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Riccardo Ricci, Domingo A. Tarzia

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A Spatially Inhomogeneous Diffusion Problem with Strong Absorption.

RICCARDO RICCI - DOMINGO A. TARZIA

Sunto. – *Si studia il comportamento asintotico delle soluzioni di un problema di diffusione non lineare con assorbimento forte. Si dimostra la convergenza alla soluzione stazionaria nella norma L^∞ usando una opportuna famiglia di sopra e sottosoluzioni. In appendice si dimostra la buona posizione del problema.*

Summary. – *We study the asymptotic behaviour ($t \to +\infty$) of the solutions of a nonlinear diffusion problem with strong absorption. We prove convergence to the stationary solution in the L^∞ by means of an appropriate family of sub and supersolutions. In appendix we prove the well posedness of the problem.*

1. – Introduction.

Some years ago we discussed the problem of the asymptotic behaviour of the non-negative solution of the Dirichlet problem for the one-dimensional nonlinear reaction-diffusion equation

$$u_t - (u^m)_{xx} + u^n = 0, \qquad x > 0,\ t > 0, \qquad u(0, t) = 1, \qquad t > 0, \tag{1}$$

with $1 \leqslant m$ and $0 \leqslant n < m$. Assuming $u(x, 0) \geqslant 0$ with compact support, we proved that the solution converges uniformly exponentially fast to the stationary solution if the power m and n satisfies the inequality

$$m + n \leqslant 2, \tag{2}$$

see [11], [12]. In particular, when $m = 1$, any $n < 1$ is allowed, so this gave a first answer to the problem of the asymptotic behaviour of the so called «dead-core» problem [13], see also [12], [10] and [4] for related results.

Recently a paper by A. Berezovsky and R. Kersner, [1], addressed the question of the asymptotic behaviour for a reaction-diffusion problem in presence of a point source, namely

$$\frac{\partial C}{\partial t} = \operatorname{div}(D(C) \operatorname{grad} C) - R(C)\, C + Q(t) \cdot \delta(x). \tag{3}$$

Here $C(x, t)$ is the concentration profile of a chemical emitted from a point source and dispersed under the combined effect of diffusion and absorption into some chemical reaction. $D(C)$ and $R(C)$ are respectively the diffusion coefficient and the reaction rate, $Q(t)$ is the emission rate, and $\delta(x)$ is the Dirac measure.

Assuming spherical symmetry of the solution (which is the case for a spherically symmetrical initial value) the authors were able to reduce (formally) the problem to a one-dimensional problem, namely to find the non-negative solutions of

$$\mathcal{L}(u) = x^{(m-1)/m} u_t - (u^m)_{xx} + x^{(m-n)/m} u^n = 0, \qquad x > 0,\ t > 0, \tag{4}$$

$$u^m(x, t) \to \varphi(t) \leq 1 \quad \text{as} \quad x \to 0^+, \qquad t > 0, \tag{5}$$

$$u(x, 0) = 0, \qquad x > 0, \tag{6}$$

where u is the adimesionalized concentration times $x^{1/m}$, x denotes now the radial coordinate, and the boundary value $u^m(0, t)$ is proportional to the emission rate $Q(t)$.

The most interesting case from the mathematical point of view is when $n < m$. This includes the case of linear diffusion ($m = 1$) with strong absorption ($0 \leq n < 1$), where the special case $n = 0$ means that the reaction term is given by the Heaviside function of the concentration (zero-order reaction.) In [1] the authors established the convergence in the L^2 norm of the solution of the parabolic problem to the stationary solution only in the case $m = 1$ (however no explicit estimate of the rate of convergence was given).

It is immediate to realize that problem (4), (5) and (6) is a generalization of the Dirichlet problem for equation (1), to the case of an equation with non homogeneous coefficients and with a non constant boundary condition.

Here we give a convergence result in the L^∞ norm using families of super and sub-solutions in the spirit of what was made in [11]. In particular we still have to restrict to values of m and n satisfying (29). Although the super and sub-solution technique is quite similar to what was done in [11], the construction of the families of super and sub-solution differs substantially. In fact in [11] they were constructed using the translational invariance of equation (1), which fails in the present case. Moreover in [11] only the case of constant boundary value was considered. Here we show how to deal with non-constant boundary conditions.

The stationary solution u_s has compact support in the space variable. Let us denote by $L = \sup\{x \mid u_s(x) > 0\}$. Under reasonable assumptions on the initial data, the solutions $u(x, t)$ of the parabolic problem have compact support too. We denote it by $s(t) = \sup\{x \mid u(x, t) > 0\}$. We also prove that $s(t)$ converges to L as $t \to \infty$. Finally, since it seems that no reference is available for the proof of the existence and uniqueness of the solution, as well as for a comparison theorem, we shortly sketch in appendix how these results can be obtained modifying similar results for simpler but similar equations.

2. – The stationary problem.

In this section we summarize some results about the stationary problem under the assumption

$$0 \leq n < m . \tag{7}$$

This problem can be formulated as a free-boundary problem, [2]: find a function $u_s(x)$ and a number $L > 0$ such that the following equations hold

$$\mathcal{L}_s(u_s) = -(u_s^m)'' + x^{(m-n)/m} u_s^n = 0 \,, \; 0 < x < L, \tag{8}$$

$$u_s(0) = 1 \,, \tag{9}$$

$$u_s(L) = 0 \,, \tag{10}$$

$$u_s'(L) = 0 \,, \tag{11}$$

where $'$ denotes the x derivative. Condition (11) is the free-boundary condition, and it determines the value of the free-boundary L. Of course, in this setting, equation (8) makes sense only if u_s is (strictly) positive in $(0, L)$.

We first observe that this problem can be reduced to a variational problem and that existence and uniqueness of the solution can be easily established using standard techniques in this area.

To do this, first put $v = u_s^m$ and rewrite equation (8) as

$$-v'' + x^p v^q = 0 \,, \tag{12}$$

with $p, q \in (0, 1)$ (in our problem we have $p = 1 - n/m$ and $q = n/m = 1 - p$, but this relation is inessential in what follows).

We now look for *non-negative* solutions of (12) for $x \in (0, M)$ with boundary conditions $v(0) = 1$ and $v(M) = 0$, where M is a (large) number. This problem has a natural variational formulation: find the minimum of the functional

$$J(w) = \frac{1}{2} \int_0^M (w')^2 \, dx + \frac{1}{q+1} \int_0^M x^p w^{q+1} \, dx \,, \tag{13}$$

in the convex set

$$K = \{ w \in H^1(0, M) \,|\, w \geq 0, \; w(0) = 1, \; w(M) = 0 \} . \tag{14}$$

Existence and uniqueness of the solution, as well as its regularity, follows immediately from the standard theory of variational problem (see the solution of the dead-core problem in [5]).

It remains to prove that we have solved our free-boundary problem. This reduces to show that if M is large enough there exists a positive L (not depending on M) such that the solution satisfies $v(x) > 0$ in $(0, L)$ and $v(x) = 0$ in (L, M). So let $M > 1 + \sqrt{2m(m+n)}/(m-n)$, then we have

LEMMA 2.1. – *The function v has compact support.*

To prove this it is sufficient to compare v with the solution v_1 of the spatially homogeneous problem

$$-(v_1)_{xx} + v_1^q = 0\,, \qquad x > 0\,, \tag{15}$$

$$v_1(1) = 1\,, \tag{16}$$

which is given by

$$v_1(x) = \left[1 - \frac{x-1}{l}\right]_+^{\frac{2m}{m-n}} \tag{17}$$

where

$$l = \frac{\sqrt{2m(m+n)}}{m-n} \tag{18}$$

In fact we have $v(1) < 1$ because of the maximum principle. Since, for $x > 1$, $x^{(m-n)/m} > 1$, the comparison principle gives $v(x) \leq v_1(x)$ for $x > 1$ and consequently $v(x) = 0$ for $x > 1 + l$. We can summarize the preceding discussion in the following

PROPOSITION 2.2. – *The free-boundary problem* (8)-(11) *has a unique solution with free-boundary $L < 1 + \sqrt{2m(m+n)}/(m-n)$.*

REMARK 2.3. – *Alternative estimates of the free boundary are in* [2], *in particular the authors showed that*

$$\left\{\frac{2(1+q)}{(1+q)^2}\right\}^{1/(3-q)} < L < \left\{\frac{(1+q)(3-q)^2}{2(1+q)^2}\right\}^{1/(3-q)} \tag{19}$$

For the construction of the super and sub-solutions of the parabolic problem we need the following estimate for the derivative u' of the solutions of the stationary equation

$$(u^m)_{xx} = x^{(m-n)/m} u^n\,, \qquad u(0) = a > 0\,. \tag{20}$$

LEMMA 2.4. – *Let u_a be the solution of* (20) *and let*

$$b = La^{\frac{m(m-n)}{3m-n}} \tag{21}$$

denote its free boundary (*i.e.* $u_a(x) > 0$ *in* $(0, b)$ *and* $u_a(x) = 0$ *for* $x > b$), *then there exists a positive constant C such that, for any* $x \in (0, b)$ *we have*

$$\frac{u_a'}{u_a^n} > -C u_a^{\frac{2-m-n}{2}} \tag{22}$$

PROOF. – First observe that (21) can be obtained from the following scaling argument: let $w(y) = a^{-1} u_a(x)$ with $y = x/d$. Then w solves

$$(w^m)_{yy} = \frac{d^{\frac{m-n}{m}}}{a^{m-n}} y^{(m-n)/m} w^n, \qquad w(0) = 1\,. \tag{23}$$

Choosing $d = a^{\frac{m(m-n)}{3m-n}}$, the function $w(y)$ coincides with the solution of (8)-(11), so $w(y) > 0$ for $y \in (0, L)$ and $w(x) = 0$ for $y \leqslant L$. Coming back to the x coordinate we have the expression (21) for the right boundary of the support of u_a.

Estimate (22) is a special case of the regularity estimate for the parabolic case, which can be obtained by the Bernstein method, see [6]. For the stationary solution the proof is elementary and we present it here for the sake of completeness.

Let $v(x) = u^m(x)$, so that

$$v'' = x^p v^q, \tag{24}$$

with $p = 1 - \frac{n}{m}$ and $q = \frac{n}{m}$. Since $v(x) \equiv 0$ for $x > B$ we have

$$v'' \leqslant b^p v^q \tag{25}$$

Moreover, a straightforward application of the Hopf and the maximum principles says that v has non-positive first derivative (strictly negative when $v(x) > 0$,) and then

$$v'' v' \geqslant b^p v^q v' \tag{26}$$

or

$$\frac{1}{2}(v'^2)' \geqslant \frac{b^p}{q+1}(v^{q+1})'. \tag{27}$$

Integrating (27) between x and L and taking into account that $v' < 0$

we have

$$v'(x) > -\sqrt{\frac{2b^p}{q+1}}\, v^{\frac{q+1}{2}}(x), \tag{28}$$

from which (22) follows.

3. – Super and sub-solutions.

In this section we construct a family of super-solutions using a suitable «deformation» of the stationary solution $u_s(x)$. Being deeply based on estimate (22), it turns out that our approach allows to construct sub and super-solutions for values of m and n which satisfy, together with inequality (7), the inequalities

$$0 \leqslant n, \quad 1 \leqslant m, \quad m+n \leqslant 2. \tag{29}$$

In case of linear diffusion, $m=1$, any strong-absorption power n is allowed.

Let $\lambda(t)$ be a given positive function and define

$$\overline{u}(x,t) = u_s(\lambda(t)\,x). \tag{30}$$

Then, for any t, $\overline{u}(x,t)$ is the solution of

$$-\frac{d^2}{dx^2}(\overline{u}^m(x,t)) + \lambda^2(t)(\lambda x)^{1-n/m}\overline{u}^n(x,t) = 0, \quad x>0, \quad \overline{u}(0,t)=1. \tag{31}$$

Let us show that it is possible to choose the function λ in such a way that $\overline{u}(x,t)$ is a super-solution of equation (4). First observe that, for any $\lambda(t)<1$, $\overline{u}(x,t)$ is a super-solution of the stationary problem, i.e. we have

$$\mathcal{L}_s\overline{u} = (1-\lambda^{3-n/m}(t))\,x^{1-n/m}\overline{u}^n(x,t) > 0 \tag{32}$$

so that

$$\overline{u}(x,t) \geqslant u_s(x), \quad x>0 \tag{33}$$

(notice that the strict inequality does not hold for any x but only for those x for which $\overline{u}(x,t)>0$).

Let now $\lambda(t_0)=\lambda_0<1$, we compute

$$\mathcal{L}\overline{u}(x,t) = x^{2-\frac{1}{m}}\,u_s'(\lambda(t)\,x)\,\dot{\lambda}(t) - (u_s^m(\lambda(t)\,x))_{xx} + x^{1-\frac{n}{m}}\,u_s^n(\lambda(t)\,x)) \tag{34}$$

where $'$ denotes the derivative with respect to the argument. Then

$$\mathcal{L}\overline{u}(x,t) = \left[x^{\frac{m+n-1}{m}}\,\frac{u_s'}{u_s^n}\,\dot{\lambda}(t) - (\lambda^{3-n/m}(t)-1)\right]x^{1-\frac{n}{m}}\,u_s^n(\lambda(t)\,x)). \tag{35}$$

Suppose now that $\dot{\lambda}(t) > 0$. We can now use estimate (22) together with our assumption $m + n < 2$ to eliminate the factor u_s'/u_s^n in (35) and we get

$$\mathcal{L}\overline{u}(x, t) \geq \left[-x^{\frac{m+n-1}{m}} \text{Const.}\, \dot{\lambda}(t) - (\lambda^{3-n/m}(t) - 1)\right] x^{1-\frac{n}{m}} u_s^n(\lambda(t)x)) \geq \tag{36}$$

$$[-K\dot{\lambda}(t) - (\lambda^{3-n/m}(t) - 1)]\, x^{1-\frac{n}{m}} u_s^n(\lambda(t)\, x)),$$

for some suitable positive constant K (notice that here we use the fact that $0 \leq u < 1$ and that $x^{\frac{m+n-1}{m}}$ is bounded because u_s vanishes identically for x large enough).

Finally we chose $\lambda(t)$ to be the solution of the o.d.e.

$$\dot{\lambda} = \frac{1}{K}(1 - \lambda^{3-n/m}), \quad \lambda(0) = \lambda_0. \tag{37}$$

Choosing $\lambda_0 < 1$, $\lambda(t)$ has positive derivative so that (36) is true and becomes

$$\mathcal{L}\overline{u}(x, t) \geq 0\,, \tag{38}$$

i.e. $\overline{u}$ is a super-solution.

Solving (37) with a $\lambda_0 > 1$ we obtain a sub-solution $\underline{u}(x, t) = u_s(\lambda(t)x)$.

We can state the results of this section in the following

PROPOSITION 3.1. – *Let $\lambda(t)$ be a solution of* (37) *with $\lambda_0 < 1$ ($\lambda_0 > 1$) then the function $\overline{u}(x, t) = u_s(\lambda(t)\, x)$ ($\underline{u}(x, t) = u_s(\lambda(t)\, x)$) is a super-solution* (*sub-solution*) *uniformly converging to the stationary solution $u_s(x)$. The convergence is exponentially fast.*

As a consequence we can give a first result on the asymptotic behaviour of the solution of the parabolic problem, in the case of special boundary and initial conditions.

THEOREM 3.2. – *Let $u(x, t)$ be a solution of* (4) *with*

$$u(0, t) = 1\,, \quad t > 0\,, \quad u(x, 0) = u_0(x)\,, \quad x > 0\,, \tag{39}$$

and suppose there exist two constants $\lambda_a > 1$ and $0 < \lambda_b < 1$ such that

$$u_s(\lambda_a x) \leq u_0(x) \leq u_s(\lambda_b x)\,, \quad x > 0\,. \tag{40}$$

Then, for $t \to \infty$, the function $u(x, t)$ converges uniformly to $u_s(x)$ and the rate of converge is exponential.

This theorem does not give a general answer to our original problem because of the requirements on both the initial and boundary conditions. In the next section we give some estimates which allow to use similar sub and super-solution to prove the convergence for general boundary and initial conditions.

4. – Estimates.

In this section we show that, for any fixed time $\bar{t}$, we can find a sub-solution of the form introduced in the previous section, whose value at time $\bar{t}$ is below $u(x, \bar{t})$. The starting point of the following estimates is the construction of a new sub-solution

PROPOSITION 4.1. – *Let $\psi(t)$ be a positive function, $\psi \in C^1([0, T])$. Then there exist a sufficiently large constant $A > 0$ such that the function*

$$v(x, t) = [\psi(t) - Ax]_+^{\frac{2}{m-n}} \tag{41}$$

satisfies

$$\mathcal{L}v \leqslant 0 \tag{42}$$

(*i.e. v is a sub-solution*).

We have

$$\mathcal{L}v = v^n \left[x^{1-\frac{1}{m}} \frac{2}{m-n} v^{2-m-n} \cdot 2\dot{\psi} - \frac{2m(m+n)}{(m-n)^2} A^2 + x^{1-\frac{n}{m}} \right]. \tag{43}$$

Because $x < \psi(t)/A$, and (7), (29), the first and third terms in the sum are uniformly bounded by a constant times $\max|\psi|$ and $\max|\dot{\psi}|$. So we can choose A large enough to have $\mathcal{L}v \leqslant 0$.

Let $u(x, t)$ be the solution of (4), (5) and (6), then the sub-solution v can be used to give an estimate of $u_x(0, t)$. We fix a time $\bar{t}$ and choose $\psi(t)$ such that

$$0 = (\phi(0))^{\frac{m-n}{2}} = \psi(0) < \psi(t) < (\phi(t))^{\frac{m-n}{2}}, \quad t < \bar{t}, \quad \psi(\bar{t}) = \phi(\bar{t}). \tag{44}$$

Then

$$u(x, t) \geqslant v(x, t), \quad 0 < t < \bar{t}, \quad x > 0, \tag{45}$$

and $u(0, \bar{t}) = v(0, \bar{t})$, from which it follows

$$u_x(0, \bar{t}) > v_x(0, \bar{t}) = -A \frac{2m}{m-n} (\phi(\bar{t}))^{\frac{m+n}{m-n}}. \tag{46}$$

Notice that A depends only on the choice of ψ which in turn depends only on the boundary value ϕ.

COROLLARY 4.2. – *Let $u(x,t)$ be the solution of* (4), (5) *and* (6), *and denote by $u_{\phi(t)}(x)$ the stationary solution of* (20) *with boundary value $a=\phi(t)$. Then for any $t>0$ there exits λ_t such that*

$$u(x,t)\geqslant u_{\phi(t)}(\lambda_t x)\ x>0\,. \tag{47}$$

PROOF. – For any fixed $\bar{t}$, estimates (45) and (46) imply that there exists a sufficiently large A such that

$$u(x,\bar{t})\geqslant\left[(\phi(\bar{t}))^{\frac{m-n}{2}}-Ax\right]_+^{\frac{2}{m-n}}=v(x,\bar{t}),\qquad x\geqslant 0\,. \tag{48}$$

Since the sub-solution $v(x,\bar{t})$ is convex with respect to x, $v(x,\bar{t})$ is bounded from below by the linear function $w(x)=\phi(\bar{t})-\dfrac{2A}{m-n}(\phi(\bar{t}))^{\frac{2-m+n}{2}}x$ and so is the solution $u(x,\bar{t})$.

Let now $u_{\phi(\bar{t})}(x)$ be the stationary solution with boundary value $u_{\phi(\bar{t})}(0)=\phi(\bar{t})$ and let $\lambda_{\bar{t}}$ be large enough to have $u_{\phi(\bar{t})}(\lambda_{\bar{t}}x)=0$ for $x\leqslant\dfrac{m-n}{2A}(\phi(\bar{t}))^{\frac{m-n}{2}}$. According to (21), it suffices to take

$$\lambda_{\bar{t}}\geqslant L\frac{2A}{m-n}\left(\phi(\bar{t})^{\frac{m-n}{3m-n}}\right)^{\frac{-(m-n)}{2}}. \tag{49}$$

Since $u_{\phi(\bar{t})}$ is convex, it is bounded, now from above, by the linear function $w(x)$, and consequently (47) is satisfied.

5. – Asymptotic behaviour.

The estimates of the previous sections allow to prove the following theorem.

THEOREM 5.1. – *Suppose that m and n satisfy* (7), (29) *and let $u(x,t)$ be the solution of* (4), (5) *and* (6), *with $\phi(t)\to 1$ as $t\to\infty$.*

Then $u(x,t)$ converges uniformly to $u_s(x)$ and $s(t)$ tends to L as $t\to\infty$.

REMARK 5.2. – *It not possible to give any estimate on the rate of convergence of u to u_s without any particular assumption on the convergence rate of ϕ to 1. However it turns out from our proof that, if the convergence of ϕ is less than exponential, then u converges to u_s «as fast as» ϕ tends to 1.*

PROOF. – It follows immediately from the comparison principle that, if $\phi(t) \leq 1$ for any t, then $u(x, t) \leq u_s(x)$ for any x and any $t > 0$. So, in this case, it remains to bound from below the solution $u(x, t)$ with a family of functions converging to the stationary solution; we restrict ourselves to this case in the rest of the proof. The general case $\phi(t) \to 1$ can be treated in the same way, by means of similar estimate from above using super-solutions.

We choose a sequence of times t_k such that $t_0 > 0$ and both t_k and the difference $t_{k+1} - t_k$ diverges to $+\infty$ as $k \to +\infty$. Moreover we choose t_k in such a way that $\phi(t_k) \leq \phi(t)$ for any $t \in (t_k, t_{k+1})$.

According to Corollary (4.2), we can find λ_{0k} such that $u(x, t_k) \geq u_{\phi(t_k)}(\lambda_{0k} x)$. Now define $\lambda_k(t)$ to be the solution of (37) with initial value $\lambda_k(t_k) = \lambda_{0k}$ and use the function $u_k(x, t) = u_{\phi(t_k)}(\lambda_k(t)\, x)$ as a sub-solution up to time $t = t_{k+1}$, when we define a new sub-solution u_{k+1} with the same construction.

Because of the construction we have $u_k(0, t_k) \to 1$ as $k \to +\infty$. Moreover, according to (49), in order to satisfy $u_k(x, t_k) \leq u(x, t_k)$ for each k, it is enough to have $\lambda_{0k} \geq \dfrac{2AL}{(m-n)\left(\phi(t_0)^{\frac{m-n}{3m-n}}\right)^{(m-n)/2}}$, which is a fixed value (larger than 1). Now, because of the condition $t_{k+1} - t_k \to +\infty$, this implies that the sequence $\lambda_k(t_{k+1})$ converges to 1.

This, in turn, implies that $u_k(x, t_{k+1})$ converges uniformly to $u_s(x)$ in $(0, L)$ and so does the solution $u(x, t)$.

Finally, let $s_k(t) = \sup\{x > 0 \,|\, u_k(x, t) > 0\}$, then $\lim_{k \to \infty} s_k(t_{k+1}) = L$, which implies the convergence of the free boundary $s(t)$ to L as $t \to \infty$.

6. – Appendix.

The peculiarity of our problem is the presence of a variable and vanishing «capacity» $x^{1-1/m}$. Because of the vanishing of this factor on the boundary, we cannot divide the equation by it and consider the equation as a nonlinear diffusion equation with variable coefficient in the spirit of [7]. In fact it seems that no existing reference covers the existence and uniqueness of the solution of (4), (5) and (6).

However the «perturbation» to known results is rather meager, and the resuts can be obtained easily starting from the proofs of the cases with constant «capacity». So we limit ourselves to give a rapid sketch of the proof of the existence and uniqueness of the solution of problem (4), (5) and (6).

We can easily define a weak solution of our problem. To avoid difficulties related to the unbounded domain, we limit ourselves to solve (4) in the domain $(0, M) \times (0, T)$ with the extra boundary condition $u(M, t) = 0$. It will be *a-posteriori* clear that, if M is large enough, we have solved our free boundary problem like in the stationary case.

DEFINITION 6.1. – *A continuous nonnegative function $u(x, t)$ is a weak solution if for any test function $\varphi(x, t) \in C^2((0, M) \times (0, T))$ such that $\varphi(0, t) = \varphi(M, t) = 0$, we have*

$$\int_0^M x^{1-1/m} u(x, t)\, \varphi(x, t)\, dx = \int_0^M x^{1-1/m} u(0, t)\, \varphi(0, t)\, dx + \tag{50}$$

$$\int_0^t \int_0^M \{x^{1-1/m} u(x, t)\, \varphi_t(x, t) + u^m(x, t)\, \varphi_{xx}(x, t) - x^{1-n/m} u^n(x, t)\, \varphi(x, t)\}\, dt\, dx -$$

$$\int_0^t \{u^m(M, t)\, \varphi_x(M, t) - u^m(0, t)\, \varphi_x(0, t)\}\, dt$$

for any $t \in (0, T)$.

Weak sub and super-solutions are defined analogously substituting the equality sign by $\leqslant$ and $\geqslant$ repectively, and using only non-negative test functions.

The difficulties arising from the nonlinear diffusion and the lack of Lipschitz continuity of the reaction term can be overcome in the standard way by adding some $\varepsilon > 0$ to boundary and initial conditions and passing to the limit.

In addition we substitute the «capacity» $x^{1-1/m}$ by $x^{1-1/m} + \varepsilon$ and solve the resulting boundary value problem. Let us denote by u_ε the solution of the regularized problem

$$(x^{1-1/m} + \varepsilon)\, u_{\varepsilon t} - (u_\varepsilon^m)_{xx} + x^{1-n/m}(u_\varepsilon^n - \varepsilon^n) = 0\,, \tag{51}$$

$$u_\varepsilon(0, t) = \phi(t) + \varepsilon\,, \quad u_\varepsilon(M, t) = \varepsilon\,, \tag{52}$$

$$u_\varepsilon(x, 0) = u_0(x) + \varepsilon\,. \tag{53}$$

Existence and uniqueness of the solution is granted by the the standard theory, [8]; moreover the solutions u_ε satisfy $\varepsilon \leqslant u_\varepsilon(x, t) \leqslant \max\{\|\phi\| + 1, \|u_0\| + 1\}$, where $\|\ \|$ indicates the C^0 norm.

Here comes the essential difference with the case of nonvanishing coefficients, because the sequence u_ε is not monotone with respect to ε as in the spatially homogeneus case, see [3], or in the case in which both diffusion and reaction rate depend on (x, t) but the coefficient of u_t is constant, see [7].

However the sequence $\{u_\varepsilon\}$ is compact, so we can pass to the limit (modulo subsequences) in (50).

To prove the compactness we start by proving that $v_\varepsilon = u_{\varepsilon t}$ is uniformly bounded.

Now it is immediate to check that v_ε solves the uniformly parabolic equation

$$(x^{1-1/m} + \varepsilon)\, v_t + (mu_\varepsilon^{m-1} v)_{xx} = -x^{1-n/m} u_\varepsilon^{n-1} v \tag{54}$$

to which the maximum principle applies. So $v(x, t)$ is bounded by its maximum on the parabolic boundary. On the lateral boundary $x = 0$ and $x = M$ v is bounded by $\|\dot{\phi}\|$. To bound $v(0, t)$ we have to require some compatibility condition on the data $\phi(t)$ and $u_0(x)$ at the corner $(0, 0)$ and, in particular that

$$\frac{(u_0^m)_{xx} - u_0^n x^{1-\frac{n}{m}}}{x^{1-\frac{1}{m}}} \tag{55}$$

is bounded as $x \to 0$ (which is trivially true for $u_0 \equiv 0$) to ensure that $v(x, 0)$ is uniformly bounded for any ε. Then $u_{\varepsilon t}$ is uniformly bounded in $(0, M) \times (0, T)$. We want now to bound the function $(u_\varepsilon^m)_x$. We start by bounding the derivative $u_{\varepsilon x}$ at the boundary $x = 0$. This can be done using the sub-solution (41) which is also a subsolution of equation (51) with an appropriate choice of the constant A (depending only on $\|\phi\|$ and $\|\dot{\phi}\|$).

Then, integrating equation (51) and using the uniform bounds for u_ε and $u_{\varepsilon t}$, we get an uniform bound for $(u_\varepsilon^m)_x$ in $(0, M) \times (0, T)$. As a consequence the sequence u_ε is uniformly bounded and equi-Lipschitz continuous, and then u_ε is uniformly bounded and equicontinuous. Then, modulo subsequences, we can pass to the limit in the weak version of (51) and prove that the limit solves (50).

To prove the uniqueness and the comparison theorem we can reproduce the Holmgren's method like in the proof of uniqueness in [7]. The difference is that we are now working in a bounded interval, which simplifies the proof, and that the equation for u_ε contains an extra term due to the regularization of the «capacity» $x^{1-1/m}$. However, this term is uniformly bounded by a constant time ε, so it vanishes as $\varepsilon \to 0$.

REFERENCES

[1] A. A. BEREZOVSKY - R. KERSNER, *On the Stabilization and Localization of Diffusion-Reaction Processes* (in Russian), in Nonlinear boundary value problems of mathematical physics and their applications, A. M. Samojlenko and A. A. Berezovsky Eds., Kiev, 1998.

[2] A. A. BEREZOVSKY - R. KERSNER - L. A. PELETIER, *A Free-boundary Problem for a Reaction-Diffusion Equation*, preprint

[3] M. BERTSCH, *A Class of Degenerate Diffusion Equation with a singular nonlinear term*, Nonlinear Anal. TMA, **7** (1983), 117-127.

[4] E. COMPARINI - R. RICCI - J. L. VAZQUEZ, *Asymptotic Behavior of the Solutions of a Nonlinear Fokker-Plank Equation with Dirichlet Boundary conditions*, J. Math. Anal. Appl., **175** (1993), 606-631.

[5] J. I. DIAZ, *Nonlinear Partial Differential Equations and Free Boundaries. I. Elliptic Problems*, Res. Notes in Math., **106**, Pitman, London (1985).

[6] A. FRIEDMAN - M. A. HERRERO, *Extinction Properties of a Semilinear Heat Equation with Strong Absorption*, J. Math. Anal. Appl., **124** (1987), 530-546.

[7] R. KERSNER - F. NICOLOSI, *The Nonlinear Heat Equation with Absorption: Effects of Variable Coefficients*, J. Math. Anal. Appl., **170** (1992), 551-566.

[8] O. A. LADYZENSKAJA - V. A. SOLONNIKOV - N. N., URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, vol. **23**, Providence R.I., American Mathematical Society (1968).

[9] YU. A. MITROPOL'SKY - A. A. BEREZOVSKY, *Free and Non-local Boundary Problems in Metallurgy, Medicine, Ecology and Material Sciences*, Nat. Acad. Sci. Ukraine, Kiev, 2000.

[10] R. RICCI, *Large Time Behavior of the Solution of the Heat Equation with Nonlinear Strong Absorption*, J. Diff. Equa., **79** (1989), 1-13.

[11] R. RICCI - D. A. TARZIA, *Asymptotic behaviour of the Solutions of Class of Diffusion-Reaction Equations*, in *Free Boundary Problems: Theory and Applications*, K. H. Hoffmann & J. Sprekels ed.s, Research Notes in Maths, **186** (1990), 719-721.

[12] R. RICCI - D. A. TARZIA, *Asymptotic Behavior of the solution of the solution of the dead-core problem*, Nonlinear Anal. T.M.A., **13** (1989), 405-411.

[13] I. STAKGOLD, *Reaction-diffusion problem in chemical engineering*, in «Nonlinear Diffusion Problem» (A. Fasano and M. Primicerio, Eds.), Lect. Notes in Math., **1224**, Springer-Verlag, Berlin, 1986.

Riccardo Ricci: Dip.to di Mat. «U. Dini», Viale Morgagni, 67/A, 50134 - Firenze
ricci@math.unifi.it

Domingo A. Tarzia: Depto Matemática - CONICET FCE, Univ. Austral
Paraguay 1950, S2000 FZF Rosario, Argentina
domingo.tarzia@fce.austral.edu.ar

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