Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **7-B** (2004), n.1, p. 109–128.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2004_8_7B_1_109_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2004.

Bollettino U. M. I.
(8) **7**-B (2004), 109-128

Global Weak Solutions for a Degenerate Parabolic System Modelling a One-Dimensional Compressible Miscible Flow in Porous Media.

Y. AMIRAT - A. ZIANI

Sunto. – *Proviamo la risolubilità di un sistema parabolico non lineare degenere costituito da due equazioni che descrivono lo spostamento di un fluido compressibile, causato da un altro fluido, completamente miscibile al primo, in un mezzo poroso unidimensionale, trascurando la diffusione molecolare. Usiamo la tecnica delle soluzioni rinormalizzate per le equazioni paraboliche al fine di ottenere stime a priori per soluzioni di tipo viscosità. Passiamo al limite nel sistema parabolico, quando il coefficiente di diffusione molecolare tende a zero, tramite metodi di compattezza per compensazione.*

Summary. – *We show the solvability of a nonlinear degenerate parabolic system of two equations describing the displacement of one compressible fluid by another, completely miscible with the first, in a one-dimensional porous medium, neglecting the molecular diffusion. We use the technique of renormalised solutions for parabolic equations in the derivation of a priori estimates for viscosity type solutions. We pass to the limit, as the molecular diffusion coefficient tends to 0, on the parabolic system, owing to compensated compactness arguments.*

1. – Statement of the problem and main result.

Let $T > 0$ and let $\Omega = (0, 1)$. We set $\Omega_T = \Omega \times (0, T)$. In [2], we investigated the existence of global weak solutions to the nonlinear initial boundary value problem of parabolic type:

$$\phi a(c)\,\partial_t p + \partial_x u = 0, \quad u = -\frac{\kappa}{\mu(c)}\partial_x p \ \text{ in } \ \Omega_T, \tag{1.1}$$

$$u(0, t) = u(1, t) = 0, \quad p(x, 0) = p_0(x), (x, t) \in \Omega_T, \tag{1.2}$$

$$\phi\partial_t c + u\partial_x c + \phi b(c)\,\partial_t p - \partial_x(\phi(d_m + d_p\,|u|)\,\partial_x c) = 0 \ \text{ in } \ \Omega_T, \tag{1.2}$$

$$(d_m + d_p\,|u|)\,\partial_x c(y, t) = 0 \ (y = 0, 1), \quad c(x, 0) = c_0(x), \ (x, t) \in \Omega_T. \tag{1.4}$$

This system describes the displacement of one compressible fluid by another, completely miscible with the first, in a one-dimensional porous medium, see for instance Douglas and Roberts [7], Bear [4], Peaceman [13], Scheidegger [15], and the references therein. Here p is the pressure, c is the concentration of one of the two components of the fluid mixture, u is the Darcy velocity, $\kappa = \kappa(x)$ is the permeability of the medium, $\phi = \phi(x)$ is the porosity, $\mu = \mu(c)$ is the concentration-dependent viscosity of the fluid mixture, d_m and d_p are the molecular diffusion and the dispersion constants, respectively. The viscosity $\mu = \mu(c)$ in (1.1) is assumed to be determined by some mixing rule. For instance, in the Koval model, μ is defined on the interval $(0, 1)$ as the «quarter power mixing rule»,

$$\mu(c) = \mu(0)\big(1 + (M^{1/4} - 1)\, c\big)^{-4} \tag{1.5}$$

where $M = \mu(0)/\mu(1)$ is the mobility ratio, $M > 1$. The functions a and b are defined on the interval $(0, 1)$ by

$$a(c) = z_2 + \Delta zc, \qquad b(c) = \Delta zc(1-c) \ \text{ with } \Delta z = z_1 - z_2,$$

where z_l is the constant compressibility factor for the l-th component $(l = 1, 2)$, $z_1 \geqslant z_2 > 0$.

In this paper we make the following assumptions:

(i) The functions p_0 and c_0 satisfy

$$p_0 \in H^1(\Omega), \quad c_0 \in L^\infty(\Omega), \quad p_0'(x) \leqslant 0, \quad 0 \leqslant c_0(x) \leqslant 1 \ \text{ a.e. in } \Omega. \tag{1.6}$$

(ii) The functions μ and $1/\mu$ are convex and such that

$$\mu \in \mathcal{C}^2([0, 1]), \quad 0 < \mu_- \leqslant \mu(c) \leqslant \mu^+ \quad \forall c \in (0, 1), \tag{1.7}$$

where μ_- and μ^+ are two fixed real numbers. Obviously, the function μ defined by (1.5) satisfies (1.7).

For the purpose of simplifying the discussion, we suppose in addition the porosity and the permeability of the medium constant and equal to 1. We have proved in [2] the global existence of solutions to the problem (1.1)-(1.1) when we take into account both diffusion and dispersion terms ($d_m > 0$, $d_p > 0$). More precisely, the following result was established in [2].

THEOREM 1.1. – *Assume* (1.6) *and* (1.7) *hold, and* $\phi = 1$, $\kappa = 1$, $d_m > 0$, *and* $d_p > 0$. *Then problem* (1.1)-(1.4) *admits a weak solution, i.e. there is a pair* (p, c) *satisfying the following conditions:*

(i) $p \in L^\infty(0, T; W^{1,1}(\Omega)) \cap W^{1,\theta}(\Omega_T)$, *for any* θ, $1 \leqslant \theta < 3/2$, *and* p *is a solution of* (1.1)-(1.2) *verified in* $L^\theta(\Omega_T)$;

(ii) $c \in L^2(0, T; H^1(\Omega))$, $0 \leq c(x, t) \leq 1$ *for almost every* (x, t) *in* Ω_T, *and* c *is a weak solution of* (1.3)-(1.4), *that is* c *satisfies the integral identity*

$$-\int_{\Omega_T} c\partial_t \varphi \, dx\, dt + \int_{\Omega_T} (u\partial_x c + b(c)\, \partial_t p)\, \varphi \, dx\, dt$$
$$+ \int_{\Omega_T} (d_m + d_p\, |u|)\, \partial_x c\, \partial_x \varphi \, dx\, dt = \int_{\Omega} c_0(x)\, \varphi(x, 0)\, dx\,,$$

for any testing function φ *in* $\mathcal{C}^1(\overline{\Omega}_T)$ *with support contained in* $\overline{\Omega} \times [0, T[$.

Let us also mention some related papers to problem (1.1)-(1.4). Neglecting the dispersion effect ($d_p = 0$ and $d_m > 0$), Feng [8] proved a local existence of strong solutions to the problem (1.1)-(1.4). Amirat, Hamdache, and Ziani [1] have obtained an existence result of global weak solutions to compressible miscible flow in three-space porous medium, in the case of viscosity independent of the concentration («unit mobility» case) but the molecular diffusion and dispersion terms may vanish.

It is well known that in porous media flow the molecular dispersion is more important physically than the molecular diffusion, see Bear [4], Chou and Li [6], Pearson and Tardy [14], Wheeler [17], and Young [18]. This motivates the study of existence of solutions to the problem (1.1)-(1.4) when the molecular diffusion term is neglected. In the sequel, we fix the dispersion constant d_p equal to 1, take the molecular diffusion $d_m = \varepsilon$ with $0 < \varepsilon \ll 1$, and denote $p = p^\varepsilon$, $u = u^\varepsilon$, and $c = c^\varepsilon$ the corresponding weak solution constructed in [2]. Then we examine the asymptotic behavior, as ε goes to zero, of the solutions $(p^\varepsilon, c^\varepsilon)$. We note, as stated in Theorem 1.1 that,

$$0 \leq c^\varepsilon(x, t) \leq 1 \text{ a.e. in } \Omega_T, \tag{1.8}$$

so according to the definition of the function a, we have

$$z_2 \leq a(c^\varepsilon(x, t)) \leq z_1 \text{ a.e. in } \Omega_T. \tag{1.9}$$

We will prove the existence of a weak solution, in the sense hereafter, to the problem (1.1)-(1.4) in the case $d_m = 0$. It is a degenerate parabolic system which writes:

$$a(c)\, \partial_t p + \partial_x u = 0, \quad u = -\frac{1}{\mu(c)}\, \partial_x p \text{ in } \Omega_T, \tag{1.10}$$

$$u(0, t) = u(1, t) = 0, \quad p(x, 0) = p_0(x),\ (x, t) \in \Omega_T, \tag{1.11}$$

$$\partial_t c + u\partial_x c + b(c)\, \partial_t p - \partial_x(|u|\, \partial_x c) = 0 \text{ in } \Omega_T, \tag{1.12}$$

$$|u|\, \partial_x c(y, t) = 0 \ (y = 0, 1), \quad c(x, 0) = c_0(x),\ (x, t) \in \Omega_T. \tag{1.13}$$

DEFINITION 1.1. – *A pair* (p, c) *is said to be a weak solution of problem* (1.10)-(1.13) *if:*

(i) $p \in L^\infty(0, T; W^{1,1}(\Omega)) \cap W^{1,\theta}(\Omega_T)$, *for any* θ, $1 \leqslant \theta < 3/2$, *and* p *is a solution of* (1.10)-(1.11) *verified in* $L^\theta(\Omega_T)$;

(ii) $c \in L^\infty(\Omega_T)$, $0 \leqslant c(x, t) \leqslant 1$ *for almost every* (x, t) *in* Ω_T *with* $|u|^{1/2} \partial_x c \in L^2(\Omega_T)$, *and* c *is a weak solution of* (1.12)-(1.13), *that is* c *satisfies the integral identity*

$$-\int_{\Omega_T} c\partial_t \varphi \, dx\, dt + \int_{\Omega_T} (u\partial_x c + b(c)\, \partial_t p)\, \varphi \, dx\, dt + \int_{\Omega_T} |u|\, \partial_x c \partial_x \varphi \, dx\, dt = \int_\Omega c_0(x)\, \varphi(x, 0)\, dx\,, \tag{1.14}$$

for any testing function φ *in* $\mathcal{C}^1(\overline{\Omega}_T)$ *with support contained in* $\overline{\Omega} \times [0, T[$.

Our main result is the following one.

THEOREM 1.2. – *Assume* (1.6) *and* (1.7) *hold, and* $\phi = 1$, $\kappa = 1$, $d_p = 1$, *and* $d_m = \varepsilon$ *in* (1.1)-(1.4) *and let* $(p^\varepsilon, c^\varepsilon)$ *denote a corresponding weak solution to* (1.1)-(1.4). *Then, there are extracted subsequences from* (c^ε) *and* (p^ε), *not relabelled for convenience, such that, as* $\varepsilon \to 0$,

$$p^\varepsilon \rightharpoonup p \ \text{weakly in}\ W^{1,\theta}(\Omega_T), \qquad c^\varepsilon \rightharpoonup c \ \text{in}\ L^\infty(\Omega_T)\ \text{weak-}*,$$

for any $1 \leqslant \theta < 3/2$, *and the pair* (p, c) *is a weak solution to* (1.10)-(1.13) *in the sense of Definition* 1.1. *Furthermore, the function* $u = -\dfrac{1}{\mu(c)}\partial_x p$ *belongs to* $L^\theta(0, T; W^{1,\theta}(\Omega)) \cap L^{2\theta}(\Omega_T)$.

The rest of the paper is devoted to the proof of this result.

2. – Proof of the result.

The proof consists in two parts. In the first one we derive some estimates for p^ε, u^ε, and c^ε, that are independent of ε. We mainly use the techniques of renormalized solutions for parabolic equations, as in Boccardo and Gallouët [5] and Murat [11]. In the second part we pass to the limit, as $\varepsilon \to 0$, using compensated compactness techniques.

2.1. *Some estimates.*

Let $(p^{\varepsilon}, c^{\varepsilon})$ be the weak solution to (1.1)-(1.4), constructed in [2]. Then the flux function $u^{\varepsilon} = -\partial_x p^{\varepsilon}/\mu(c^{\varepsilon})$ is a weak solution to:

$$\partial_t(\mu(c^{\varepsilon})\, u^{\varepsilon}) - \partial_x\left(\frac{1}{a(c^{\varepsilon})}\,\partial_x u^{\varepsilon}\right) = 0 \ \text{ in } \Omega_{\mathrm{T}}, \tag{2.1}$$

$$u^{\varepsilon}(0, t) = 0, \quad u^{\varepsilon}(1, t) = 0 \ \text{ for } t \in (0, T), \tag{2.2}$$

$$(\mu(c^{\varepsilon})\, u^{\varepsilon})(x, 0) = -p_0'(x) \ \text{ for } x \in \Omega\,. \tag{2.3}$$

In the sequel, we will use often C to represent a generic positive constant depending only on fixed data.

We first have the following result.

LEMMA 2.1. – *The function u^{ε} is nonnegative and the sequence (u^{ε}) is bounded in $L^{\infty}(0, T; L^1(\Omega))$.*

We sketch formally the proof. We can make precise the arguments employed by considering a sequence (c_m^{ε}) of regularized functions that converges, as $m \to \infty$, to c^{ε} in $L^{\infty}(0, T; L^2(\Omega))$ as done in [2].

Let us prove that (u^{ε}) is bounded in $L^{\infty}(0, T; L^1(\Omega))$. Let $\eta > 0$ and let $\mathrm{sign}_{\eta}(s)$ and $\mathrm{sign}\,(s)$ denote the following real-valued functions of the real variable s:

$$\mathrm{sign}_{\eta}(s) = \frac{s}{(s^2 + \eta)^{1/2}}, \qquad \mathrm{sign}\,(s) = \begin{cases} 1 & \text{if } s \geq 0, \\ -1 & \text{if } s < 0\,. \end{cases}$$

Obviously,

$$\mathrm{sign}\,(s) = \lim_{\eta \to 0^+} \mathrm{sign}_{\eta}(s) \quad \text{for } \neq 0\,.$$

We multiply (2.1) by $\mathrm{sign}_{\eta}(u^{\varepsilon})$ and integrate over Ω. This gives

$$\int_{\Omega} \partial_t(\mu(c^{\varepsilon})\, u^{\varepsilon})\, \mathrm{sign}_{\eta}(u^{\varepsilon})\, dx + \eta \int_{\Omega} \frac{1}{a(c^{\varepsilon})}\, \frac{|\partial_x u^{\varepsilon}|^2}{(|u^{\varepsilon}|^2 + \eta)^{3/2}}\, dx = 0.$$

Since the second term is nonnegative, we have

$$\int_{\Omega} \partial_t(\mu(c^{\varepsilon})\, u^{\varepsilon})\, \mathrm{sign}_{\eta}(u^{\varepsilon})\, dx \leq 0.$$

Letting $\eta \to 0$, yields

$$\int_{\Omega} \partial_t(\mu(c^{\varepsilon})\, u^{\varepsilon})\, \mathrm{sign}\,(u^{\varepsilon})\, dx \leq 0.$$

Since $\mu(c^\varepsilon) > 0$ a.e. in Ω_T, $\operatorname{sign}(u^\varepsilon) = \operatorname{sign}(\mu(c^\varepsilon)\, u^\varepsilon)$ so that the latter inequality gives

$$\frac{d}{dt}\int_\Omega (\mu(c^\varepsilon)\,|u^\varepsilon|)\, dx \leqslant 0$$

from which follows, using (1.7),

$$\mu_- \int_\Omega |u^\varepsilon(x,t)|\, dx \leqslant \int_\Omega |p_0'(x)|\, dx \qquad \text{for a.e. } t \in (0,T).$$

Therefore (u^ε) is bounded in $L^\infty(0,T;L^1(\Omega))$.

As usual, we denote

$$\operatorname{sign}_\eta^-(s) = \begin{cases} 0 & \text{if } s \geqslant 0, \\ -\operatorname{sign}_\eta(s) & \text{if } s < 0. \end{cases}$$

Then, multiplication of (2.1) by $\operatorname{sign}_\eta^-(u^\varepsilon)$ and integration over $\Omega_t = \Omega \times (0,t)$, $0 < t < T$, yields

$$\int_{\Omega_t} \partial_t(\mu(c^\varepsilon)\, u^\varepsilon)\operatorname{sign}_\eta^-(u^\varepsilon)\, dx\, ds + \eta\int_{\Omega_t} \frac{1}{a(c^\varepsilon)}\, \frac{|\partial_x (u^\varepsilon)^-|^2}{((u^\varepsilon)^2+\eta)^{3/2}}\, dx\, ds = 0\,.$$

Since the second term is nonnegative, we have

$$\int_{\Omega_t} \partial_t(\mu(c^\varepsilon)\, u^\varepsilon)\operatorname{sign}_\eta^-(u^\varepsilon)\, dx\, ds \leqslant 0.$$

Letting $\eta \to 0$, yields

$$\int_{\Omega_t} \partial_t(\mu(c^\varepsilon)\, u^\varepsilon)\operatorname{sign}^-(u^\varepsilon)\, dx\, ds \leqslant 0.$$

Since $\operatorname{sign}^-(u^\varepsilon) = \operatorname{sign}^-(\mu(c^\varepsilon)\, u^\varepsilon)$, the latter inequality gives

$$\frac{d}{dt}\int_{\Omega_t} \mu(c^\varepsilon)(u^\varepsilon)^-\, dx\, ds \leqslant 0.$$

Then, owing to (1.6) since $u_0(x) = -\frac{1}{\mu(c_0(x))} p_0'(x)$ for a.e. $t \in (0,T)$, we get

$$\int_\Omega \mu(c^\varepsilon(x,t))(u^\varepsilon)^-(x,t)\, dx \leqslant \int_\Omega \mu(c^\varepsilon(x,0))\, u_0^-(x)\, dx = 0 \ \text{ a.e. in } (0,T).$$

It follows $(u^\varepsilon)^- = 0$ a.e. in Ω_T, *i.e.* $u^\varepsilon \geqslant 0$ a.e. in Ω_T. The proof is complete. ∎

Hereafter, we drop the absolute value of u^ε. Let us now establish the following estimates. We will use often C to represent a generic constant estimated in terms of known quantities and differing from one inequality to another.

LEMMA 2.2. – *The following properties hold:*

(i) *the sequences* $(\sqrt{\varepsilon}\partial_x c^\varepsilon)$ *and* $((u^\varepsilon)^{1/2}\partial_x c^\varepsilon)$ *are bounded in* $L^2(\Omega_T)$;

(ii) *the sequence* (u^ε) *is bounded in* $L^\theta(0, T; W^{1,\theta}(\Omega)) \cap L^{2\theta}(\Omega_T)$;

(iii) *the sequence* (p^ε) *is bounded in* $W^{1,\theta}(\Omega_T)$;

(iv) *the sequences* (u^ε) *and* $(\partial_x p^\varepsilon)$ *are bounded in* $L^\sigma(0, T; L^s(\Omega))$, *where* $s \geqslant 1$ *is arbitrary and* $\sigma = s(2\theta - 1)/(s-1)$.

In (ii)-(iv), $1 \leqslant \theta < 3/2$ *is arbitrary.*

(i) We multiply equation (1.3) by c^ε and integrate over Ω. This gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |c^\varepsilon|^2 dx + \int_\Omega (\varepsilon + u^\varepsilon)|\partial_x c^\varepsilon|^2 dx + \int_\Omega c^\varepsilon u^\varepsilon \partial_x c^\varepsilon dx + \int_\Omega c^\varepsilon b(c^\varepsilon)\,\partial_t p^\varepsilon dx = 0.$$

According to (1.1) and $0 \leqslant c^\varepsilon(x, t) \leqslant 1$, it follows

$$\int_\Omega c^\varepsilon b(c^\varepsilon)\,\partial_t p^\varepsilon dx = -\int_\Omega g(c^\varepsilon)\,\partial_x u^\varepsilon dx = \int_\Omega g'(c^\varepsilon)\,u^\varepsilon \partial_x c^\varepsilon dx,$$

with $g(c^\varepsilon) = c^\varepsilon b(c^\varepsilon)/a(c^\varepsilon)$, and then $|g'(c^\varepsilon)| \leqslant C$. Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |c^\varepsilon|^2 dx + \int_\Omega (\varepsilon + u^\varepsilon)|\partial_x c^\varepsilon|^2 dx &\leqslant (C+1)\int_\Omega u^\varepsilon |\partial_x c^\varepsilon| dx \\ &\leqslant \delta\int_\Omega u^\varepsilon |\partial_x c^\varepsilon|^2 dx + C(\delta)\int_\Omega u^\varepsilon dx,\end{aligned}$$

for any $\delta > 0$, with $C(\delta) = (C+1)/4\delta$. Integrating with respect to t and choosing δ small enough, we conclude by Lemma 2.1 that the sequence $((\varepsilon + u^\varepsilon)^{1/2}\partial_x c^\varepsilon)$ is bounded in $L^2(\Omega_T)$.

(ii) We use (i) and Lemma 2.1 to improve the estimates for the flux function u^ε. The main tool is the technique of renormalized solutions for parabolic equations, following Boccardo and Gallouët [5] and Murat [11].

Let $m \geqslant 0$ be an integer. We define the odd function S_m on $\mathbb{R}$ by

$$S_m(z) = \begin{cases} 0 & \text{if } 0 \leqslant z \leqslant 2^m, \\ z - 2^m & \text{if } 2^m \leqslant z \leqslant 2^{m+1}, \\ 2^m & \text{if } z \geqslant 2^{m+1}, \end{cases}$$

and we define B_m^ε as the set

$$B_m^\varepsilon = \{(x, t) \in \Omega_T;\ 2^m \leqslant |\mu(c^\varepsilon(x, t))\, u^\varepsilon(x, t)| \leqslant 2^{m+1}\}.$$

We multiply (2.1) by $S_m(\mu(c^\varepsilon)\, u^\varepsilon)$ and integrate over Ω. It follows

$$\int_\Omega \partial_t(\mu(c^\varepsilon)\, u^\varepsilon)\, S_m(\mu(c^\varepsilon)\, u^\varepsilon)\, dx + \int_\Omega \frac{\mu(c^\varepsilon)}{a(c^\varepsilon)} S_m'(\mu(c^\varepsilon)\, u^\varepsilon)\, |\partial_x u^\varepsilon|^2\, dx =$$

$$-\int_\Omega \frac{\mu'(c^\varepsilon)}{a(c^\varepsilon)} S_m'(\mu(c^\varepsilon)\, u^\varepsilon)\, u^\varepsilon\, \partial_x u^\varepsilon\, \partial_x c^\varepsilon\, dx \text{ in } (0, T).$$

Let us introduce the function $\tilde{S}_m \colon \mathbb{R} \to \mathbb{R}$, defined as $\tilde{S}_m(z) = \int_0^z S_m(\xi)\, d\xi$. Integrating the latter relation over $(0, T)$ we obtain

$$\int_\Omega \tilde{S}_m(\mu(c^\varepsilon)\, u^\varepsilon)(x, T)\, dx + \int_{B_m^\varepsilon} \frac{\mu(c^\varepsilon)}{a(c^\varepsilon)} S_m'(\mu(c^\varepsilon)\, u^\varepsilon)\, |\partial_x u^\varepsilon|^2\, dx\, dt =$$

$$\int_\Omega \tilde{S}_m(\mu(c^\varepsilon)\, u^\varepsilon)(x, 0)\, dx - \int_{B_m^\varepsilon} \frac{\mu'(c^\varepsilon)}{a(c^\varepsilon)} S_m'(\mu(c^\varepsilon)\, u^\varepsilon)\, u^\varepsilon\, \partial_x u^\varepsilon\, \partial_x c^\varepsilon\, dx\, dt\,.$$

Since $\tilde{S}_m(\mu(c^\varepsilon)\, u^\varepsilon)(x, t) \geqslant 0$ and $(\mu(c^\varepsilon)\, u^\varepsilon)(x, 0) = -p_0'(x)$ in Ω, and $|\tilde{S}_m(z)| \leqslant 2^m |z|$, we have

$$\left| \int_\Omega \tilde{S}_m(\mu(c^\varepsilon)\, u^\varepsilon)(x, 0)\, dx \right| \leqslant 2^m \int_\Omega |p_0'(x)|\, dx.$$

Using the properties of S_m, $\mu(c^\varepsilon)$ and $a(c^\varepsilon)$, we obtain

$$\int_\Omega \tilde{S}_m(\mu(c^\varepsilon)\, u^\varepsilon)(x, T)\, dx + \int_{B_m^\varepsilon} \frac{\mu(c^\varepsilon)}{a(c^\varepsilon)} S_m'(\mu(c^\varepsilon)\, u^\varepsilon)\, |\partial_x u^\varepsilon|^2\, dx\, dt \leqslant$$

$$2^m \int_\Omega |p_0'(x)|\, dx + C \int_{B_m^\varepsilon} |u^\varepsilon|\, |\partial_x u^\varepsilon|\, |\partial_x c^\varepsilon|\, dx\, dt \leqslant$$

$$2^m \int_\Omega |p_0'(x)|\, dx + \delta \int_{B_m^\varepsilon} |\partial_x u^\varepsilon|^2\, dx\, dt + C(\delta) \int_{B_m^\varepsilon} |u^\varepsilon|^2\, |\partial_x c^\varepsilon|^2\, dx\, dt \leqslant$$

$$2^m \int_\Omega |p_0'(x)|\, dx + \delta \int_{B_m^\varepsilon} |\partial_x u^\varepsilon|^2\, dx\, dt + 2^m C(\delta) \int_{B_m^\varepsilon} |u^\varepsilon|\, |\partial_x c^\varepsilon|^2\, dx\, dt$$

for any real number $\delta > 0$, with $C(\delta) = C/4\delta$. This implies

$$\int_{B_m^\varepsilon} |\partial_x u^\varepsilon|^2 \, dx\, dt \leqslant 2^m \left(\int_{\Omega} |p_0'(x)|\, dx + C \int_{B_m^\varepsilon} |u^\varepsilon|\, |\partial_x c^\varepsilon|^2 \, dx\, dt \right),$$

and then, using (1.6) and the inequality $\int_{\Omega_T} |u^\varepsilon|\, |\partial_x c^\varepsilon|^2 \, dx\, dt \leqslant C$, it follows

$$\frac{1}{2^m} \int_{B_m^\varepsilon} |\partial_x u^\varepsilon|^2 \, dx\, dt \leqslant C. \tag{2.4}$$

Now, let θ be a real number, $1 < \theta < 2$. Using the Hölder inequality, we get

$$\int_{B_m^\varepsilon} |\partial_x u^\varepsilon|^\theta \, dx\, dt \leqslant \left(\int_{B_m^\varepsilon} |\partial_x u^\varepsilon|^2 \, dx\, dt \right)^{\theta/2} |B_m^\varepsilon|^{1-\theta/2},$$

where $|B_m^\varepsilon|$ denotes the measure of B_m^ε. We note that on B_m^ε we have $|u^\varepsilon| \geqslant C2^m$ where the constant C depends only on variations of the function μ. Therefore,

$$|B_m^\varepsilon| \leqslant \frac{C}{2^m} \int_{B_m^\varepsilon} |u^\varepsilon|\, dx\, dt.$$

By the Hölder inequality we also have

$$\int_{B_m^\varepsilon} |u^\varepsilon|\, dx\, dt \leqslant \left(\int_{B_m^\varepsilon} |u^\varepsilon|^s \, dx\, dt \right)^{1/s} |B_m^\varepsilon|^{1/s'},$$

for any s, $s' \geqslant 1$ with $1/s + 1/s' = 1$. Thus

$$|B_m^\varepsilon| \leqslant \frac{C}{2^{ms}} \left(\int_{B_m^\varepsilon} |u^\varepsilon|^s \, dx\, dt \right).$$

Therefore, in view of (2.4),

$$\int_{B_m^\varepsilon} |\partial_x u^\varepsilon|^\theta \, dx\, dt \leqslant \frac{C}{2^{m(s(1-\theta/2)-\theta/2)}} \left(\int_{B_m^\varepsilon} |u^\varepsilon|^s \, dx\, dt \right)^{1-\theta/2}.$$

Then, choosing

$$s > \frac{\theta}{2-\theta} \tag{2.5}$$

to get the serie convergent, we have

$$\sum_{m\geqslant 0}\int_{B_m^\varepsilon}|\partial_x u^\varepsilon|^\theta\,dx\,dt\leqslant C\sum_{m\geqslant 0}\frac{1}{2^{m(s(1-\theta/2)-\theta/2)}}\left(\int_{B_m^\varepsilon}|u^\varepsilon|^s\,dx\,dt\right)^{1-\theta/2}. \tag{2.6}$$

Using the discrete Hölder inequality

$$\sum_m a_m b_m\leqslant\Big(\sum_m a_m^{r'}\Big)^{1/r'}\Big(\sum_m b_m^r\Big)^{1/r},\qquad r,\ r'\geqslant 1,\ \frac{1}{r}+\frac{1}{r'}=1,$$

the right-hand side of (2.6) is majorized as follows:

$$\sum_{m\geqslant 0}\frac{1}{2^{m(s(1-\theta/2)-\theta/2)}}\left(\int_{B_m^\varepsilon}|u^\varepsilon|^s\,dx\,dt\right)^{1-\theta/2}\leqslant$$

$$\left(\sum_{m\geqslant 0}\frac{1}{2^{mr(s(1-\theta/2)-\theta/2)}}\right)^{1/r}\left(\sum_{m\geqslant 0}\left(\int_{B_m^\varepsilon}|u^\varepsilon|^s\,dx\,dt\right)^{r'(1-\theta/2)}\right)^{1/r'}.$$

We choose $r'=\dfrac{2}{2-\theta}$. We infer from (2.6)

$$\sum_{m\geqslant 0}\int_{B_m^\varepsilon}|\partial_x u^\varepsilon|^\theta\,dx\,dt\leqslant C\left(\sum_{m\geqslant 0}\int_{B_m^\varepsilon}|u^\varepsilon|^s\,dx\,dt\right)^{1-\theta/2}.$$

Now we define $\widetilde{B}_\varepsilon$ as the set

$$\widetilde{B}_\varepsilon=\{(x,t)\in\Omega_T;\ 0\leqslant|\mu(c^\varepsilon(x,t))\,u^\varepsilon(x,t)|\leqslant 1\}$$

so that $\Omega_T=\widetilde{B}_\varepsilon\cup\Big(\bigcup_{m\geqslant 0}B_m^\varepsilon\Big)$. We have to estimate $\int_{\widetilde{B}_\varepsilon}|\partial_x u^\varepsilon|^\theta\,dx\,dt$. To this purpose, we introduce the functions $R,\widetilde{R}:\mathbb{R}\to\mathbb{R}$ defined as:

$$R(z)=\begin{cases}1 & \text{if } z\geqslant 1,\\ z & \text{if } -1\leqslant z\leqslant 1,\\ -1 & \text{if } z\leqslant -1,\end{cases}\qquad \widetilde{R}(z)=\int_0^z R(\xi)\,d\xi.$$

Multiplying (2.1) by $R(\mu(c^\varepsilon)\,u^\varepsilon)$ and integrating over Ω_T, we get

$$\int_\Omega\widetilde{R}(\mu(c^\varepsilon)\,u^\varepsilon)(x,T)\,dx+\int_{\Omega_T}\frac{\mu'(c^\varepsilon)}{a(c^\varepsilon)}R'(\mu(c^\varepsilon)\,u^\varepsilon)\,|\partial_x u^\varepsilon|^2\,dx\,dt=$$

$$\int_\Omega\widetilde{R}(\mu(c^\varepsilon)\,u^\varepsilon)(x,0)\,dx-\int_{\Omega_T}\frac{\mu'(c^\varepsilon)}{a(c^\varepsilon)}R'(\mu(c^\varepsilon)\,u^\varepsilon)\,u^\varepsilon\,\partial_x u^\varepsilon\,\partial_x c^\varepsilon\,dx\,dt.$$

We have

$$\left| \int_{\Omega} \widetilde{R}(\mu(c^{\varepsilon})\, u^{\varepsilon})(x,\, 0)\, dx \right| \leqslant \int_{\Omega} |p_0'(x)|\, dx,$$

then, using the properties of $\widetilde{R}$, $\mu(c^{\varepsilon})$ and $a(c^{\varepsilon})$, it follows

$$\int_{\Omega} \widetilde{R}(\mu(c^{\varepsilon})\, u^{\varepsilon})(x,\, T)\, dx + \int_{\widetilde{B}} \frac{\mu'(c^{\varepsilon})}{a(c^{\varepsilon})} R'(\mu(c^{\varepsilon})\, u^{\varepsilon})\, |\partial_x u^{\varepsilon}|^2\, dx\, dt \leqslant$$

$$\int_{\Omega} |p_0'(x)|\, dx + \delta \int_{\widetilde{B}_{\varepsilon}} |\partial_x u^{\varepsilon}|^2\, dx\, dt + C(\delta) \int_{\widetilde{B}_{\varepsilon}} |u^{\varepsilon}|\, |\partial_x c^{\varepsilon}|^2\, dx\, dt$$

for any real number $\delta > 0$, with $C(\delta) = C/4\delta$. This implies

$$\int_{\widetilde{B}_{\varepsilon}} |\partial_x u^{\varepsilon}|^2\, dx\, dt \leqslant C.$$

We have established the following estimate

$$\int_{\Omega_T} |\partial_x u^{\varepsilon}|^{\theta}\, dx\, dt \leqslant C \left(1 + \left(\int_{\Omega_T} |u^{\varepsilon}|^{s}\, dx\, dt \right)^{1 - \theta/2} \right), \tag{2.7}$$

for any $s > \dfrac{\theta}{2 - \theta}$. We now use the Gagliardo-Nirenberg multiplicative embedding inequality for the flux function $u^{\varepsilon}(\cdot,\, t)$, which satisfies $u^{\varepsilon}(0,\, t) = u^{\varepsilon}(1,\, t) = 0$ for almost every t in $(0,\, T)$. We have

$$\left(\int_{\Omega} |u^{\varepsilon}|^{s}\, dx \right)^{1/s} \leqslant C \left(\int_{\Omega} |\partial_x u^{\varepsilon}|^{\theta}\, dx \right)^{\lambda/\theta} \left(\int_{\Omega} |u^{\varepsilon}|^{r}\, dx \right)^{(1-\lambda)/r}$$

with $r \geqslant 1$, $0 \leqslant \lambda \leqslant 1$, and such that

$$\lambda = \left(\frac{1}{r} - \frac{1}{s} \right) \left(1 - \frac{1}{\theta} + \frac{1}{r} \right)^{-1}.$$

We take $r = 1$. Then, since $\|u^{\varepsilon}\|_{L^{\infty}(0,\, T;\, L^1(\Omega))} \leqslant C$,

$$\|u^{\varepsilon}(t)\|_{L^{s}(\Omega)} \leqslant C \|\partial_x u^{\varepsilon}(t)\|^{\lambda}_{L^{\theta}(\Omega)} \quad \text{for a.e.} \quad t \in (0,\, T), \tag{2.8}$$

where

$$\lambda = \left(1 - \frac{1}{s} \right) \left(2 - \frac{1}{\theta} \right)^{-1},$$

for arbitrary $s \geqslant 1$ such that $0 \leqslant \lambda \leqslant 1$. Raising (2.8) to the power s and then integrating over $(0, T)$, for

$$s \leqslant 2\theta \,, \tag{2.9}$$

we can apply the Hölder inequality to the right-hand side. This yields

$$\int_0^T \int_\Omega |u^\varepsilon|^s \, dx\, dt \leqslant C \left(\int_0^T \int_\Omega |\partial_x u^\varepsilon|^\theta \, dx\, dt \right)^{(s-1)/(2\theta - 1)} .$$

Then, from (2.7) follows

$$\int_{\Omega_T} |\partial_x u^\varepsilon|^\theta \, dx\, dt \leqslant C \left(1 + \left(\int_{\Omega_T} |\partial_x u^\varepsilon|^\theta \, dx\, dt \right)^{\overline{\theta}} \right) \quad \text{with } \overline{\theta} = \frac{(s-1)(2-\theta)}{2(2\theta - 1)} \,,$$

for any s satisfying (2.5) and (2.9) which requires $\dfrac{\theta}{2-\theta} < 2\theta$, *i.e.* $1 \leqslant \theta < \dfrac{3}{2}$. We note that $0 < \overline{\theta} < 1$. Hence

$$\|\partial_x u^\varepsilon\|_{L^\theta(\Omega_T)} \leqslant C, \qquad \|u^\varepsilon\|_{L^{2\theta}(\Omega_T)} \leqslant C \ \text{ for any } 1 \leqslant \theta < 3/2. \tag{2.10}$$

Point (ii) is established.

(iii) The estimate of (p^ε) follows from (1.1) and (2.10).

(iv) In view of (2.10), $\|\partial_x u^\varepsilon(t)\|^\lambda_{L^\theta(\Omega)}$ belongs to $L^{\theta/\lambda}(0, T)$ where $\lambda = \dfrac{\theta(s-1)}{s(2\theta - 1)}$ and $s \geqslant 1$ is arbitrary. Then, using (2.8), (u^ε) and $(\partial_x p^\varepsilon)$ are bounded in the space $L^\sigma(0, T; L^s(\Omega))$, with

$$\sigma = \frac{s(2\theta - 1)}{s - 1} \quad \text{and } 1 \leqslant \theta < 3/2.$$

The proof of the lemma is finished. ∎

Now we can prove the following estimates.

LEMMA 2.3. – *The sequences* $(c^\varepsilon u^\varepsilon)$ *and* $(\partial_x(c^\varepsilon u^\varepsilon))$ *are bounded in* $L^{2\theta}(\Omega_T)$ *and* $L^\theta(\Omega_T)$ *respectively, while the sequence* $(u^\varepsilon \partial_x c^\varepsilon)$ *is bounded in* $L^{4\theta/(2\theta+1)}(\Omega_T)$, *for any* $1 \leqslant \theta < 3/2$.

The estimates of $(c^\varepsilon u^\varepsilon)$ and $(c^\varepsilon \partial_x u^\varepsilon)$ follow from Lemma 2.2 and (1.8). Writing

$$u^\varepsilon \partial_x c^\varepsilon = (u^\varepsilon)^{1/2} (u^\varepsilon)^{1/2} \partial_x c^\varepsilon$$

and using Lemma 2.2, and the Hölder's inequality, we find that $(u^\varepsilon \partial_x c^\varepsilon)$ is bounded in $L^{4\theta/(2\theta+1)}(\Omega_T)$. We note that

$$\frac{3}{2} > \frac{4\theta}{2\theta + 1} > \theta \ \text{ for any } 1 \leqslant \theta < 3/2,$$

and then $(u^\varepsilon \partial_x c^\varepsilon)$ and $(\partial_x(c^\varepsilon u^\varepsilon))$ are bounded in $L^\theta(\Omega_T)$. Lemma 2.3 is proved. ■

2.2. *Passing to the limit as* $\varepsilon \to 0$.

In view of Lemma 2.2 and (1.8), there exist functions p, u, and c in the spaces $L^\infty(0, T; W^{1,1}(\Omega)) \cap W^{1,\theta}(\Omega_T)$, $L^\theta(0, T; W^{1,\theta}(\Omega))$, and $L^\infty(\Omega_T)$ respectively, such that, for appropriate subsequences, we have

$$
\begin{aligned}
&p^\varepsilon \rightharpoonup p \text{ in } W^{1,\theta}(\Omega_T) \text{ weakly,}\\
&u^\varepsilon \rightharpoonup u \text{ in } L^\theta(0, T; W_0^{1,\theta}(\Omega)) \text{ weakly,}\\
&c^\varepsilon \rightharpoonup c \text{ in } L^\infty(\Omega_T) \text{ weak-}*.
\end{aligned}
$$

In the sequel, for convenience, the convergent extracted subsequences are not relabelled, they are denoted likewise the original sequences.

Let us first prove the following result.

LEMMA 2.4. – *The sequences* $(\partial_x p^\varepsilon)$ *and* (u^ε) *are sequentially compact in* $L^{2\theta}(\Omega_T)$, *for any* θ, $1 \leqslant \theta < 3/2$.

In view of Lemma 2.2, the sequences $(\partial_x p^\varepsilon)$ and (u^ε) are bounded in $L^{2\theta}(\Omega_T)$. Then it is enough to show that theses sequences are sequentially compact in $L^\theta(\Omega_T)$, for any θ, $1 \leqslant \theta < 3/2$. As a consequence of Lemmas 2.2 and 2.3, writing equation (1.1) in the form

$$
a(c^\varepsilon)\, \partial_t p^\varepsilon - \frac{1}{\mu(c^\varepsilon)} \partial_x^2 p^\varepsilon - \frac{\mu'(c^\varepsilon)}{\mu(c^\varepsilon)}\, u^\varepsilon \partial_x c^\varepsilon = 0,
$$

it follows that $(\partial_x^2 p^\varepsilon)$ is bounded in $L^\theta(\Omega_T)$ and then $(\partial_x p^\varepsilon)$ is bounded in $L^\theta(0, T; W^{1,\theta}(\Omega))$. Owing to (2.1), $(\partial_t(\partial_x p^\varepsilon))$ is bounded in $L^\theta(0, T; W^{-1,\theta}(\Omega))$. Applying a classical compactness argument of Aubin [3], see also Lions ([10], Théorème 5.1, pp. 58), we obtain, for an appropriate subsequence,

$$
\partial_x p^\varepsilon \to \partial_x p \text{ strongly in } L^\theta(\Omega_T), \tag{2.11}
$$

and therefore $(\partial_x p^\varepsilon)$ is sequentially compact in $L^\theta(\Omega_T)$, for any θ, $1 \leqslant \theta < 3/2$.

The sequence $(\mu(c^\varepsilon))$ is bounded in $L^\infty(\Omega_T)$. According to (1.7), there are $\overline{\mu}$ and μ_{-1} in $L^\infty(\Omega_T)$ such that, for extracted subsequences,

$$
\mu(c^\varepsilon) \rightharpoonup \overline{\mu} \quad \text{and} \quad \frac{1}{\mu(c^\varepsilon)} \rightharpoonup \frac{1}{\mu_{-1}} \quad \text{in } L^\infty(\Omega_T) \text{ weak-}*.
$$

Now, we multiply (1.3) by $\mu'(c^\varepsilon)$. This writes in the form

$$\partial_t \mu(c^\varepsilon) + \partial_x (u^\varepsilon \mu(c^\varepsilon) - (\varepsilon + u^\varepsilon)\, \partial_x \mu(c^\varepsilon)) = \\ - b(c^\varepsilon)\mu'(c^\varepsilon)\, \partial_t p^\varepsilon + \mu(c^\varepsilon)\, \partial_x u^\varepsilon + \mu''(c^\varepsilon)(\varepsilon + u^\varepsilon)\, |\partial_x c^\varepsilon|^2. \tag{2.12}$$

In view of Lemmas 2.2 and 2.3, the right-hand side is bounded in $L^1(\Omega_T)$. Let us consider the two sequences (u^ε) and $(\mu(c^\varepsilon))$. The sequence (u^ε) is bounded in $L^\theta(0, T; W^{1,\theta}(\Omega)) \cap L^{2\theta}(\Omega_T)$, the sequence $(\mu(c^\varepsilon))$ is bounded in $L^\infty(\Omega_T)$, and we have, extracting subsequences if necessary,

$$u^\varepsilon \rightharpoonup u \text{ in } L^\theta(0, T; L^\theta(\Omega)) \text{ weak},$$
$$\mu(c^\varepsilon) \rightharpoonup \bar{\mu} \text{ in } L^q(0, T; L^{\theta'}(\Omega)) \text{ weak}, \quad \forall q \geq \theta',$$

where θ' is the conjugate of θ, $\theta' = \theta/(\theta - 1)$. Furthermore, $(\partial_t(\mu(c^\varepsilon)))$ is bounded in the space $L^1(0, T; (W^{1,\theta'}(\Omega))')$. Then, using a compensated compactness argument, see Kazhikhov ([9], Lemma 6, pp. 36), Murat [12], Tartar [16], we obtain, for an extracted subsequence,

$$\mu(c^\varepsilon)\, u^\varepsilon \rightharpoonup \bar{\mu} u \quad \text{ in } \mathscr{D}'(0, T, \mathscr{D}'(\Omega)).$$

This implies $u = -\partial_x p/\bar{\mu}$. Owing to (2.11), $u = -\partial_x p/\mu_{-1}$ and then $\bar{\mu} u = \mu_{-1} u$. Then, following Tartar [16], the convexity of the function μ gives $\bar{\mu} \geq \mu(c)$. By considering the function $1/\mu$ we get $1/\mu_{-1} \geq 1/\mu(c)$. Then $\bar{\mu} \geq \mu(c) \geq \mu_{-1}$ and therefore

$$\bar{\mu} u = \mu(c)\, u = \mu_{-1} u, \qquad u = -\partial_x p/\mu(c). \tag{2.13}$$

We observe also, similarly to (2.12), that $1/\mu(c^\varepsilon)$ satisfies

$$\partial_t(1/\mu(c^\varepsilon)) + \partial_x(u^\varepsilon/\mu(c^\varepsilon) - (\varepsilon + u^\varepsilon)\, \partial_x(1/\mu(c^\varepsilon))) = \mu'(c^\varepsilon)\, b(c^\varepsilon)\, \partial_t p^\varepsilon/(\mu(c^\varepsilon)^2) + \\ \partial_x u^\varepsilon/\mu(c^\varepsilon) + (\mu''(c^\varepsilon)\, \mu(c^\varepsilon) - 2\mu'(c^\varepsilon)^2)(\varepsilon + u^\varepsilon)\, |\partial_x c^\varepsilon|^2/\mu(c^\varepsilon)^3.$$

The sequence $(1/\mu(c^\varepsilon))$ has the same properties than $(\mu(c^\varepsilon))$; the same analysis applies. We obtain, for an appropriate subsequence,

$$u^\varepsilon/\mu(c^\varepsilon) \rightharpoonup u/\mu_{-1} = u/\mu(c) \quad \text{ in } \mathscr{D}'(0, T, \mathscr{D}'(\Omega)).$$

Writing $(u^\varepsilon)^2 = (u^\varepsilon/\mu(c^\varepsilon))(-\partial_x p^\varepsilon)$ and using (2.11) and (2.13), we deduce, extracting a subsequence if necessary,

$$(u^\varepsilon)^2 \rightharpoonup -u\partial_x p/\mu(c) = u^2 \quad \text{ in } L^\theta(\Omega_T) \text{ weak}. \tag{2.14}$$

Here, we have used the fact that (u^ε) is bounded in $L^{2\theta}(\Omega_T)$. Consequently, the sequence (u^ε) is sequentially compact in $L^{2\theta}(\Omega_T)$. The lemma is now proved. ■

Let us now prove that the pair (p, c) is a weak solution to (1.10)-(1.13). We first note that system (1.1)-(1.4) is equivalent to the one where (1.3) is replaced by the conservative equation:

$$\partial_t(c^\varepsilon - \alpha p^\varepsilon) + \partial_x((c^\varepsilon - \beta)\, u^\varepsilon - (\varepsilon + u^\varepsilon)\, \partial_x c^\varepsilon) = 0, \tag{2.15}$$

with $\alpha = z_1 z_2/\Delta z$ and $\beta = z_1/\Delta z$. To pass to the limit, as $\varepsilon \to 0$, in the pressure equation

$$a(c^\varepsilon)\, \partial_t p^\varepsilon + \partial_x u^\varepsilon = 0, \tag{2.16}$$

we have to determine the weak limit of the product of the two weakly convergent sequences (c^ε) and $(\partial_t p^\varepsilon)$. This is the purpose of the following lemma.

LEMMA 2.5. – *There are extracted subsequences from (c^ε) and (p^ε), not relabelled for convenience, such that, for any $1 \leq \theta < 3/2$, as $\varepsilon \to 0$,*

$$c^\varepsilon\, \partial_t p^\varepsilon \rightharpoonup c\partial_t p \quad \textit{in } L^\theta(\Omega_T) \textit{ weak}.$$

From (1.8) we deduce that there exists $c_2 \in L^\infty(\Omega_T)$ such that, for an appropriate subsequence,

$$(c^\varepsilon)^2 \rightharpoonup c_2 \text{ in } L^\infty(\Omega_T) \text{ weak-}*.$$

Considering the two sequences (c^ε) and $(c^\varepsilon u^\varepsilon)$ and using a compensated compactness argument, we obtain, for an appropriate subsequence,

$$(c^\varepsilon)^2 u^\varepsilon \rightharpoonup c^2 u \text{ weakly in } L^{2\theta}(\Omega_T).$$

According to the strong convergence of (u^ε), we get $c_2 u = c^2 u$. As a consequence, for an extracted subsequence,

$$(c^\varepsilon - c)^2 u^\varepsilon \to 0 \text{ strongly in } L^1(\Omega_T).$$

This follows from the identity

$$(c^\varepsilon - c)^2 u^\varepsilon = (c^\varepsilon)^2 u^\varepsilon + c^2 u^\varepsilon - 2cc^\varepsilon u^\varepsilon,$$

and the previous convergent results. Now, let f be a convex $\mathcal{C}^2$ function on $[0, 1]$. Writing

$$f(c^\varepsilon)\, u^\varepsilon = f(c)\, u^\varepsilon + (c^\varepsilon - c)\, u^\varepsilon f'(c) + \frac{1}{2}(c^\varepsilon - c)^2 u^\varepsilon f''(d^\varepsilon),$$

where $d^\varepsilon \in (0, 1)$, we deduce, for an appropriate subsequence,

$$f(c^\varepsilon)\, u^\varepsilon \to f(c)\, u \text{ strongly in } L^1(\Omega_T). \tag{2.17}$$

Let us now multiply the pressure equation (2.16) by $\mathrm{sign}_\eta(u^\varepsilon)$, with $\eta > 0$

fixed. This gives

$$\frac{u^{\varepsilon} a(c^{\varepsilon})}{((u^{\varepsilon})^2+\eta)^{1/2}} \partial_t p^{\varepsilon} + \partial_x ((u^{\varepsilon})^2+\eta)^{1/2} = 0. \tag{2.18}$$

We note that $0 \leqslant \mathrm{sign}_\eta(u^{\varepsilon}) < 1$. Due to the strong convergence of (u^{ε}) and $(c^{\varepsilon} u^{\varepsilon})$ in $L^{2\theta}(\Omega_T)$, we have, extracting subsequences eventually,

$$\frac{u^{\varepsilon} a(c^{\varepsilon})}{((u^{\varepsilon})^2+\eta)^{1/2}} \rightarrow \frac{ua(c)}{(u^2+\eta)^{1/2}} \text{strongly in } L^{\tau}(\Omega_T) \text{ for any } 1 \leqslant \tau < \infty .$$

Therefore, sending $\varepsilon \to 0$ in (2.18), yields

$$\frac{ua(c)}{(u^2+\eta)^{1/2}} \partial_t p + \partial_x (u^2+\eta)^{1/2} = 0,$$

which writes also

$$\frac{u}{(u^2+\eta)^{1/2}} (a(c)\, \partial_t p + \partial_x u) = 0. \tag{2.19}$$

We observe that

$$\frac{c^{\varepsilon}}{a(c^{\varepsilon})} \partial_x u^{\varepsilon} = \frac{1}{\Delta z} \left(1 - \frac{z_2}{a(c^{\varepsilon})} \right) \partial_x u^{\varepsilon},$$

then, using equation (2.16),

$$\frac{c^{\varepsilon}}{a(c^{\varepsilon})} \partial_x u^{\varepsilon} = \frac{1}{\Delta z} \partial_x u^{\varepsilon} + \frac{z_2}{\Delta z} \partial_t p^{\varepsilon}. \tag{2.20}$$

We also have

$$\frac{c^{\varepsilon}}{a(c^{\varepsilon})} \partial_x u^{\varepsilon} = \partial_x \left(\frac{c^{\varepsilon}}{a(c^{\varepsilon})} u^{\varepsilon} \right) - z_2 \frac{u^{\varepsilon}}{a(c^{\varepsilon})^2} \partial_x c^{\varepsilon}. \tag{2.21}$$

Thanks to Lemma 2.3 and (1.9), the sequence $\left(\frac{u^{\varepsilon}}{a(c^{\varepsilon})^2} \partial_x c^{\varepsilon} \right)$ is bounded in $L^{\theta}(\Omega_T)$. We write

$$\frac{u^{\varepsilon}}{a(c^{\varepsilon})^2} \partial_x c^{\varepsilon} = (u^{\varepsilon})^{1/2} \xi^{\varepsilon} \text{ with } \xi^{\varepsilon} = \frac{(u^{\varepsilon})^{1/2}}{a(c^{\varepsilon})^2} \partial_x c^{\varepsilon}.$$

First, using the part (i) of Lemma 2.3 and (1.9) the sequence (ξ^{ε}) is bounded in $L^2(\Omega_T)$. Then, for a subsequence, $\xi^{\varepsilon} \rightharpoonup \xi$ weakly in $L^2(\Omega_T)$. Next, Lemma 2.4 insures that $((u^{\varepsilon})^{1/2})$ is sequentially compact in $L^{4\theta}(\Omega_T)$. Consequently, for an extracted subsequence,

$$\frac{u^{\varepsilon}}{a(c^{\varepsilon})^2}\,\partial_x c^{\varepsilon} \rightharpoonup u^{1/2}\xi \text{ in } L^{\theta}(\Omega_T) \text{ weak.}$$

Using the sequentially compactness of $(c^{\varepsilon} u^{\varepsilon})$ in $L^{2\theta}(\Omega_T)$, the weak convergence of (c^{ε}) in $L^{\infty}(\Omega_T)$ weak-$*$ *, and (1.9) we have for an appropriate subsequence,

$$\frac{c^{\varepsilon}}{a(c^{\varepsilon})}\,u^{\varepsilon} \rightharpoonup \frac{c}{a(c)}\,u \text{ weakly in } L^{2\theta}(\Omega_T).$$

Using (2.20) and (2.21), we have

$$\partial_x\left(\frac{c^{\varepsilon}}{a(c^{\varepsilon})}\,u^{\varepsilon}\right) = \frac{1}{\Delta z}\,\partial_x u^{\varepsilon} + \frac{z_2}{\Delta z}\,\partial_t p^{\varepsilon} + z_2\,\frac{u^{\varepsilon}}{a(c^{\varepsilon})^2}\,\partial_x c^{\varepsilon}.$$

We deduce by passing to the limit, as $\varepsilon \to 0$, in the latter relation that

$$\partial_x\left(\frac{c}{a(c)}\,u\right) - z_2 u^{1/2}\xi = \frac{1}{\Delta z}\,\partial_x u + \frac{z_2}{\Delta z}\,\partial_t p$$

which writes also

$$\Delta z a(c)\left(z_2\,\frac{u}{a(c)^2}\,\partial_x c - z_2 u^{1/2}\xi\right) = a(c)\,\partial_t p + \partial_x u.$$

Then, from (2.19) and the latter inequality follows

$$a(c)\,\partial_t p + \partial_x u = 0.$$

The lemma is proved. ■

It remains to show that the function c is a solution to the concentration equation (1.12). After multiplication of (2.15) by a test function φ in $\mathscr{C}^1(\overline{\Omega}_T)$ with compact support contained in $\overline{\Omega} \times [0, T[$ and integration by parts, we obtain

$$\text{(2.22)} \quad \int_{\Omega_T} \{(c^{\varepsilon} - a p^{\varepsilon})\,\partial_t\varphi + (u^{\varepsilon}(c^{\varepsilon} - \beta) - (\varepsilon + u^{\varepsilon})\,\partial_x c^{\varepsilon})\,\partial_x\varphi\}\,dx\,dt = - \int_{\Omega} (c_0(x) - a p_0(x))\,\varphi(x, 0)\,dx\,.$$

To pass to the limit, as $\varepsilon \to 0$, in (2.22), thanks to Lemmas 2.2 and 2.4, we have to characterize the weak limit of the sequence $(u^{\varepsilon}\,\partial_x c^{\varepsilon})$. This is the purpose of the following lemma.

LEMMA 2.6. – *There are extracted sequences from* (c^{ε}) *and* (u^{ε}), *not relabelled for convenience, such that, as* $\varepsilon \to 0$,

$$u^{\varepsilon} \partial_x c^{\varepsilon} \rightharpoonup u \partial_x c \quad \text{in } L^{\theta}(\Omega_T) \text{ weak.}$$

Let us denote

$$R^{\varepsilon} = \partial_x \left(\frac{u^{\varepsilon}}{a(c^{\varepsilon})} \right) - \frac{1}{a(c^{\varepsilon})} \partial_x u^{\varepsilon}.$$

Choosing $f(c) = 1/a(c)$ in (2.17), we obtain (for an appropriate subsequence)

$$\frac{u^{\varepsilon}}{a(c^{\varepsilon})} \to \frac{u}{a(c)} \text{ strongly in } L^{2\theta}(\Omega_T).$$

From equation (2.16) and Lemma 2.5 follows

$$\frac{1}{a(c^{\varepsilon})} \partial_x u^{\varepsilon} \rightharpoonup \frac{1}{a(c)} \partial_x u \quad \text{in } L^{\theta}(\Omega_T) \text{ weak.}$$

Then

$$R^{\varepsilon} \rightharpoonup R = \partial_x \left(\frac{u}{a(c)} \right) - \frac{1}{a(c)} \partial_x u \text{ weakly in } L^{\theta}(0, T; W^{-1,\theta}(\Omega)).$$

In view of Lemma 2.2, the sequence $((u^{\varepsilon})^{1/2} \partial_x c^{\varepsilon})$ is bounded in $L^2(\Omega_T)$. Then, there is $\overline{\xi} \in L^2(\Omega_T)$ such that, extracting a subsequence if necessary,

$$(u^{\varepsilon})^{1/2} \partial_x c^{\varepsilon} \rightharpoonup \overline{\xi} \text{ weakly in } L^2(\Omega_T).$$

The strong convergence of (u^{ε}) together with the above convergence implies

$$u^{\varepsilon} \partial_x c^{\varepsilon} \rightharpoonup u^{1/2} \overline{\xi} \text{ in } L^{\theta}(\Omega_T) \text{ weak.}$$

Choosing $f(c) = 1/(a(c))^4$ in (2.17), we have

$$\frac{u^{\varepsilon}}{a(c^{\varepsilon})^4} \to \frac{u}{a(c)^4} \text{ strongly in } L^1(\Omega_T)$$

from which follows

$$\frac{(u^{\varepsilon})^{1/2}}{a(c^{\varepsilon})^2} \to \frac{u^{1/2}}{a(c)^2} \text{ strongly in } L^2(\Omega_T).$$

By identification of the limit R,

$$\frac{u^{1/2}}{(a(c))^2}\,\overline{\xi} = \frac{u}{(a(c))^2}\,\partial_x c,$$

which gives

$$u^{1/2}\overline{\xi} = u\partial_x c.$$

Consequently, for an appropriate subsequence,

$$u^{\varepsilon}\,\partial_x c^{\varepsilon} \rightharpoonup u\partial_x c \text{ in } L^{\theta}(\Omega_T) \text{ weak}.$$

This ends the proof of the lemma. ∎

To conclude, we send ε to 0 in (2.22). We obtain

$$\int_{\Omega_T} \{(c-ap)\,\partial_t \varphi + ((c-\beta)\,u - u\partial_x c)\,\partial_x \varphi\}\,dx\,dt = -\int_{\Omega} (c_0(x) - ap_0(x))\,\varphi(x,0)\,dx,$$

for any testing function φ in $\mathscr{C}^1(\overline{\Omega}_T)$ with compact support contained in $\overline{\Omega} \times [0, T[$. Combining with equation (1.1) we obtain (1.14). This completes the proof of Theorem 1.2. ∎

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Pervenuta in Redazione
il 6 aprile 2002 e in forma rivista il 29 gennaio 2003