_{-1}^{1} \frac{1}{\sqrt{1-u^2}}\,\frac{1}{\sqrt{u+\lambda}}\,du = 2\int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{u+\lambda}} + 2\int_{-1}^{0}\frac{du}{\sqrt{1-u^2}\sqrt{u+\lambda}}$$

$$= (I) + (II).$$

Since $r\in(r_0,t-\varrho)$ and λ is a decreasing function of r, we have $\lambda(r)\in(1,2)$; so

$$(I)\leqslant\pi\,. \tag{15}$$

For (II) we have:

$$\begin{aligned}(II)\ &\leqslant 2\int_0^1 \frac{1}{\sqrt{u}\sqrt{u+\lambda-1}}\,du\\ &\leqslant 2\int_0^{1/(\lambda-1)} \frac{1}{\sqrt{s(s+1)}}\,ds\,.\end{aligned}$$

Since $\lambda>1$, we have $\dfrac{1}{\lambda-1}\in(1,\,+\infty)$. Then

$$(II)\ \leqslant 2\int_0^1\frac{ds}{\sqrt{s}}+2\int_1^{1/(\lambda-1)}\frac{ds}{s}\leqslant 4+2\int_1^{1/(\lambda-1)}\frac{ds}{s}\,.$$

Thus

$$(II)\ \leqslant 4+2\log\frac{1}{\lambda-1}\,. \tag{16}$$

(15) and (16) then give

$$J\leqslant 4+\pi+2\log\frac{1}{\lambda-1}\,. \tag{17}$$

If we come back to (b) and use the definition of r_0 we have by (17)

$$\begin{aligned}(b)\ &\leqslant\frac{1}{\varrho}\left[(4+\pi)^2(t-\varrho-r_0)+4\int_{r_0}^{t-\varrho}\left(\log\frac{1}{\lambda-1}\right)^2dr\right]\\ &\leqslant C_2+\frac{4}{\varrho}\int_{r_0+\varrho}^{t}\left(\log\frac{2\varrho(u-\varrho)}{t^2-u^2}\right)^2du\\ &\leqslant C_2+\frac{4}{\varrho}\int_{r_0+\varrho}^{t}\left(\log\frac{2\varrho(t-\varrho)}{t(t-u)}\right)^2du\\ &\leqslant C_2+\frac{4}{\varrho}\left[\frac{2\varrho(t-\varrho)}{t}\right]\int_0^1(\log v)^2\,dv,\end{aligned}$$

so

$$(b) \leqslant C_3 \tag{18}$$

Estimation of (*c*). Here we have $r \in (t - \varrho, \sqrt{t^2 - \varrho^2})$ so $\lambda(r) \in (0, 1)$ and

$$J = 2 \int_0^{\alpha} (\cos\theta + \lambda)^{-1/2} \, d\theta$$

where $\cos\alpha = -\lambda = \dfrac{\varrho^2 + r^2 - t^2}{2r\varrho}$ is in $(-1, 0)$. Making the following change of variables $\cos\theta = \dfrac{1 + \cos\alpha}{2} + s\left(\dfrac{1 - \cos\alpha}{2}\right)$, we obtain:

$$J = \frac{2\sqrt{2}}{\sqrt{3 + \cos\alpha}} \int_{-1}^{1} \frac{ds}{\sqrt{1 + \gamma s}\sqrt{1 - s^2}},$$

with $\gamma = \dfrac{1 - \cos\alpha}{3 + \cos\alpha}$. For $s \in [0, 1]$, since $\cos\alpha \in (-1, 0)$ we deduce that $\gamma \in \left(\dfrac{1}{3}, 1\right)$, $1 + \gamma s \geqslant 1$ and

$$\frac{2\sqrt{2}}{\sqrt{3 + \cos\alpha}} \int_0^1 \frac{ds}{\sqrt{1 + \gamma s}\sqrt{1 - s^2}} \leqslant 2 \int_0^1 \frac{ds}{\sqrt{1 - s^2}} = \pi.$$

We also have

$$\frac{2\sqrt{2}}{\sqrt{3 + \cos\alpha}} \int_{-1}^{0} \frac{ds}{\sqrt{1 + \gamma s}\sqrt{1 - s^2}} \leqslant 2 \int_0^1 \frac{ds}{\sqrt{1 - \gamma s}\sqrt{1 - s^2}}.$$

Combining the last two inequalities we obtain:

$$J \leqslant \pi + 2 \int_0^1 \frac{ds}{\sqrt{1 - \gamma s}\sqrt{1 - s^2}}. \tag{19}$$

In the following we will estimate the integral in the right hand side of (19).

Let us introduce the variable u such that

$$u = \frac{1-\varepsilon}{\varepsilon}(1-s), \quad \text{where } \varepsilon = 1-\gamma$$

$$2\int_0^1 \frac{ds}{\sqrt{1-\gamma s}\sqrt{1-s^2}} \leqslant \frac{2}{\sqrt{1-\varepsilon}} \int_0^{\frac{1-\varepsilon}{\varepsilon}} \frac{du}{\sqrt{u(u+1)}},$$

with $\varepsilon \in \left]0, \frac{2}{3}\right[$ and $\frac{1-\varepsilon}{\varepsilon} \geqslant \frac{1}{2}$, so

$$2\int_0^1 \frac{ds}{\sqrt{1-\gamma s}\sqrt{1-s^2}} \leqslant 2\sqrt{6} + 2\sqrt{3}\log 2\left(\frac{1-\varepsilon}{\varepsilon}\right)$$

and by (19)

$$J \leqslant \pi + 2\sqrt{6} + 2\sqrt{3}\log 2\left(\frac{1-\varepsilon}{\varepsilon}\right). \tag{20}$$

We now estimate the term (c). Using (20) we have

$$(c) \;\leqslant \frac{C_4}{\varrho}\left[\left(\sqrt{t^2-\varrho^2} - (t-\varrho)\right) + \int_{t-\varrho}^{\sqrt{t^2-\varrho^2}} \left(\log\frac{2}{\varepsilon}(1-\varepsilon)\right)^2 dr\right]$$

$$\leqslant C_4\left(1 + \frac{1}{\varrho}\int_t^{\varrho+\sqrt{t^2-\varrho^2}} \left(\log\frac{u-t}{\mu}\right)^2 du\right)$$

where

$$\mu = \frac{2\varrho}{t}\left(t - \varrho + \sqrt{t^2-\varrho^2}\right).$$

Since $\varrho < t$ it is easy to find that

$$(c) \;\leqslant C_5. \tag{21}$$

Estimation of (d). In this case we have $r \in (\sqrt{t^2-\varrho^2}, t+\varrho)$ and $\lambda(r) = -\cos\alpha$ is in $(-1, 0)$, so by making the same change of variables as in the previous paragraph we obtain

$$J = \frac{2\sqrt{2}}{\sqrt{3+\cos\alpha}} \int_{-1}^{1} \frac{ds}{\sqrt{1-s^2}\sqrt{1+\gamma s}}$$

with $\cos\alpha \in (0, 1)$ and $\gamma \in \left(0, \frac{1}{3}\right)$. We then easily have

$$\begin{aligned} J &\leqslant 2\int_{-1}^{1} \frac{ds}{\sqrt{1-s^2}} \\ &\leqslant 2\pi \end{aligned}$$

and

$$\begin{aligned} (d) &= \frac{1}{2\varrho}\int_{\sqrt{t^2-\varrho^2}}^{t+\varrho} J^2\, dr \\ &\leqslant \frac{2\pi^2}{\varrho}\left(t+\varrho-\sqrt{t^2-\varrho^2}\right), \end{aligned}$$

so

$$(d) \leqslant C_6. \tag{22}$$

Finally combining (14), (18), (21) and (22) we obtain (13), and this ends the proof of Lemma 2.5. ■

Next, Lemma 2.5 and inequality (12) give (10) and complete the proof of Lemma 2.4. ■

End of the proof of Proposition 2.2:

According to Lemmas 2.3 and 2.4 and since the support of $v(t, \cdot)$ is contained in $B(0, t+R)$ for all positive t, we have for every positive α and every x in $\mathbb{R}^2$,

$$e^{\alpha v^2(t,x)} \leqslant e^{\alpha M}\left(\frac{t}{\varrho}\right)^{\alpha/4\pi} \qquad \text{if } 0<\varrho<t$$

and

$$e^{\alpha v^2(t,x)} \leqslant e^{\alpha/2\pi} \qquad \text{if } \varrho \geqslant t\,. \tag{23}$$

So by (23) and since $\alpha < 8\pi$ we obtain for every positive t:

$$\begin{aligned} \int_{B(0,t+R)} e^{\alpha v^2(t,x)}\, dx &\leqslant 2\pi\left[\int_0^t \varrho\left(\frac{t}{\varrho}\right)^{\alpha/4\pi} e^{\alpha M}\, d\varrho + \int_t^{t+R} \varrho e^{\alpha/2\pi}\, d\varrho\right] \\ &\leqslant C_1\left(\frac{t^2}{8\pi-\alpha} + (t+R)^2\right) \end{aligned}$$

and Proposition 2.2 is proved. ■

3. – Local existence for the semilinear problem.

In this section we will prove the Theorem 1.1 by using the estimates proved in section 2.

PROOF OF THE THEOREM 1. – To prove the existence of the solution we use a fixed point method in an appropriate space. We define the following space

$$B_0 = \left\{ \begin{array}{l} w \in \mathcal{C}^0([0, T_0], H^1),\ \partial_t w \in \mathcal{C}^0([0, T_0], L^2) \text{ such that} \\ \operatorname{supp} w(t, .) \subset B(0, t+R) \text{ for every } t \in [0, T_0], \\ w_{|t=0} = \partial_t w_{|t=0} = 0, \ \sup\limits_{t \in [0, T_0]} (\|\partial_t w(t, \cdot)\|_{L^2} + \|\nabla_x w(t, \cdot)\|_{L^2}) \leq \delta_0 \\ \text{and } \sup\limits_{t \in [0, T_0]} \int_{B(0, t+R)} (e^{\beta w^2} - 1)\, dx \leq \delta_1 \end{array} \right\},$$

where T_0, δ_0, β and δ_1 are positive constants which will be chosen later on. We then define

$$\varphi : B_0 \to B_0$$

$$w \mapsto \tilde{w}$$

where $\tilde{w}$ is the solution of

$$\left\{ \begin{array}{ll} \Box \tilde{w} & = -(v+w)\, e^{\alpha_0 (v+w)^2} = g \\ \tilde{w}_{|t=0} = \partial_t \tilde{w}_{|t=0} = 0 \end{array} \right. \tag{24}$$

and v is the solution of the linear problem (4). First of all, by using standard methods in solving linear evolution problems, it is well known that, for each w in B_0, the problem (24) has a unique solution $\tilde{w} \in \mathcal{C}^0([0, T_0], H^1) \cap \mathcal{C}^1([0, T_0], L^2)$ which satisfies the energy estimate

$$\sup_{t \in [0, T_0]} (\|\partial_t \tilde{w}(t, \cdot))\|_{L^2} + \|\nabla \tilde{w}(t, \cdot)\|_{L^2} \leq C\|g\|_{L^1([0, T_0], L^2)}, \tag{25}$$

where C is an absolute constant.

The constants appearing in the definition of B_0 will be chosen such that φ is well defined and is a contraction. The proof is divided in three steps.

STEP 1. – Energy estimate:
According to the definition of g in (24), we have for every positive t and every w

in B_0:

$$\|g(t,\cdot)\|_{L^2}^2 = \int_{B(0,\,t+R)} (v+w)^2 e^{2\alpha_0 (v+w)^2}\,dx.$$

Let γ be fixed in $(2\alpha_0, 8\pi)$ and ε so small such that $\dfrac{8\pi}{\gamma(1+\varepsilon^2)} > 1$. Then we take p and β satisfying

$$1 < p < \frac{8\pi}{\gamma(1+\varepsilon^2)} \quad \text{and } \beta > \frac{p}{p-1}\gamma\left(1+\frac{1}{\varepsilon^2}\right). \tag{26}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
\|g(t,\cdot)\|_{L^2}^2 &\leqslant \frac{1}{e(\gamma-2\alpha_0)} \int_{B(0,\,t+R)} e^{\gamma(v+w)^2}\,dx \\
&\leqslant K \int_{B(0,\,t+R)} e^{\gamma[(1+\varepsilon^2)v^2 + (1+\frac{1}{\varepsilon^2})w^2]}\,dx \\
&\leqslant K\left[\int_{B(0,\,t+R)} e^{\gamma(1+\varepsilon^2)v^2}(e^{\gamma(1+\frac{1}{\varepsilon^2})w^2}-1)\,dx + \int_{B(0,\,t+R)} e^{\gamma(1+\varepsilon^2)v^2}\,dx\right] \\
&\leqslant K\left[\left(\int_{B(0,\,t+R)} e^{\gamma p(1+\varepsilon^2)v^2}\,dx\right)^{1/p}\left(\int_{B(0,\,t+R)} (e^{\gamma(1+\frac{1}{\varepsilon^2})w^2}-1)^q\,dx\right)^{1/q}\right. \\
&\qquad \left. + \int_{B(0,\,t+R)} e^{\gamma(1+\varepsilon^2)v^2}\,dx\right],
\end{aligned}$$

where $K = \dfrac{1}{e(\gamma-2\alpha_0)}$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. Using the fact that

$$(e^{\gamma(1+\frac{1}{\varepsilon^2})w^2}-1)^q \leqslant e^{\gamma q(1+\frac{1}{\varepsilon^2})w^2}-1,$$

we obtain:

$$\begin{aligned}
\|g(t,\cdot)\|_{L^2}^2 \leqslant K&\left[\left(\int_{B(0,\,t+R)} e^{\gamma p(1+\varepsilon^2)v^2}\,dx\right)^{1/p}\left(\int_{B(0,\,t+R)} (e^{\gamma q(1+\frac{1}{\varepsilon^2})w^2}-1)\,dx\right)^{1/q}\right. \\
&\left. + \int_{B(0,\,t+R)} e^{\gamma(1+\varepsilon^2)v^2}\,dx\right].
\end{aligned}$$

We take $T_0 \in (0,1)$ and set $\delta_1 = 2\pi e^{4\pi M} + (e-1)\pi(1+R)^2$ where M is the constant appearing in Lemma 2.4. Since $w \in B_0$, it follows by (5) and (26) that

$$\int_{B(0,\,t+R)} e^{\gamma p(1+\varepsilon^2)v^2}\,dx \leqslant C_1\left(\frac{t^2}{8\pi-\gamma p(1+\varepsilon^2)}+(t+R)^2\right)$$

and

$$\int_{B(0,\,t+R)} (e^{\gamma q(1+\frac{1}{\varepsilon^2})w^2}-1)\,dx \leqslant \delta_1.$$

So

$$\|g(t,\cdot)\|_{L^2}^2 \leqslant KC_1\left[\left(\frac{t^2}{8\pi-\gamma p(1+\varepsilon^2)}+(t+R)^2\right)^{1/p}\delta_1^{1/q}+\frac{t^2}{8\pi-\gamma(1+\varepsilon^2)}+(t+R)^2\right]$$

$$KC_1\left[\left(\frac{1}{8\pi-\gamma p(1+\varepsilon^2)}+(1+R)^2\right)^{1/p}\delta_1^{1/q}+\frac{1}{8\pi-\gamma(1+\varepsilon^2)}+(1+R)^2\right].$$

Thus

$$\|g(t,\cdot)\|_{L^2}^2 \leqslant KC_1K' \tag{27}$$

and

$$\|g\|_{L^1([0,\,T_0],\,L^2)} \leqslant T_0(KK'C_1)^{1/2}. \tag{28}$$

We finally choose $\delta_0^2 = T_0$ and T_0 so small that

$$(KK'C_1T_0)^{1/2} \leqslant \frac{1}{C} \tag{29}$$

where C is the constant appearing in the energy inequality (25). Therefore (25), (28) and (29) imply

$$\|g\|_{L^1([0,\,T_0],\,L^2)} \leqslant \frac{\sqrt{T_0}}{C} = \frac{\delta_0}{C}$$

and

$$\sup_{[0,\,T_0]}(\|\partial_t\widetilde{w}(t,\cdot)\|_{L^2}+\|\nabla_x\widetilde{w}(t,\cdot)\|_{L^2}) \leqslant \delta_0. \tag{30}$$

STEP 2. – In this paragraph we will prove that the solution $\widetilde{w}$ satisfies

$$\sup_{[0,\,T_0]}\int_{B(0,\,t+R)}(e^{\beta\widetilde{w}^2}-1)\,dx \leqslant \delta_1.$$

On one hand, according to (27) and (29) we have:

$$\|g(t,\cdot)\|_{L^2} \leqslant \frac{1}{C\sqrt{T_0}}. \tag{31}$$

On the other hand, using the Hadamard representation, the solution $\widetilde{w}$ of the problem (24) is given by

$$\widetilde{w}(t,x) = \int_0^t E(t-s,x) * g(s,x)\, ds,$$

where

$$E(t,x) = \begin{cases} \dfrac{1}{2\pi}(t^2-|x|^2)^{-1/2}, & \text{if } |x|<t \\ 0 & \text{if } |x| \geqslant t. \end{cases}$$

Applying Lemmas 2.4 and 2.5 to the function $h(t,s,x) = E(t-s) * g(s,x)$ instead of v, we get for $0<s<t$

$$h(t,s,x) = \begin{cases} \|g(s,\cdot)\|_{L^2}\left(M + \dfrac{1}{4\pi}\log\dfrac{t-s}{\varrho}\right)^{1/2} & \text{if } \varrho < t-s \\ \|g(s,\cdot)\|_{L^2}\left(\dfrac{t-s}{2\pi\varrho}\right)^{1/2} & \text{if } \varrho \geqslant t-s. \end{cases}$$

So for $\varrho < t$ and using (31) we have

$$\begin{aligned} |\widetilde{w}(t,x)| &\leqslant \sup_{s\in[0,t]} \|g(s,\cdot)\|_{L^2}\left[\int_0^{t-\varrho}\left(M+\frac{1}{4\pi}\log\frac{t-s}{\varrho}\right)^{1/2} ds + \int_{t-\varrho}^{t}\left(\frac{t-s}{2\pi\varrho}\right)^{1/2} ds\right] \\ &\leqslant \frac{\varrho}{C\sqrt{T_0}}\left[\int_1^{t/\varrho}\left(M+\frac{1}{4\pi}\log y\right)^{1/2} dy + \frac{1}{\sqrt{2\pi}}\right] \\ &\leqslant \frac{\varrho}{C\sqrt{T_0}}\left[M^{1/2}\left(\frac{t}{\varrho}-1\right) + \frac{t}{\varrho\sqrt{4\pi}}\left(\log\frac{t}{\varrho}\right)^{1/2} + \frac{1}{\sqrt{2\pi}}\right]. \end{aligned}$$

We deduce that

$$|\widetilde{w}(t,x)| \leqslant \frac{t}{C\sqrt{T_0}}\left[\frac{1}{\sqrt{4\pi}}\left(\log\frac{t}{\varrho}\right)^{1/2} + M^{1/2}\right]. \tag{32}$$

For $\varrho \geqslant t$,

$$|\widetilde{w}(t, x)| \leqslant \sup_{s \in [0, t]} \|g(s, \cdot)\|_{L^2} \int_0^t \left(\frac{t-s}{2\pi\varrho} \right)^{1/2} ds .$$

So

$$|\widetilde{w}(t, x)| \leqslant \frac{\sqrt{2}}{3C\sqrt{T_0 \pi \varrho}} t^{3/2}. \tag{33}$$

(31), (32) and (33) then yield for $\varrho < t \leqslant T_0$:

$$e^{\beta \widetilde{w}^2} \leqslant e^{\frac{2\beta T_0}{C^2}[\frac{1}{4\pi}(\log \frac{t}{\varrho} + M)]}$$

and

$$e^{\beta \widetilde{w}^2} \leqslant e^{\frac{2M\beta T_0}{C^2}} \left(\frac{t}{\varrho} \right)^{\beta T_0 / 2\pi C^2} . \tag{34}$$

For $\varrho \geqslant t$

$$e^{\beta \widetilde{w}^2} \leqslant e^{\frac{2\beta T_0}{9\pi C^2}} . \tag{35}$$

We choose again T_0 so small that

$$\frac{\beta T_0}{2\pi C^2} < 1 , \tag{36}$$

so (34), (35), (36), $T_0 < 1$ and the definition of δ_1 imply

$$\begin{aligned} \int_{B(0, t+R)} (e^{\beta \widetilde{w}^2} - 1)\, dx &\leqslant 2\pi \int_0^t e^{4\pi M} t d\varrho + 2\pi e \int_t^{t+R} \varrho d\varrho \\ &\quad - \pi(t+R)^2 \leqslant 2\pi e^{4\pi M} + (e-1)\,\pi(1+R)^2 = \delta_1 . \end{aligned}$$

STEP 3. – In this section we prove that φ is a contraction. Let us define the energy norm by

$$\|u\|_{\mathcal{E}} = \sup_{t \in [0, T_0]} (\|\partial_t u(t, \cdot)\|_{L^2} + \|\nabla_x u(t, \cdot)\|_{L^2}).$$

For every w_1 and w_2 in B_0 we have by the energy inequality

$$\|\varphi(w_1) - \varphi(w_2)\|_{\mathcal{E}} \leqslant C \|(v + w_1)\, e^{\alpha_0 (v + w_1)^2} - (v + w_2)\, e^{\alpha_0 (v + w_2)^2}\|_{L^1(L^2)} . \tag{37}$$

If we note for $i = 1, 2$

$$g_i = (v + w_i)\, e^{\alpha_0 (u + w_i)^2},$$

we then have

$$\|g_1 - g_2\|_{L^2}^2 = \int_{B(0,\, t+R)} (w_1 - w_2)^2 (1 + 2\alpha \overline{w}^2)^2 e^{2\alpha_0 \overline{w}^2}\, dx\,,$$

where $\overline{w} = v + \theta w_1 + (1 - \theta)\, w_2,\ \theta \in [0, 1]$.

So for γ fixed in $(2\alpha, 8\pi)$, p' and q' in $(1, +\infty)$ such that $\frac{1}{p'} + \frac{1}{q'} = 1$, we obtain by Hölder inequality

$$\|g_1 - g_2\|_{L^2}^2 \leqslant K_1 \|w_1 - w_2\|_{L^{2p'}}^2 \left(\int_{B(0,\, t+R)} e^{\gamma q' \overline{w}^2}\, dx \right)^{1/q'}, \tag{38}$$

where K_1 is an absolute constant.

On the other hand using again the Hölder inequality, we have for every positive ε

$$\begin{aligned}
\int_{B(0,\, t+R)} e^{\gamma q' \overline{w}^2}\, dx \leqslant & \int_{B(0,\, t+R)} e^{\gamma q' (1+\varepsilon^2) v^2} \Big[\big(e^{2\gamma q' (1 + \frac{1}{\varepsilon^2}) w_1^2} - 1 \big) \big(e^{2\gamma q' (1 + \frac{1}{\varepsilon^2}) w_2^2} - 1 \big) \Big] \\
& + \sum_{i=1}^{2} \int_{B(0,\, t+R)} e^{\gamma q' (1+\varepsilon^2) v^2 + 2\gamma q' (1 + \frac{1}{\varepsilon^2}) w_i^2}\, dx \\
& \leqslant \| e^{\gamma q' (1+\varepsilon^2) v^2} \|_{L^s} \Bigg[\left(\int_{B(0,\, t+R)} \big(e^{2\gamma q' r (1 + \frac{1}{\varepsilon^2}) w_1^2} - 1 \big)\, dx \right)^{1/r} \\
& \times \left(\int_{B(0,\, t+R)} \big(e^{2\gamma q' l (1 + \frac{1}{\varepsilon^2}) w_2^2} - 1 \big)\, dx \right)^{1/l} \\
& + \sum_{i=1}^{2} \left(\int_{B(0,\, t+R)} \big(e^{2\gamma q' k (1 + \frac{1}{\varepsilon^2}) w_i^2} - 1 \big)\, dx \right)^{1/k} \Bigg] \\
& + 2 \int_{B(0,\, t+R)} e^{\gamma q' (1+\varepsilon^2) v^2}\, dx,
\end{aligned}$$

where $\frac{1}{s} + \frac{1}{l} + \frac{1}{r} = 1$ and $\frac{1}{k} = \frac{1}{l} + \frac{1}{r}$.

We then choose $q' s = p$, defined in step 1 and we take β again sufficiently large in order that:

$$\begin{cases} 2\gamma q'\left(1+\dfrac{1}{\varepsilon^2}\right) r & <\beta \\ 2\gamma q'\left(1+\dfrac{1}{\varepsilon^2}\right) l & <\beta . \end{cases}$$

Using (5) and the fact that w_1 and w_2 are in B_0 we obtain:

(39) $$\int_{B(0,\, t+R)} e^{\gamma q' \overline{w}^2}\, dx \leqslant$$

$$C_1\left[3\left(\frac{T_0^2}{8\pi-\gamma p(1+\varepsilon^2)}+(T_0+R)^2\right)\delta_1^{1/k}+2\left(\frac{T_0^2}{8\pi-\gamma q'(1+\varepsilon^2)}+(T_0+R)^2\right)\right]=C_1 K_2$$

So by (38) and (39) we have

(40) $$\|g_1-g_2\|_{L^2}^2 \leqslant K_1 (C_1 K_2)^{1/q'} \|w_1-w_2\|_{L^{2p'}}^2 .$$

Since $\operatorname{supp} w_i \subset B(0,\, T_0+R) \subset B(0,\, 1+R)$, we have for $0<t\leqslant 1+R$

(41) $$\|w_1-w_2\|_{L^{2p'}} \leqslant C_{p'} \|\nabla_x (w_1-w_2)\|_{L^2},$$

where $C_{p'}$ depends only on R.

Combining (40) and (41) we then have for positive t

$$\|g_1-g_2\|_{L^2} \leqslant C_{p'} K_1^{1/2} (C_1 K_2)^{1/2q'} \|\nabla_x (w_1-w_2)\|_{L^2}$$

and

(42) $$\|g_1-g_2\|_{L^1([0,\, T_0],\, L^2)} \leqslant T_0 C_{p'} K_1^{1/2} (C_1 K_2)^{1/2q'} \|\nabla_x (w_1-w_2)\|_{L^2}.$$

Then, (37) and (42) imply

$$\|\varphi(w_1)-\varphi(w_2)\|_8 \leqslant T_0 C C_{p'} \sqrt{K_1} (C_1 K_2)^{1/2q'} \|w_1-w_2\|_8 .$$

Chosing again T_0 sufficiently small such that

$$T_0 C C_{p'} \sqrt{K_1} (C_1 K_2)^{1/2q'} < 1,$$

φ is a contraction and Theorem 1 is proved. ■

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