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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **6-B** (2003), n.3, p. 739–748.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2003_8_6B_3_739_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2003.

Bollettino U. M. I.
(8) **6**-B (2003), 739-748

On the Simple Connectivity at Infinity of Groups (*)

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Sunto. – *In questo articolo si definisce e si studia la nozione di semplice connessione all'infinito dei gruppi di presentazione finita, dando poi, in un caso particolare, una prova geometrica della sua invarianza per quasi-isometrie.*

Summary. – *We study the simple connectivity at infinity of groups of finite presentation, and we give a geometric proof of its invariance under quasi-isometry in a special case.*

Keywords. – π_1^∞, Cayley complex, quasi-isometry.
MSC Subject. – 20 F 32.

1. – Introduction.

In this paper we define the simple connectivity at infinity of groups, and we prove that (under some conditions) it is a geometric property of finitely presented groups.

Definition 1. – *A connected, locally compact topological space X is simply connected at infinity* (*and one writes $\pi_1^\infty X = 0$*) *if for each compact subset $k \subseteq X$ there exists a larger compact subset $k \subseteq K \subseteq X$ such that any closed null-homotopic loop in $X - K$ is null homotopic in $X - k$* (*otherwise we shall write $\pi_1^\infty X \neq 0$*).

The Euclidean space $\mathbb{R}^2$ is not simply connected at infinity (by dimensional arguments), while in dimension three the most familiar example of a contractible manifold which is not simply connected at infinity is the Whitehead 3-manifold Wh^3 (see [17]).

A related problem is to decide whether the universal covering of a manifold is $\mathbb{R}^n$ (i.e. to find conditions on $\pi_1 M$ implying that $\pi_1^\infty \widetilde{M} = 0$). J. Stallings ([14]) proved that, if $n \geqslant 5$, contractible manifolds which are simply connected at infinity (s.c.i.) are homeomorphic to $\mathbb{R}^n$. Lee and Raymond (see [8]) showed that the universal covering of a closed, aspherical manifold M of dimension > 4

(*) Partially Supported by G.N.S.A.G.A.

whose fundamental group contains a finitely generated (non trivial) abelian subgroup is $\mathbb{R}^n$ (aspherical means that $\widetilde{M}$ is contractible).

In 1983 Davis proved that for every $n > 3$ there exists a closed aspherical n-manifold $M = K(\pi_1, 1)$ such that $\pi_1^\infty \widetilde{M} \neq 0$ (see [1]). The 3-dimensional case became the so-called «covering conjecture».

CONJECTURE. – *The universal covering of a closed, irreducible 3-manifold having infinite fundamental group is* $\mathbb{R}^3$.

This conjecture was proved for a manifold having a geometric structure in the sense of Thurston, or under several different additional assumptions on $\pi_1 M$ (see [16], [3] and [7]).

McMillan and Thickstun pointed out in [10] that there exist examples of contractible 3-manifolds which do not cover closed, aspherical 3-manifolds, since there are uncountably many contractible open 3-manifolds, but there are only countably many contractible closed 3-manifolds and therefore only countably many contractible open 3-manifolds that cover closed 3-manifolds. In [11] one finds concrete examples of such manifolds: *the genus one Whitehead manifolds* (a generalization of the original Wh^3, namely a sequence of solid tori V_n such that for any $n : V_n \subseteq \operatorname{int}(V_{n+1})$, the inclusion $i : V_n \to V_{n+1}$ null-homotopic, and $X_n := V_{n+1} - \operatorname{int}(V_n)$ irreducible). These manifolds admit no nontrivial free properly discontinuous group actions, hence they cannot cover nontrivialy even a non-compact 3-manifold.

The covering conjecture was finally proved in 2000 by Poénaru (see [12] and [13]).

In this paper we address the simple connectivity at infinity of groups; we are also interested in knowing whether it is a geometric property of groups.

We now turn to groups and recall some notions. The basic idea is that a group has, together with its algebraic structure, a geometric structure, namely a distance. Let $G = < S | R >$ be a group (we will always suppose G of finite presentation such that $S = S^{-1}$ and $e \notin S$ where e is the identity of G), for every $g \in G$ let $l_S(g)$ (the length of g with respect to S) be the minimal number of elements of S required to write g. Put $d_S(g, h) = l_S(g^{-1}h)$ (the distance between g and h with respect to S). It is easy to check that d_S is a distance on G (called the word metric).

The Cayley graph of G, noted by $\mathcal{C}(G)$, is a graph of which the vertices are the elements of G where g is joined to h if $d_S(g, h) = 1$. Any segment can be endowed with a Riemannian metric, providing a distance on $\mathcal{C}(G)$ as the minimum of the lengths of the arcs joining two points. In this way one has a path-connected, geodesic space in which G is embedded (we recall that a metric space X is said geodesic if for all $x, y \in X$ there exists an isometry $g : [0, d(x, y) = a] \to X$ such that $g(0) = x$ and $g(a) = y$).

Even if this construction depends on S, Cayley graphs associated to different presentations look alike seen from afar. The following definition realizes this idea:

DEFINITION 2. – *The metric spaces* (X, d_X) *and* (Y, d_Y) *are quasi-isometric (in the sense of Gromov-Margulis) if there are constants* λ, C *and maps* $f: X \to Y$ *and* $g: Y \to X$ *(called* (λ, C)*-quasi-isometries) so that, for all* $x, x_1, x_2 \in X$ *and* $y, y_1, y_2 \in Y$, *the following holds:*

$$d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C$$

$$d_X(g(y_1), g(y_2)) \leq \lambda d_Y(y_1, y_2) + C$$

$$d_X(fg(x), x) \leq C$$

$$d_Y(gf(y), y) \leq C$$

EXAMPLE: $\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric (the map $f(x) := [x]$ for $x \in \mathbb{R}$ is a quasi-isometry).

The key observation is that the Cayley graphs corresponding to distinct presentations of G are quasi-isometric (it is sufficient to generalize the map of the example above), and thus one can associate to any group G a metric space well defined up to quasi-isometry. Hence every quasi-isometry invariant of $\mathcal{C}(G)$ determines an invariant of G.

Group theoretical properties that are invariant under quasi-isometry are called geometric, for example Gromov's word hyperbolicity, being of finite presentation or polynomial growth, and the number of ends, are «geometric» concepts (see [6] for an extensive discussion on this topic).

2. – Definitions and examples.

We now turn to the definition of the simple connectivity at infinity of groups by recalling some details . Let $P = (x_1, x_2, \ldots, x_n \,|\, R_1, R_2, \ldots, R_t)$ be a presentation of a group G, where $x_1, x_2, \ldots, x_n$ are the generators of G and $R_1, R_2, \ldots, R_t$ are the relators of P. The standard two-complex $K(G)$, corresponding to P, is the finite complex constructed as follows. Consider B a bouquet of n oriented circles (where n is the number of generators of G). For each relator R_i of P, attach to B a 2-cell by identifying its boundary with the circuit on B corresponding to R_i. This yields a compact 2-complex, K, having G as fundamental group. Its universal covering $\widetilde{K(P)}$ is called the Cayley complex of G (associated to the presentation P).

DEFINITION 3. – *A finitely presented group G is said to be simply connected at infinity (or s.c.i.) if its Cayley complex $\widetilde{K(P)}$, associated to some presentation P of G, is simply connected at infinity.*

We will show that being s.c.i. only depends on the group. The first observation is that it only depends on the 2-skeleton of X.

PROPOSITION 1. – *Let X be a compact connected polyhedron and $X^{(2)}$ the 2-skeleton. Then $\pi_1^\infty \widetilde{X} = 0$ if and only if $\pi_1^\infty \widetilde{X}^{(2)} = 0$.*

PROOF. – Let k be a compact subset of $X_2 = \widetilde{X}^{(2)} \subseteq \widetilde{X}$. Suppose that $\widetilde{X}$ is s.c.i., then there exists $K \supseteq k$ a compact subset of $\widetilde{X}$ verifying definition 1. The result follows by taking the 2-skeleton K_2 of K. Let γ be a loop in $X_2 - K_2$. γ is contained in $\widetilde{X} - K$, so it bounds a disk D satisfying $D \cap k = \emptyset$. Up to homotopy, D is contained in $X_2 - k$. Hence X_2 is s.c.i.

Conversely, suppose X_2 s.c.i., and let c be a compact subset of $\widetilde{X}$. The 2-skeleton c_2 of c is a compact subset of X_2, thus there exists $C_2 \supseteq c_2$ satisfying definition 1. The set $C = C_2 \cup \{n\text{-cells of } c,\ n \geqslant 3\}$ is a compact subset of $\widetilde{X}$ containing c. If γ is a loop in $\widetilde{X} - C$, then it is homotopically equivalent to a loop in $X_2 - C_2$. Since X_2 is s.c.i., the proof is achieved. ■

LEMMA 1. – *If X and Y are two compact, connected 2-dimensional polyhedra with isomorphic π_1's, then there exists a compact polyhedron M and compact subpolyhedra X_1 and Y_1, such that M collapses onto each of X_1 and Y_1; furthermore, X_1 is the wedge of X and a finite number of S^2's, and similarly $Y_1 = Y \vee S^2 \vee \ldots \vee S^2$.*

REMARK 1. – *This result goes back to J.H.C. Whitehead* ([17]). *His proof involved looking at certain moves changing one group presentation into another presentation of the same group: the Tietze transformations T_i.*

- T_1: add r, a consequence of the relators, to the relators,
- T_2: the inverse of T_1,
- T_3: add a new generator y and a new relator yu^{-1} where u is a word in the old generators,
- T_4: the inverse of T_3.

Explicitly, the collapsing referred to here involves simplicial structures. One says that A collapses to B, when there is a triangulation of A with B covered by sub complexes, and there is a sequence of elementary collapses leading from A to B. An elementary collapse from A to B involves some simplex σ of A not in B, and a face τ of σ which is a face of no other simplex of A (namely a proper face); one then removes the interior of

σ and τ to get B. The inverse operation is called an elementary dilatation.

PROPOSITION 2. – *If X and Y are two compact connected polyhedra with isomorphic π_1's, then $\widetilde{X}$ is s.c.i. if and only if $\widetilde{Y}$ is also s.c.i.*

PROOF. – By the previous proposition, we can restrict our to 2-dimensional polyhedra, and, by the lemma, we need to consider only two cases:

- Y is the wedge product of X with a 2-sphere,
- Y collapses to X.

In the first case $\widetilde{Y}$ is the wedge product of $\widetilde{X}$ and an infinite number of S^2's. Thus one direction is obvious. On the other hand, suppose $\widetilde{Y}$ s.c.i., let k be a compact subset of $\widetilde{X}$, then there exists K a compact subset of $\widetilde{Y}$ such that any loop outside K bounds a disk outside k. If we consider $K \cap \widetilde{X}$, we obtain a compact subset of $\widetilde{X}$. Let take a loop not in this subset. It is a loop of $\widetilde{Y}$ not in K, so it bounds a disk. This disk, after removal of some S^2's, is contained in $\widetilde{X}$ and thus the claim is proved. The second case can be reduced, by induction, to one elementary collapse. If Y collapses to X by an elementary collapse at the simplex Δ, then $\widetilde{Y} = \widetilde{X}$ with an infinite numbers of Δ's. These simplexes Δ_i are properly embedded in $\widetilde{Y}$, i.e. they are two by two disjoint and every compact subset intersects only a finite number of them. Let k be a compact subset of $\widetilde{Y}$ and $k_1 = k \cap \widetilde{X}$. Suppose that $\pi_1^\infty \widetilde{X} = 0$, then there exists a compact subset K_1 such that any loop not in K_1 is null-homotopic outside k_1. Let $K = K_1 \cup A$ where A is the set of all Δ_i having non-empty intersection with k. Let γ be a loop outside this compact subset K (since A contains a finite number of elements). This loop is homotopically equivalent (with a homotopy of $\widetilde{Y} - K$) to a loop in $\widetilde{X} - K_1$. Hence it is null-homotopic in $\widetilde{X} - k_1$ and so in $\widetilde{Y} - k$. This proves the first direction.

On the other hand, let c be a compact subset of $\widetilde{X}$. It is also a compact subset of $\widetilde{Y}$, and so there exists a compact subset C in $\widetilde{Y}$ such that any loop outside C bounds a disk outside c. Let be $C_1 = C \cap \widetilde{X}$. It is a compact subset of $\widetilde{X}$ and any loop outside C_1, since $\pi_1^\infty \widetilde{Y} = 0$, bounds a disk of $\widetilde{X} - c$ (after removal of some Δ_i's). ■

Hence, it follows that if $G = \pi_1 X$ for some compact polyhedron such that $\pi_1^\infty \widetilde{X} = 0$, we can conclude that for every compact polyhedron B with $\pi_1 B = G$, $\widetilde{B}$ is also s.c.i. Thus, $\pi_1^\infty G = 0$ is a well defined group notion. (The same result is proved in [15] by showing that Tietze transformations do not affect the simple connectivity at infinity).

Now we study this class of groups, by giving some examples. We start with an easy result.

COROLLARY 1. – *If H is a finite index subgroup of G, then* $\pi_1^\infty H = 0 \Leftrightarrow \pi_1^\infty G = 0$.

PROOF. – Let X be a compact polyhedron such that $\pi_1 X = G$ with universal covering $\widetilde{X}$. $G = \pi_1 X$ acts on $\widetilde{X}$ and so, by restriction, on H. The space $X_1 = \widetilde{X}/H$ is compact (because the index of H is finite), and the commutative diagram:

$$\begin{array}{ccc} & \widetilde{X} & \\ \swarrow & & \searrow \\ \widetilde{X}/G = X & \leftarrow & X_1 = \widetilde{X}/H \end{array}$$

shows that X and X_1 have the same universal covering $\widetilde{X}$ and so $\pi_1^\infty H = 0 \Leftrightarrow \pi_1^\infty \widetilde{X} = 0 \Leftrightarrow \pi_1^\infty G = 0$. ■

Examples of groups simply connected at infinity

1: If X is an abelian group, then there exists a finite index subgroup $H < G$ such that $H = \mathbb{Z} + Z \ldots + Z$ and so $\pi_1^\infty G = \pi_1^\infty H = \pi_1^\infty (\mathbb{Z} + Z + \ldots Z) = \pi_1^\infty \mathbb{R}^n = 0$ (iff $n > 2$).

2: If $F_n = \mathbb{Z} * Z * \ldots * Z$ (the free group of rank n), then the space $Y =$ the n-connected sum of $(S^1 \times S^2)$ has $\pi_1 Y = F_n$ and $\pi_1^\infty \widetilde{Y} = 0$, because $\widetilde{Y} = \mathbb{R}^n - \{\text{tame Cantor set}\}$. (A Cantor set of a manifold M is said tame if it can be embedded into a smooth arc of M).

We observe also that all F_n are quasi-isometric (for $n > 1$).

3: If G is the fundamental group of a closed 3-manifold, then $\pi_1^\infty G = 0$.

4: A group G quasi-isometric to $\mathbb{Z}$ contains a subgroup isomorphic to $\mathbb{Z}$ (see [4]), and so $\pi_1^\infty G = \pi_1^\infty \mathbb{Z} = 0$. The same holds if G is quasi-isometric to $\mathbb{Z}^n$.

5: $\pi_1^\infty G = 0$ if G is finite (because its Cayley complex is compact), and all finite groups are quasi-isometric.

REMARK 2. – *The s.c.i. is not a quasi-isometry invariant for topological spaces, as the following example shows.*

EXAMPLE. – Consider $X = (S^1 \times \mathbb{R}) \bigcup_{S^1 \times \mathbb{Z}} D^2$ and $Y = (S^1 \times \mathbb{R}) \bigcup_{S^1 \times \{0\}} D^2$.

Obviously $\pi_1 X = \pi_1 Y = 0$ and Y and X are two quasi-isometric spaces (in fact any disk D^2 can be split into its boundary by a quasi-isometry). They are not both s.c.i.: $\pi_1^\infty X = 0$ and $\pi_1^\infty Y \neq 0$.

For every compact subset $k \subset X$, there exists another compact subset $k \subset K \subset X$ such that every closed loop in $X - K$ is null-homotopic in $X - k$ (it is sufficient to take $K = (S^1 \times [-n, n]) \bigcup_{S^1 \times [-n, n]} D^2$ with n sufficiently large).

This is not true for Y, because if $k=D^2$, then the loop $\gamma=S^1\times\{n\}$, $(n\neq 0)$, that is null-homotopic in Y, is not null-homotopic in $Y-K$ (for no $K\supseteq k$).

Now we prove a weak version of our main statement:

THEOREM 1. – *Let G_1 and G_2 be the fundamental groups of compact Riemannian manifolds, M_1 and M_2 respectively, and let $\varepsilon=\min(i_1, i_2)/3$ where for each $\alpha=1, 2$, i_α is the injectivity radius of the universal covering $\widetilde{M}_\alpha$ of M_α. If $\widetilde{M}_1$ and $\widetilde{M}_2$ are quasi-isometric with a $(1, \varepsilon)$ quasi-isometry, then $\pi_1^\infty G_1=0 \Leftrightarrow \pi_1^\infty G_2=0$.*

REMARK 3. – *Any finite presentation group is isomorphic to the fundamental group of a (Riemannian) manifold of dimension $\geqslant 5$.*

PROOF. – Let us construct the 2-complex $K(P)$ associated to a presentation P of G as before, embed $K(P)$ into $\mathbb{R}^5$ and now take a regular neighborhood N of $K(P)$. We see that $\pi_1 N=G$ and N is a manifold with boundary ∂N having G as fundamental group. Hence the double manifold $2N=N\bigcup_{\partial N} N$ (glued along the common boundary) is the required manifold. ∎

REMARK 4. – *Without loss of generality, we assume $\widetilde{M}_1$ and $\widetilde{M}_2$ have infinite diameter (otherwise $\widetilde{M}_1$ and $\widetilde{M}_2$ will be compact).*

REMARK 5. – *If $G_1=\pi_1 M_1$ and $G_2=\pi_1 M_2$, then G_1 is quasi-isometric to $\widetilde{M}_1$ and G_2 to $\widetilde{M}_2$ (see* [6]*), so if G_1 and G_2 are quasi-isometric, then so are $\widetilde{M}_1$ and $\widetilde{M}_2$.*

Before proving the theorem, we need the following lemma:

LEMMA 2. – *Let X be a simply connected, complete manifold of infinite diameter. If for every compact subset $k\subset X$ there exists $K\supseteq k$ a compact subset of X such that every loop in $X-K$ at distance from $K\geqslant C$ (with $C=$constant) is null-homotopic in $X-k$, then $\pi_1^\infty X=0$.*

PROOF. – Suppose that $\pi_1^\infty X\neq 0$. Then there exists a compact subset $\bar{k}$ such that for any compact subset $K\supseteq\bar{k}$, there exists a loop $\bar{\lambda}\subset X-K$ non contractible in $X-\bar{k}$. But, by hypothesis, there exists $\bar{K}$ (depending on $\bar{k}$) such that every loop in $X-\bar{K}$ at a distance $\geqslant C$ from $\bar{K}$ is null-homotopic in $X-\bar{k}$.

Now, $\bar{K}$ is a compact subset and so it is contained into some ball $B(x, r)$ of X, hence, since X is complete, the ball $K=B(x, r+C)$ is a compact subset containing $\bar{k}$. Therefore there exists a loop $\bar{\lambda}$ in $X-K$ not null-homotopic in $X-\bar{k}$. But the distance between $\bar{\lambda}$ and $\bar{K}$ is $\geqslant C$, so $\bar{\lambda}$ is null-homotopic in $X-\bar{k}$. It follows that X must be s.c.i. ∎

3. – Proof of the theorem.

We will prove that $\pi_1^\infty \widetilde{M}_2 = 0$ assuming that $\pi_1^\infty \widetilde{M}_1 = 0$. Let k_2 be a compact subset of $\widetilde{M}_2$, we must find another compact subset, H, satisfying definition 1.

Let $B_1(x, R)$ be a ball in $\widetilde{M}_2$ containing k_2 (such a ball exists since k_2 is compact). By the properties of the quasi-isometry, $g(B_1)$ is contained in the ball $B_2(g(x), R+\varepsilon)$ of $\widetilde{M}_1$. Let k_1 be the closure of the ball $B(g(x), R+8\varepsilon)$: this is a compact subset of $\widetilde{M}_1$ (complete manifold). By hypothesis there exists T containing k_1 such that every loop in $\widetilde{M}_1 - T$ is null-homotopic in $\widetilde{M}_1 - k_1$. T is a compact subset, and so it is contained in some ball $B_3(c, S)$, and $f(B_3)$ is contained in another ball $B_4(f(c), S+\varepsilon)$ of $\widetilde{M}_2$. The statement follows by taking H as the closure of B_4.

Let λ be a loop in $\widetilde{M}_2 - H$ with $d(\lambda, H) \geqslant 5\varepsilon$, we will find a disk in $\widetilde{M}_2 - k_2$ bounding λ.

The loop λ can be covered by a collection of balls $Q_i(q_i, \varepsilon)$ such that any two consecutive balls have non empty intersection. Using g we can «transport» this necklace with the same property:

$$g(Q_i) \subset P_i = B_i(g(q_i), 2\varepsilon) \text{ and } Q_i \cap Q_{i+1} \neq \emptyset \text{ implies } P_i \cap P_{i+1} \neq \emptyset.$$

Let us choose a point e_i in each intersection of two consecutive balls. Any center c_i of P_i can be joined with e_i and e_{i+1} by a geodesic, so to construct a loop $g\lambda$ unique up to homotopy (because the geodesics are in balls with radii equal to the injectivity radius, i.e. contractible balls).

We have chosen λ with $d(\lambda, H) \geqslant 5\varepsilon$ so that P_i is contained in $\widetilde{M}_1 - T$ (and so $g\lambda$ is also). In fact, if there exists $p \in P_i \cap T$, then the distance $d(p, g(q_i))$ will be $< 2\varepsilon$ and $d(p, c) < S$, and so $d(g(q_i), c) < S + 2\varepsilon$ which implies that $d(fg(q_i), f(c)) < S + 3\varepsilon$. This is absurd because $d(\lambda, H) \geqslant 5\varepsilon$.

Now, the loop $g\lambda$ is contained in $\widetilde{M}_1 - T$, and so, by hypothesis, it bounds a disk D^2 in $\widetilde{M}_1 - k_1$. We will «transport» this disk to give us a disk in $\widetilde{M}_2$ bounding λ in $\widetilde{M}_2 - k_2$.

The disk D^2 can be covered by a collection of balls $D_i^1(d_i^1, 2\varepsilon)$ such that any three «consecutive» balls have non empty intersection.

D_i^1 is a covering $\mathcal{U}$ of a disk, and it is known that there exists $\mathcal{U}^1$ a subcovering of $\mathcal{U}$ such that its nerve $N(\mathcal{U}^1) = D^2$ and $N(\mathcal{U}^2) = S^1 = \partial D^2$, where $\mathcal{U}^2$ is constituted of the elements of $\mathcal{U}^1$ that cover ∂D^2. (We recall that the nerve of a covering $\mathcal{U}$ is a simplicial complex the vertices v_i of which correspond to the elements of the covering, and $v_1 \ldots v_n$ span a n-simplex if the corresponding elements of $\mathcal{U}$ have non empty intersection).

Let us consider $D_i^2 = B(f(d_i^1), 3\varepsilon) \subset \widetilde{M}_2$, we have a collection of balls the nerve of which is a disk and the nerve of $f(\mathcal{U}^2)$ is S^1 (because the nerve only depends on intersections). Moreover these balls are contained in $\widetilde{M}_2 - B_1$, in fact if there exists $y \in D_i^2 \cap B_1$, then $d(y, f(d_i^1)) < 3\varepsilon$ and $d(y, x) < R$ and so $d(gf(d_i^1), g(x)) < 4\varepsilon + R$ and hence $d(d_i^1, g(x)) < 5\varepsilon + R$ which implies that $d_i^1 \in k_1$ which is absurd because $d_i^1 \in D^2 \subset \widetilde{M}_1 - k_1$.

So we have in $\widetilde{M}_2 - B_1$ a collection of balls with the same property as the collection D_i^1, and having radii $\leqslant$ injectivity radius. It follows that these are «true» topological balls, and so one can fill all the balls to construct a singular disk.

Let u_i be the center of the ball D_i^2 and a a point of the intersection of three «consecutive» balls D_i^2, D_{i+1}^2 and D_{i+2}^2. We know that there exists a unique geodesic joining u_i, u_{i+1} and u_{i+2} with a. Let us take a point, say a_i, in any double intersection of these balls. Then there exists a unique geodesic joining u_i with a_i, and a with a_i. In this way we obtain 6 geodesic triangles, each of them contained in a contractible ball, so they can be filled, and, filling all the triangles in each ball, we obtain a singular disk. The boundary of this disk is, up to homotopy, λ, since λ is contained in this (contractible) disk.

4. – Final comments.

In [2] we have completed the proof of the quasi-isometry invariance of the s.c.i. of groups in the general case.

Now, an interesting problem would be to define the fundamental group at infinity for any finite presented group G. Hopf's theorem says that the number of ends $b(G)$ of G is equal to 0, 1, 2 or is infinite.

If $b(G) = 0$ then G is finite. If $b(G) = 2$ then G is either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Stalling's theorem says that if $b(G)$ is infinite and if G is torsion free, then it is a free product. Looking at free factors one has $b(G_i) = 1$ or $b(G_j) = 2$. Hence it is sufficient to give a definition of the fundamental group at infinity for the case $b(G) = 1$.

We finish as giving some open questions.

Let G be a one ended group, X a finite simplicial complex with fundamental group G, $\widetilde{X}$ its universal covering and r a proper ray of $\widetilde{X}$.

QUESTION 1. – *Is* $\pi_1^\infty(G) = \varprojlim\{\pi_1(\widetilde{X} - L, r \cap L)$, *such that* L *is a compact subset of* $\widetilde{X}\}$ *independent on* r *and on the presentation of* G*?* (*See* [5] *for details about the fundamental group at infinity*).

QUESTION 2. – *Is* $\pi_1^\infty G$ *a geometric property of* G*?*

Acknowledgements. I thank Valentin Poénaru for suggesting me this problem. Thanks are also due to Louis Funar and Pierre Pansu for helpful conversations, and finally I want to thank Giancarlo Passante for his help during my first years of university.

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Pervenuta in Redazione
il 21 gennaio 2002