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Entire Elliptic Hankel Convolution Equations (*).

M. BELHADJ - J. J. BETANCOR

Sunto. – *In questo lavoro caratterizziamo gli operatori di convoluzione di Hankel ellittici interi su distribuzioni temperate in termini della crescita delle loro trasformate di Hankel.*

Summary. – *In this paper we characterize the entire elliptic Hankel convolutors on tempered distributions in terms of the growth of their Hankel transforms.*

1. – Introduction and preliminaries.

The Hankel transformation is usually defined by ([18])

$$h_\mu(f)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)\, f(x) x^{2\mu+1}\, dx, \qquad y > 0.$$

Here J_μ denotes the Bessel function of the first kind and order μ. Throughout this paper we will assume that $\mu > -\frac{1}{2}$.

The Hankel transformation h_μ has been studied in spaces of distributions of slow growth by G. Altenburg [1]. Altenburg's investigation was inspired in the studies of A. H. Zemanian ([26] and [28]) about the variant $\mathcal{H}_\mu$ of the Hankel transformation defined through

$$\mathcal{H}_\mu(f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\, f(x)\, dx, \qquad y > 0.$$

It is clear that h_μ and $\mathcal{H}_\mu$ are closely connected.

G. Altenburg [1] introduced the space H constituted by all those complex valued and smooth functions ϕ on $(0, \infty)$ such that, for every $m, n \in \boldsymbol{N}$,

$$\gamma_{m,n}(\phi) = \sup_{x \in (0,\infty)} (1+x^2)^m \left| \left(\frac{1}{x} \frac{d}{dx} \right)^n \phi(x) \right| < \infty.$$

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On H it considers the topology associated with the family $\{\gamma_{m,n}\}_{m,n\in\boldsymbol{N}}$ of seminorms. Thus H is a Fréchet space and h_μ is an automorphism of H ([1, Satz 5]). According to [12, p. 85] the space H coincides with the space S_{even} constituted by all the even functions in the Schwartz space S. From [3, Theorem 2.3] it is immediately deduced that a function f defined on $(0, \infty)$ is a pointwise multiplier of H, write $f\in\mathcal{O}$, if, and only if, f is smooth on $(0, \infty)$ and, for every $k\in\boldsymbol{N}$, there exists $m\in\boldsymbol{N}$ for which $(1+x^2)^{-n}\left(\frac{1}{x}\frac{d}{dx}\right)^k f(x)$ is bounded on $(0, \infty)$.

The dual space of H, is, as usual represented by H'. If f is a measurable function on $(0, \infty)$ such that $(1+x^2)^{-n}f(x)$ is a bounded function on $(0, \infty)$, for some $n\in\boldsymbol{N}$, then f generates an element of H', that we continue calling f, by

$$\langle f, \phi\rangle = \int_0^\infty f(x)\,\phi(x)\,\frac{x^{2\mu+1}}{2^\mu\,\Gamma(\mu+1)}\,dx, \qquad \phi\in H.$$

The Hankel transformation h'_μ is defined on H' as the transpose of h_μ-transformation of H. That is, if $T\in H'$ the Hankel transformation $h'_\mu T$ is the element of H' given through

$$\langle h'_\mu T, \phi\rangle = \langle T, h_\mu\phi\rangle, \qquad \phi\in H.$$

Thus h'_μ is an automorphism of H' when on H' it considers the weak $*$ or the strong topologies.

Also in [1] G. Altenburg considered, for every $a>0$ the space $\mathcal{B}_a$ constituted by all those functions ϕ in H such that $\phi(x)=0$, $x\geqslant a$. $\mathcal{B}_a$ is endowed with the topology induced on it by H. The Hankel transform $h_\mu(\mathcal{B}_a)$ of $\mathcal{B}_a$ can be characterized by invoking [27, Theorem 1]. The union space $\mathcal{B}=\bigcup_{a>0}\mathcal{B}_a$ is equipped with the inductive topology. The dual spaces of $\mathcal{B}_a$, $a>0$, and $\mathcal{B}$ are denoted, as usual, by $\mathcal{B}'_a$, $a>0$, and $\mathcal{B}'$, respectively.

In [24] K. Trimèche introduced, for every $a>0$, the space $\mathcal{D}_{*,a}$ constituted by all those smooth and even functions ϕ on $\boldsymbol{R}$ such that $\phi(x)=0$, $|x|\geqslant a$. Also he considered the union space $\mathcal{D}_*=\bigcup_{a>0}\mathcal{D}_{*,a}$. According to [12, p. 85], the spaces $\mathcal{B}_a$, $a>0$, and $\mathcal{B}$, coincides with the spaces $\mathcal{D}_{*,a}$, $a>0$, and $\mathcal{D}_*$, respectively.

F. M. Cholewinski [10], D. T. Haimo [17] and I. I. Hirschman [19] investigated the convolution operation of the Hankel transformation h_μ on Lebesgue spaces. We say that a measurable function f is in $L_{1,\mu}$ when

$$\int_0^\infty |f(x)|\,x^{2\mu+1}\,dx<\infty.$$

If $f, g \in L_{1,\mu}$ the Hankel convolution $f\#_\mu g$ of f and g is defined by

$$(f\#_\mu g)(x) = \int_0^\infty f(y)({}_\mu\tau_x g)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy, \quad a.e. \quad x \in (0, \infty),$$

where the Hankel translated ${}_\mu\tau_x g$, $x \in (0, \infty)$, is given through

$$(1.1) \quad ({}_\mu\tau_x g)(y) = \int_0^\infty g(z)\, D_\mu(x, y, z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dz, \quad a.e. \quad y \in (0, \infty),$$

and being

$$D_\mu(x, y, z) = (2^\mu \Gamma(\mu+1))^2 \int_0^\infty (xt)^{-\mu} J_\mu(xt)(yt)^{-\mu} J_\mu(yt)(zt)^{-\mu} J_\mu(zt)\, t^{2\mu+1} dt,$$

$$x, y, z \in (0, \infty).$$

Here *a.e.* is understood respect to the Lebesgue mesure on $(0, \infty)$.

The Hankel transformation h_μ and the Hankel convolution $\#_\mu$ are related by ([19, Theorem 2.d])

$$h_\mu(f\#_\mu g) = h_\mu(f)\, h_\mu(g), \quad f, g \in L_{1,\mu}.$$

Since we think no confusion will appear, in the sequel we will write $\#$, τ_x, $x \in (0, \infty)$, and D instead of $\#_\mu$, ${}_\mu\tau_x$, $x \in (0, \infty)$, and D_μ, respectively.

As it was mentioned the transformations $\mathcal{H}_\mu$ and h_μ are closely connected. After a straightforward manipulation it can be deduced from $\#$ a form for the convolution operation $*$ for the Hankel transformation $\mathcal{H}_\mu$.

The investigation of the $*$ convolution on the distribution spaces was began by J. de Sousa-Pinto [23]. He considered the 0-order transformation $\mathcal{H}_0$ and compact support distributions on $(0, \infty)$. More recently in a series of papers J. J. Betancor and I. Marrero ([4], [5], [6], [7] and [21]) have extended the studies of J. de Sousa-Pinto. They defined the $*$ convolution of the Hankel transformation $\mathcal{H}_\mu$ on Zemanian distribution spaces of slow growth ([21]) and rapid growth ([4]). J. J. Betancor and L. Rodríguez-Mesa ([9]) studied the hypoellipticity of Hankel $*$ convolution on Zemanian distribution spaces.

The main aspects of the distributional theory developed by the $*$ convolution can be transplanted to the $\#$ convolution. Our objective in this paper is to analyze the entire ellipticity of the $\#$ convolution operators on the spaces H and H'.

For every $x \in (0, \infty)$, the Hankel translated τ_x defines a continuous linear mapping from H into itself ([21, Proposition 2.1]). For every $T \in H'$ and $\phi \in H$

the Hankel convolution $T\#\phi$ of T and ϕ is defined by

$$(T\#\phi)(x) = \langle T, \tau_x \phi\rangle, \quad x \in (0, \infty).$$

By [21, Proposition 3.5], $T\#\phi$ is a multiplier of H, for each $T \in H'$ and $\phi \in H$. In general $T\#\phi$ is not in H when $T \in H'$ and $\phi \in H$. Indeed, if we define the functional T on H by

$$\langle T, \phi\rangle = \int_0^\infty \phi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dx, \qquad \phi \in H,$$

then $T \in H'$ and, for every $\phi \in H$,

$$(T\#\phi)(x) = \int_0^\infty (\tau_x \phi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy = \int_0^\infty \phi(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy, \quad x \in (0, \infty).$$

Hence $T\#\phi \notin H$ when $\int_0^\infty \phi(y)\, y^{2\mu+1} dy \neq 0$. According to [21, Proposition 4.2] we can characterize the subspace constituted by all those $T \in H'$ such that $T\#\phi \in H$, for every $\phi \in H$. Let $m \in \boldsymbol{Z}$. We say that a complex valued and smooth function ϕ on $(0, \infty)$ is in $O_{\mu, m, \#}$ if and only if, for every $k \in \boldsymbol{N}$,

$$w^k_{m,\mu}(\phi) = \sup_{x \in (0,\infty)} (1+x^2)^m \, |\Delta^k_\mu \phi(x)| < \infty,$$

where Δ_μ denotes the Bessel operator $x^{-2\mu-1} D x^{2\mu+1} D$. $O_{\mu, m, \#}$ is a Fréchet space when it is endowed with the topology associated with the system $\{w^k_{m,\mu}\}_{k \in \boldsymbol{N}}$ of seminorms. It is clear that H is contained in $O_{\mu, m, \#}$. We denote by $\mathcal{O}_{\mu, m, \#}$ the closure of H in $O_{\mu, m, \#}$. By $\mathcal{O}_{\mu, \#}$ we represent the inductive limit space $\bigcup_{m \in \boldsymbol{Z}} \mathcal{O}_{\mu, m, \#}$. The dual space $\mathcal{O}'_{\mu, \#}$ of $\mathcal{O}_{\mu, \#}$ can be characterized as the subspace of H' of #-convolution operators on H ([5, Proposition 2.5]). Moreover, by defining on $\mathcal{O}'_{\mu, \#}$ the topology associated with the family $\{\eta_{m,k,\phi}\}_{m,k \in \boldsymbol{N}, \phi \in H}$ of seminorms, where, for each $m, k \in \boldsymbol{N}$ and $\phi \in H$,

$$\eta_{m,k,\phi}(T) = w^k_{m,\mu}(T\#\phi), \qquad T \in \mathcal{O}'_{\mu, \#},$$

and by considering on $\mathcal{O}$ the topology induced by the simple topology of the space $\mathcal{L}(H)$ of the linear and continuous mappings from H into itself, the Hankel transformation h'_μ is an isomorphism from $\mathcal{O}'_{\mu, \#}$ onto $\mathcal{O}$.

The Hankel convolution $T\#S$ of $T \in H'$ and $S \in \mathcal{O}'_{\mu, \#}$ is defined by

$$\langle T\#S, \phi\rangle = \langle T, S\#\phi\rangle, \qquad \phi \in H.$$

Thus $T\#S \in H'$, for each $T \in H'$ and $S \in \mathcal{O}'_{\mu, \#}$.

In [9] J. J. Betancor and L. Rodríguez-Mesa investigated the hypoellipticity of the $*$-Hankel convolution equations on Zemanian spaces. Results as in

[9] can be obtained for the #-Hankel convolutions. A distribution $S \in \mathcal{O}'_{\mu,\#}$ is said to be hypoelliptic in H' when the following property holds: $T \in \mathcal{O}_{\mu,\#}$ provided that $T \in H'$ and $T\#S \in \mathcal{O}_{\mu,\#}$. From [9, Proposition 3.3] it infers that $S \in \mathcal{O}'_{\mu,\#}$ is hypoelliptic in H' when, and only when, there exist $b, B > 0$ such that

$$|h'_\mu(S)(y)| \geq y^{-b}, \qquad y \geq B.$$

Motivated by the celebrated paper of L. Ehrenpreis [14] and the investigations of Z. Zielezny [29], we study in this paper the entire elliptic Hankel convolution equations on H'.

By $\boldsymbol{H}_e$ we represent the space of even and entire functions. It is equipped, as usual, with the topology of the uniform convergence of the bounded sets of $\boldsymbol{C}$.

We will say that $f \in \boldsymbol{H}_e$ is in $\mathcal{E}H'$ if, and only if, for every $l, n \in \boldsymbol{N}$, there exist $C > 0$ and $k \in \boldsymbol{N}$ for which

$$|\tau_{z_1}\tau_{z_2}\dots\tau_{z_n}(f)(z)| \leq$$
$$C((1+|z|)(1+|z_1|)\dots(1+|z_n|))^k, \qquad z, z_1, z_2, \dots, z_n \in I_l,$$

where $I_l = \{w \in \boldsymbol{C} : |\operatorname{Im} w| \leq l\}$.

Here the complex Hankel translation operator τ_z, $z \in \boldsymbol{C}$, must be understood as in [11]. If $f \in \boldsymbol{H}_e$ and $f(z) = \sum_{k=0}^{\infty} a_k z^{2k}$, $z \in \boldsymbol{C}$, then

$$(\tau_w f)(z) = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n+\mu+1)\,\Gamma(\mu+1)}{\Gamma(n-k+\mu+1)\,\Gamma(k+\mu+1)} z^{2(n-k)} w^{2k}, \qquad z, w \in \boldsymbol{C}.$$

Thus, the Hankel translation operator is extended to the complex plane.

A distribution $S \in \mathcal{O}'_{\mu,\#}$ will say to be entire elliptic in H' when the following property holds: $T \in \mathcal{E}H'$ provided that $T \in H'$ and $T\#S \in \mathcal{E}H'$.

We will start Section 2 proving that the space $\mathcal{O}'_{\mu,\#}$ of Hankel convolution operators of H is really not depending on μ. Also, in Section 2 we obtain a characterization for the entire elliptic elements of $\mathcal{O}'_{\mu,\#}$ in terms of the growth of their Hankel transforms. We will prove that $S \in \mathcal{O}'_{\mu,\#}$ is entire elliptic on H' if, and only if, there exist $a, A > 0$ such that

$$|h'_\mu(S)(y)| \geq e^{-ay}, \qquad y \geq A.$$

Throughtout this paper by C we always represent a suitable positive constant that can change from a line to the other one.

2. – Entire elliptic Hankel convolution equations in H'.

We firstly prove that the space $\mathcal{O}'_{\mu,\#}$ of Hankel convolution operators is really not depending on μ.

Let $m \in \boldsymbol{Z}$, $m \leq 0$. We denote by $O_{m,\#}$ the space constituted by all those smooth functions ϕ on $(0, \infty)$ for which there exists an even and smooth function ψ such that $\psi(x) = \phi(x)$, $x \in (0, \infty)$, and that

$$\gamma_m^k(\phi) = \sup_{x \in (0, \infty)} (1+x^2)^m \, |D^k \phi(x)| < \infty ,$$

for every $k \in \boldsymbol{N}$. $O_{m,\#}$ is endowed with the topology associated with the family $\{\gamma_m^k\}_{k \in \boldsymbol{N}}$ of seminorms. Thus, $O_{m,\#}$ is a Fréchet space. By $\mathcal{O}_{m,\#}$ we understood the closure of $\mathcal{D}_*$ in $O_{m,\#}$. It is clear that $\mathcal{O}_{m,\#}$ is a Fréchet space. Moreover, $\mathcal{O}_{m,\#}$ contains continuously $\mathcal{O}_{m+1,\#}$. The union space $\bigcup_{m \in \boldsymbol{Z}, m \leq 0} \mathcal{O}_{m,\#}$ is denoted by $\mathcal{O}_{\#}$ and it is contained in the space $\mathcal{O}$ of the pointwise multipliers of H.

Note that, for every $m \in \boldsymbol{Z}$, $m \leq 0$, a function $\phi \in \mathcal{O}_{m,\#}$ if, and only if, ϕ can be extended to an even function ψ that is in the space S_m studied in [20] and [22]. Hence an even and smooth function ϕ on $\boldsymbol{R}$ is in $\mathcal{O}_{m,\#}$ when, and only when, for every $k \in \boldsymbol{N}$, $\lim_{x \to \infty} (1+x^2)^m D^k \phi(x) = 0$.

PROPOSITION 2.1. – *Let $m \in \boldsymbol{Z}$, $m \leq 0$. The spaces $\mathcal{O}_{\mu,m,\#}$ and $\mathcal{O}_{m,\#}$ coincide topologically and algebraically.*

PROOF. – Assume that $\phi \in \mathcal{O}_{\mu,m,\#}$. There exists a sequence $\{\phi_n\}_{n \in \boldsymbol{N}}$ in $\mathcal{D}_*$ such that $\phi_n \to \phi$, as $n \to \infty$, in $O_{\mu,m,\#}$.

Let $k \in \boldsymbol{N}$. We choose a function $\alpha \in \mathcal{D}_{*,2k}$, such that $\alpha(x) = 1$, $x \in (-k, k)$. Then, since $\{\phi_n\}_{n \in \boldsymbol{N}}$ is a Cauchy sequence in $O_{\mu,m,\#}$, $\{\phi_n \alpha\}_{n \in \boldsymbol{N}}$ is a Cauchy sequence in $\mathcal{D}_{*,2k}$. Hence, there exists $\psi \in \mathcal{D}_{*,2k}$ for which $\phi_n \alpha \to \psi$, as $n \to \infty$, in $\mathcal{D}_{*,2k}$. Since the convergence in $O_{\mu,m,\#}$ implies the pointwise convergence on $(0, \infty)$, we conclude that ϕ admits an even and smooth extension to $\boldsymbol{R}$.

We can write

$$\left(\frac{1}{x} D\right) \phi(x) = x^{-2\mu-2} \int_0^x \Delta_\mu \phi(t)\, t^{2\mu+1}\, dt, \qquad x \in (0, \infty).$$

Hence, it obtains

$$\sup_{x \in (0, \infty)} (1+x^2)^m \left| \left(\frac{1}{x} D\right) \phi(x) \right| \leq C \sup_{x \in (0,\infty)} (1+x^2)^m \, |\Delta_\mu \phi(x)| .$$

Moreover, since

$$\Delta_\mu \phi(x) = D^2 \phi(x) + \frac{2\mu + 1}{x} D\phi(x), \qquad x \in (0, \infty),$$

we have that

$$\sup_{x \in (0,\infty)} (1 + x^2)^m |D^2 \phi(x)| \leq C \sup_{x \in (0,\infty)} (1 + x^2)^m |\Delta_\mu \phi(x)|. \tag{2.1}$$

On the other hand, a straightforward manipulation allows to get

$$\int_x^{x+1} (x + 1 - t) D^2 \phi(t)\, dt = -D\phi(x) + \phi(x + 1) - \phi(x), \qquad x \in (0, \infty). \tag{2.2}$$

Hence, we deduce from (2.1) and (2.2) that

$$\sup_{x \in (0,\infty)} (1 + x^2)^m |D\phi(x)| \leq C\left(\sup_{x \in (0,\infty)} (1 + x^2)^m |D^2 \phi(x)| + \sup_{x \in (0,\infty)} (1 + x^2)^m |\phi(x)|\right). \tag{2.3}$$

Also we have that

$$D\Delta_\mu \phi(x) = D^3 \phi(x) + (2\mu + 1)\, x \left(\frac{1}{x} D\right)^2 \phi(x), \qquad x \in (0, \infty). \tag{2.4}$$

The family $\{w_{m,\mu}^k\}_{m,k \in \mathbf{N}}$ generates the topology of H. Then, we can find $k \in \mathbf{N}$ such that

$$\sup_{x \in (0,1)} \left| \left(\frac{1}{x} D\right)^2 \phi(x) \right| \leq \sup_{x \in (0,1)} \left| \left(\frac{1}{x} D\right)^2 (\phi(x)\, \alpha(x)) \right| \leq C \sup_{x \in (0,2)} |D_\mu^k (\phi(x)\, \alpha(x))|,$$

where $\alpha \in \mathcal{D}_{*,2}$ and $\alpha(x) = 1$, $|x| \leq 1$.

Hence from (2.1), (2.3) and (2.4), since $\sup_{x \in (0,\infty)} (1 + x^2)^m |D\Delta_\mu \phi(x)| < \infty$, it is deduced that

$$\sup_{x \in (0,\infty)} (1 + x^2)^m |D^3 \phi(x)| < \infty.$$

By repeating the above procedure we can prove that $\phi \in O_{m,\#}$.

Moreover, since $\phi_n \to \phi$, as $n \to \infty$, in $O_{\mu,m,\#}$, the above arguments allows us to conclude that $(1 + x^2)^m |D^k \phi(x)| \to 0$, as $x \to \infty$, for every $k \in \mathbf{N}$. Thus we show that $\phi \in \mathcal{O}_{m,\#}$.

Suppose now that $\phi \in \mathcal{O}_{m,\#}$. Let $k \in \boldsymbol{N}$. It is not hard to see that

$$(2.5) \qquad |\Delta_\mu^k \phi(x)| \leq C \sum_{j=0}^{2k} |D^j \phi(x)|, \qquad x \geq 1.$$

Moreover, by choosing a function $\alpha \in \mathcal{D}_{*,2}$, since $\{w_{l,\mu}^j\}_{l,j \in \boldsymbol{N}}$ generates the topology of H, we can find $l \in \boldsymbol{N}$ such that

$$(2.6) \qquad \sup_{x \in (0,1)} (1+x^2)^m |D_\mu^k \phi(x)| \leq \sup_{x \in (0,1)} |D_\mu^k(\phi(x)\,\alpha(x))|$$

$$\leq C \sum_{j=0}^{l} \sup_{x \in (0,2)} |D^j(\phi(x)\,\alpha(x))|$$

$$\leq C \sum_{j=0}^{l} \sup_{x \in (0,\infty)} (1+x^2)^m |D^j \phi(x)|.$$

By combining (2.5) and (2.6) we obtain that $\phi \in O_{\mu,m,\#}$. Also, we can see that if $\{\phi_n\}_{n \in \boldsymbol{N}} \subset \mathcal{D}_*$ and $\phi_n \to \phi$, as $n \to \infty$, in $O_{m,\#}$, then $\phi_n \to \phi$, as $n \to \infty$, in $O_{\mu,m,\#}$. Hence we deduce that $\phi \in \mathcal{O}_{\mu,m,\#}$.

Thus we proved that $\mathcal{O}_{\mu,m,\#} = \mathcal{O}_{m,\#}$. Moreover (2.5) and (2.6) imply that the topology generated by $\{\gamma_m^k\}_{k \in \boldsymbol{N}}$ is stronger than the one induced by $\{w_{m,\mu}^k\}_{k \in \boldsymbol{N}}$. Then the open mapping theorem allows to conclude that the topologies defined by $\{\gamma_k^m\}_{k \in \boldsymbol{N}}$ and $\{w_{m,\mu}^k\}_{k \in \boldsymbol{N}}$ coincide.

Thus the proof is finished. ■

From Proposition 2.1 we infer that $\mathcal{O}_\# = \mathcal{O}_{\mu,\#}$. Hence the space of Hankel convolution operators $\mathcal{O}'_{\mu,\#}$, $\mu > -\frac{1}{2}$, coincides with the dual space $\mathcal{O}'_\#$ of $\mathcal{O}_\#$.

Althought, according to Proposition 2.1, the space of Hankel convolution operators is not depending on μ, the representation given in [21, Proposition 4.2] that involves the Bessel operator Δ_μ is very useful.

Our next objective is to obtain a characterization of the entire elliptic elements of $\mathcal{O}_\#$ involving the Hankel transformation.

Firstly some properties of the elements of $\mathcal{E}H'$ are established.

PROPOSITION 2.2. – *Let $f \in \mathcal{E}H'$. Then, for every $l \in \boldsymbol{N}$, there exists $C > 0$ and $r \in \boldsymbol{N}$, such that, for each $0 < R < l$,*

$$|\Delta_\mu^k f(z)| \leq C \left(\frac{2}{R}\right)^{2k} k!\,\Gamma(\mu+k+1)(1+|z|)^r (1+R)^r, \quad z \in I_l \text{ and } k \in \boldsymbol{N}.$$

PROOF. – Since f is an even and entire function, according to [11], we can write

$$(\tau_z f)(w) = \sum_{k=0}^{\infty} \frac{w^{2k}}{2^{2k} k!\, \Gamma(\mu + k + 1)} (\Delta_\mu^k f)(z), \qquad w, z \in \boldsymbol{C}.$$

Hence, for every $k \in \boldsymbol{N}$, $R > 0$ and $z \in \boldsymbol{C}$, it has

$$(\Delta_\mu^k f)(z) = \frac{2^{2k} k!\, \Gamma(\mu + k + 1)}{2\pi i} \int_{C_R} \frac{(\tau_z f)(w)}{w^{2k+1}} dw. \tag{2.7}$$

Here C_R denotes the circle having as a parametric representation to $w(t) = Re^{it}$, $T \in [0, 2\pi)$. Then, for every $l \in \boldsymbol{N}$ and $0 < R < l$, there exists $C > 0$ and $r \in \boldsymbol{N}$, for which

$$|\Delta_\mu^k f(z)| \leq C\left(\frac{2}{R}\right)^{2k} k!\, \Gamma(\mu+k+1)(1+|z|)^r (1+R)^r, \quad z \in I_l \text{ and } k \in \boldsymbol{N}. \qquad \blacksquare$$

A consequence of Proposition 2.2 is the following one.

COROLLARY 2.3. – *Let* $f \in \mathcal{E}H'$. *Then* $f \in \mathcal{O}_\#$.

PROOF. – To see that $f \in \mathcal{O}_\#$ it is sufficient to use Proposition 2.2 and to argue as in the proof of Proposition 2.1. ■

By proceeding as in [16, Proposition 5.2] (see also [2, Proposition 3.5]) we can prove that if L is a continuous linear mapping from $\boldsymbol{H}_e$ into itself that commutes with Hankel translations, that is, $\tau_z L = L\tau_z$, for every $z \in \boldsymbol{C}$, then there exists an even and entire function Φ of exponential type such that, for every $f \in \boldsymbol{H}_e$,

$$Lf(z) = \sum_{k=0}^{\infty} a_k \Delta_\mu^k f(z), \qquad z \in \boldsymbol{C},$$

where $\Phi(w) = \sum_{k=0}^{\infty} a_k w^{2k}$, $w \in \boldsymbol{C}$.

In the sequel, if Φ is an even and entire function admiting the representation $\Phi(w) = \sum_{k=0}^{\infty} a_k w^{2k}$, $w \in \boldsymbol{C}$, we will understand by $\Phi(\Delta_\mu)$ the operator defined by

$$\Phi(\Delta_\mu) f = \sum_{k=0}^{\infty} a_k \Delta_\mu^k f, \qquad f \in D_\Phi.$$

Here the domain D_Φ of $\Phi(\Delta_\mu)$ is constituted by all those even and entire functions f such that the series $\sum_{k=0}^{\infty} a_k \Delta_\mu^k f(z)$ converges for every $z \in \boldsymbol{C}$. In particular, if $r > 0$ and

$$\Phi_{r,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (rz)^{2k}}{2^{2k} k! \, \Gamma(\mu + k + 1)}, \qquad z \in \boldsymbol{C},$$

from Proposition 2.2 we deduce that $\mathcal{E}H'$ is contained in $D_{\Phi_{r,\mu}}$. Note that the function $\Phi_{r,\mu}$, $r > 0$, is closely connected with the Bessel function J_μ of the first kind and order μ (see [25]).

PROPOSITION 2.4. – *Let $f \in \mathcal{E}H'$. Then $\Delta_\mu f \in \mathcal{E}H'$. Moreover $\Phi_{r,\mu}(\Delta_\mu) f$ is in $\mathcal{E}H'$, for every $r > 0$.*

PROOF. – Assume that $z_1, z_2, \ldots, z_n \in \boldsymbol{C}$ with $n \in \boldsymbol{N}$. By taking into account that the operators Δ_μ and τ_z, $z \in \boldsymbol{C}$, commute on $\boldsymbol{H}_e$, (2.7) leads to

$$\text{(2.8)} \qquad \tau_{z_1} \tau_{z_2} \ldots \tau_{z_n} (\Delta_\mu f)(z) = \frac{2\Gamma(\mu + 2)}{\pi i} \int_{C_1} \frac{(\tau_{z_1} \ldots \tau_{z_n} \tau_z f)(w)}{w^3} dw, \qquad z \in \boldsymbol{C}.$$

Here C_1 denotes the circle with parametric representation $w = e^{it}$, $t \in [0, 2\pi)$.

Since $f \in \mathcal{E}H'$, $\Delta_\mu f$ is an even and entire function and, by (2.8), for every $n, l \in \boldsymbol{N}$ there exist $C > 0$ and $r \in \boldsymbol{N}$ such that

$$|\tau_{z_1} \tau_{z_2} \ldots \tau_{z_n} (\Delta_\mu f)(z)| \leqslant$$
$$C((1 + |z_1|)(1 + |z_2|) \ldots (1 + |z_n|)(1 + |z|))^r, \qquad z_1, z_2, \ldots, z_n, z \in I_l.$$

Hence $\Delta_\mu f \in \mathcal{E}H'$.

Let now $r > 0$. As it was mentioned $\mathcal{E}H'$ is contained in $D_{\Phi_{r,\mu}}$. Moreover, by Proposition 2.2, the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k} k! \, \Gamma(\mu + k + 1)} \Delta_\mu^k f(z)$$

is convergent in $\boldsymbol{H}_e$. Hence, according to (2.7), we can write

$$\tau_{z_1} \tau_{z_2} \ldots \tau_{z_n} (\Phi_{r,\mu}(\Delta_\mu) f)(z) = \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2\pi i} \int_{C_{2r}} \frac{\tau_{z_1} \tau_{z_2} \ldots \tau_{z_n} \tau_z (f)(w)}{w^{2k+1}} dw,$$

for every $z, z_1, \ldots, z_n \in \boldsymbol{C}$, where C_{2r} represents the circle with parametric representation $w = 2re^{it}$, $t \in [0, 2\pi)$. Then, since $f \in \mathcal{E}H'$, we conclude that $\Phi_{r,\mu}(\Delta_\mu) f \in \mathcal{E}H'$. ∎

We now establish that the Hankel convolution maps $\mathcal{O}'_{\#} \times \mathcal{E}H'$ into $\mathcal{E}H'$.

PROPOSITION 2.5. – *Let* $S \in \mathcal{O}'_{\#}$ *and* $f \in \mathcal{E}H'$. *Then* $S\#f \in \mathcal{E}H'$.

PROOF. – According to [21, Proposition 4.2], for every $m \in \boldsymbol{N}$ there exist $k \in \boldsymbol{N}$ and continuous functions f_j on $(0, \infty)$ such that $(1+x^2)^{m+1} x^{2\mu+1} f_j(x)$ is bounded on $(0, \infty)$, $j = 0, 1, \ldots, k$, and

$$\langle S, \phi \rangle = \sum_{j=0}^{k} \int_0^\infty f_j(x)\, \Delta_\mu^j \phi(x)\, x^{2\mu+1}\, dx, \qquad \phi \in \mathcal{O}_{-m,\#}.$$

Let $l \in \boldsymbol{N}$. Since $f \in \mathcal{E}H'$, by Proposition 2.2, there exist $C > 0$ and $r \in \boldsymbol{N}$ for which

$$|\Delta_\mu^j(\tau_z f)(x)| \leq C((1+x)(1+|z|))^r,$$

when $x \in (0, \infty)$, $j \in \boldsymbol{N}$ and $z \in I_l$. Here C can be depending on j but r is not depending on j.

We choose $m \in \boldsymbol{N}$ such that $f \in \mathcal{O}_{-m,\#}$ and that $2m+1 > r$. Then

$$(S\#f)(z) = \sum_{j=0}^{k} \int_0^\infty f_j(x)\, \tau_z(\Delta_\mu^j f)(x)\, x^{2\mu+1}\, dx, \qquad z \in (0, \infty).$$

Moreover, since for every $j = 0, 1, \ldots, k$ the function $\tau_z(\Delta_\mu^j f)(x)$ is continuous on the set $\{(x, z) : x \in (0, \infty),\ z \in \boldsymbol{C}\}$, $S\#f$ can be continuously extended to $\boldsymbol{C}$ as an even function.

Let $j \in \boldsymbol{N}$, $0 \leq j \leq k$. We can write

$$\frac{d}{dz} \tau_z(\Delta_\mu^j f)(x) = z^{-2\mu-1} \int_0^z w^{2\mu+1} \Delta_{\mu,w} \tau_w(\Delta_\mu^j f)(x)\, dw, \qquad z \in \boldsymbol{C} \setminus \{0\}.$$

The last integral is extended on the segment from 0 to z.

Then if $l \in \boldsymbol{N}$, for a certain $r \in \boldsymbol{N}$ it has

$$\begin{aligned} \left| \frac{d}{dz} \tau_z(\Delta_\mu^j f)(x) \right| &\leq |z|^{-2\mu-1} \int_0^z |w|^{2\mu+1} |\tau_w(\Delta_\mu^{j+1} f)(x)|\, |dw| \\ &\leq C(1+|z|)^{r+1}(1+x)^r, \qquad x \in (0, \infty) \text{ and } z \in I_l \setminus \{0\}. \end{aligned}$$

Hence, $S\#f$ is a holomorphic function on $I_l \setminus \{0\}$ and

$$\frac{d}{dz}(S\#f)(z) = \sum_{j=0}^{k} \int_0^\infty f_j(x) \frac{d}{dz} \tau_z(\Delta_\mu^j f)(x)\, x^{2\mu+1}\, dx, \qquad z \in I_l \setminus \{0\}.$$

Since $S\#f$ is continuous on $\boldsymbol{C}$, Riemann theorem implies that $S\#f$ is holomorphic on I_l. Arbitrariness of l allows to conclude that $S\#f$ is an entire function.

Also, for every $w \in \boldsymbol{C}$, the function $\tau_w(S\#f)$ is even and entire.

By choosing a suitable representation (according to [21, Proposition 4.2]) for S and by proceeding as above we can see that, for every $l, n \in \boldsymbol{N}$, there exist $C > 0$ and $s \in \boldsymbol{N}$, for which

$$|\tau_{z_1}\tau_{z_2}\dots\tau_{z_n}(S\#f)(z)| \leq$$
$$C((1+|z|)(1+|z_1|)\dots(1+|z_n|))^s, \qquad z, z_1, z_2, \dots, z_n \in I_l.$$

Thus we conclude that $S\#f \in \mathcal{E}H'$. ■

Next result will be very useful in the sequel. Similar results can be encountered in [9, Proposition 3.2] and [29, Lemma 1]

PROPOSITION 2.6. – *Assume that* $\{\xi_j\}_{j\in\boldsymbol{N}}$ *is a sequence of positive real numbers being* $\xi_0 > 1$ *and* $\xi_{j+1} - \xi_j > 1$, *for every* $j \in \boldsymbol{N}$, *and that* $\{a_j\}_{j\in\boldsymbol{N}}$ *is a sequence of complex numbers for which there exists a positive real number* γ *verifying that* $|a_j| = O(e^{-\gamma\xi_j})$, *as* $j \to \infty$. *Then the series*

$$\sum_{j=0}^{\infty} a_j \tau_{\xi_j}\delta$$

converges in the weak $*$ *topology of* H', *where* δ *denotes, as usual, the Dirac functional. Moreover,* $h'_\mu\left(\sum_{j=0}^{\infty} a_j\tau_{\xi_j}\delta\right)$ *is in* $\mathcal{E}H'$ *if, and only if, for every* $\eta > 0$, $|a_j| = O(e^{-\eta\xi_j})$, *as* $j \to \infty$.

PROOF. – Let $\phi \in H$. For every $n, m \in \boldsymbol{N}$, $n > m$, we can write

$$\left|\sum_{j=m}^{n} a_j\langle\tau_{\xi_j}\delta, \phi\rangle\right| \leq \sum_{j=m}^{n}|a_j||\phi(\xi_j)| \leq C\sum_{j=m}^{n} e^{-\gamma j}.$$

Hence, the series $\sum_{j=0}^{\infty} a_j\langle\tau_{\xi_j}\delta, \phi\rangle$ converges in $\boldsymbol{C}$. Thus we proved that the series $\sum_{j=0}^{\infty} a_j\tau_{\xi_j}\delta$ converges in the weak $*$ topology of H'.

According to [6, Lemma 2.1] we have that

$$h'_\mu\left(\sum_{j=0}^{\infty} a_j\tau_{\xi_j}\delta\right) = 2^\mu\Gamma(\mu+1)\sum_{j=0}^{\infty} a_j(.\,\xi_j)^{-\mu}J_\mu(.\,\xi_j),$$

where the convergence of the last series is understood in the weak $*$ topology of H'. Moreover, by taking into account [13, (5.3.a)] the last series defines a holomorphic function in the interior of the strip I_γ. Indeed, for every $n, m \in \boldsymbol{N}$, being $n > m$, it has

$$\left|\sum_{j=m}^{n} a_j(z\xi_j)^{-\mu}J_\mu(z\xi_j)\right| \leq C\sum_{j=m}^{n} e^{-(\gamma-|\mathrm{Im}\, z|)\xi_j}, \qquad |\mathrm{Im}\, z| < \gamma.$$

We now define

$$F(z) = \sum_{j=0}^{\infty} a_j (z\xi_j)^{-\mu} J_\mu(z\xi_j), \qquad |\mathrm{Im}\, z| < \gamma.$$

Suppose that $|a_j| = O(e^{-\eta\xi_j})$, as $j \to \infty$, for each $\eta > 0$. Then, by proceeding as above, we can see that F is an even and entire function that is bounded in I_l, for each $l \in \boldsymbol{N}$. Since the series defining F converges in $\boldsymbol{H}_e$, by [19, 2, (1)], we get

$$\tau_{z_1}\tau_{z_2}\dots\tau_{z_n}(F)(z) =$$

$$(2^\mu \Gamma(\mu+1))^n \sum_{j=0}^{\infty} a_j (z\xi_j)^{-\mu} J_\mu(z\xi_j)(z_1\xi_j)^{-\mu} J_\mu(z_1\xi_j)\dots(z_n\xi_j)^{-\mu} J_\mu(z_n\xi_j),$$

for every $z, z_1, z_2, \dots, z_n \in \boldsymbol{C}$. By invoking again [13, (5.3.a)] we can see that $F \in \mathcal{E}H'$.

Assume now that $F \in \mathcal{E}H'$. Let $r > 0$. By Proposition 2.4, $\Phi_{r,\mu}(\Delta_\mu)F \in \mathcal{E}H'$. Moreover, for every $l \in \boldsymbol{N}$ there exists $m \in \boldsymbol{N}$ such that

$$(1+|z|)^{-m} \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k} k!\, \Gamma(\mu+k+1)} \Delta_\mu^k F(z)$$

converges uniformly in I_l.

According to [4, (3.1)], we can write, for every $\phi \in H$,

$$2^\mu \Gamma(\mu+1) \int_0^\infty (xy)^{-\mu} J_\mu(xy)\, \Phi_{r,\mu}(\Delta_\mu)\, F(x)\, \phi(x) x^{2\mu+1}\, dx$$

$$= \int_0^\infty \Phi_{r,\mu}(\Delta_\mu)\, F(x)\, h_\mu(\tau_y(h_\mu \phi))(x)\, x^{2\mu+1}\, dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k} k!\, \Gamma(\mu+k+1)} \int_0^\infty \Delta_\mu^k F(x)\, h_\mu(\tau_y(h_\mu\phi))(x)\, x^{2\mu+1}\, dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k} k!\, \Gamma(\mu+k+1)} \int_0^\infty F(x)\, \Delta_\mu^k h_\mu(\tau_y(h_\mu\phi))(x)\, x^{2\mu+1}\, dx$$

$$= \sum_{k=0}^{\infty} \frac{r^{2k}}{2^{2k} k!\, \Gamma(\mu+k+1)} \langle h'_\mu(F)(x), x^{2k} \tau_y(h_\mu\phi)(x)\rangle$$

$$= \sum_{k=0}^{\infty} \frac{r^{2k}}{2^{2k} k!\, \Gamma(\mu+k+1)} \sum_{j=0}^{\infty} a_j \xi_j^{2k} \tau_y(h_\mu\phi)(\xi_j), \qquad y \in (0, \infty).$$

By invoking Proposition 2.4 and Corollary 2.3, $\Phi_{r,\mu}(\Delta_\mu)\, F$ is a multiplier of

H. From [1, Satz 5] it follows that, for every $\phi \in H$ and $l \in \boldsymbol{N}$,

$$(2.9)\quad y^l \int_0^\infty (xy)^{-\mu} J_\mu(xy)\, \Phi_{r,\mu}(\Delta_\mu)\, F(x)\, \phi(x)\, x^{2\mu+1} dx \to 0, \quad \text{as } y \to \infty .$$

We now choose a function $\phi \in H$ such that $h_\mu(\phi)(x) \geqslant 0$, $x \in (0, \infty)$, $h_\mu(\phi)(x) = 0$, $x \notin (0, 1)$, and $h_\mu(\phi)(x) > \frac{1}{2}$, $x \in \left(0, \frac{1}{2}\right)$. Note that such a function can be easily found.

If $x, y \in (0, \infty)$ and $x - y > 1$, by using [15, 8.11, (31)] (see also [19, p. 308, (2)]) then

$$(2.10)\qquad \tau_y(h_\mu \phi)(x) = \int_{x-y}^{x+y} D(x, y, z)\, h_\mu(\phi)(z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dz = 0 .$$

On the other hand, according to again [15, 8.11, (31)], we can write

$$(2.11)\qquad \tau_x(h_\mu \phi)(x) = \int_0^{2x} D(x, x, z)\, h_\mu(\phi)(z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dz$$

$$= C \int_0^{2x} x^{-4\mu} z^{2\mu} (4x^2 - z^2)^{\mu - 1/2} h_\mu(\phi)(z)\, dz$$

$$= C \int_0^{1} x^{-4\mu} z^{2\mu} (4x^2 - z^2)^{\mu - 1/2} h_\mu(\phi)(z)\, dz$$

$$= C \int_0^{1/2x} u^{2\mu} (1 - u^2)^{\mu - 1/2} h_\mu(\phi)(2xu)\, du$$

$$\geqslant C \int_{1/8x}^{1/4x} u^{2\mu} (1 - u^2)^{\mu - 1/2} h_\mu(\phi)(2xu)\, du$$

$$\geqslant C \int_{1/8x}^{1/4x} u^{2\mu} (1 - u^2)^{\mu - 1/2} du$$

$$\geqslant C x^{-2\mu - 1}, \qquad x \geqslant \frac{1}{2} .$$

From (2.10) we deduce that

$$2^{\mu}\Gamma(\mu+1)\int_0^{\infty}(x\xi_l)^{-\mu}J_{\mu}(x\xi_l)\,\Phi_{r,\mu}(\Delta_{\mu})\,F(x)\,\phi(x)\,x^{2\mu+1}dx$$

$$=\sum_{k=0}^{\infty}\frac{r^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\xi_l^{2k}a_l\tau_{\xi_l}(h_{\mu}\phi)(\xi_l)$$

$$=\Phi_{r,\mu}(i\xi_l)\,a_l\tau_{\xi_l}(h_{\mu}\phi)(\xi_l),\qquad l\in\boldsymbol{N}.$$

Hence, (2.9) and (2.11) imply that

$$a_l\Phi_{r,\mu}(i\xi_l)\to 0,\qquad \text{as } l\to\infty.$$

By taking into account $\Phi_{r,\mu}(iz)=2^{\mu}(rz)^{-\mu}\mathbb{I}_{\mu}(rz)$, $z\in\boldsymbol{C}$ and $r>0$, where $\mathbb{I}_{\mu}$ denotes the modified Bessel function of the first kind and order μ, from [26, (5), 6.2] (see also [25, p. 203, (2) and (3)]) it infers that

$$\Phi_{r,\mu}(ir\xi_l)\geq C(r\xi_l)^{-\mu-1/2}e^{r\xi_l},\qquad l\in\boldsymbol{N}.$$

Hence, it is conclude that $|a_l|=O(e^{-r\xi_l})$, as $l\to\infty$, for every $r>0$. Thus the proof is finished. ■

The last proposition allows us to obtain necessary conditions in order that a distribution $T\in\mathcal{O}'_{\#}$ is entire elliptic in H'.

PROPOSITION 2.7. – *Let $S\in\mathcal{O}'_{\#}$. If S is entire elliptic in H' then, there exist positive constants a and A such that*

$$|h'_{\mu}(S)(y)|\geq e^{-ay},\qquad y>A. \tag{2.12}$$

PROOF. – Suppose that we can not find $a, A>0$ for which (2.12) holds. Then there exists a sequence $\{\xi_j\}_{j\in\boldsymbol{N}}\subset(0,\infty)$ such that $\xi_0>1$, $\xi_j-\xi_{j-1}>1$, for every $j\in\boldsymbol{N}\backslash\{0\}$, and $|h'_{\mu}(S)(\xi_j)|<e^{-j\xi_j}$, for each $j\in\boldsymbol{N}$.

We define the distribution

$$T=2^{\mu}\Gamma(\mu+1)\sum_{j=0}^{\infty}(.\,\xi_j)^{-\mu}J_{\mu}(.\,\xi_j).$$

It is not hard to see that the series defining T converges in H'. Moreover, Proposition 2.6 implies that $T\notin\mathcal{E}H'$. On the other hand, by the interchange formula for the distributional Hankel transformation ([21, Proposition 4.5]), we have

$$h'_{\mu}(T\#S)=h'_{\mu}(T)\,h'_{\mu}(S)$$

$$=\sum_{j=0}^{\infty}h'_{\mu}(S)(\xi_j)\,\tau_{\xi_j}\delta.$$

Hence,

$$T\#S = 2^{\mu}\,\Gamma(\mu+1)\sum_{j=0}^{\infty} h'_{\mu}(S)(\xi_j)(.\,\xi_j)^{-\mu} J_{\mu}(.\,\xi_j),$$

and by taking into account Proposition 2.6, $T\#S \in \mathcal{E}H'$.

Thus we conclude that S is not entire elliptic on H'. ■

In the next proposition we prove that the condition (2.12) implies the entire ellipticity of the element S of $\mathcal{O}'_{\#}$.

PROPOSITION 2.8. – *Let $S \in \mathcal{O}'_{\#}$. If there exist $a, A > 0$ such that* (2.12) *holds for S, then S is entire elliptic on H'.*

PROOF. – We first take a function $\phi \in H$ such that $\phi(x) = 1$, $x \leqslant A$, and $\phi(x) = 0$, $x > A+1$. We define the function g by

$$g(x) = 0, \quad 0 < x \leqslant A, \quad \text{and } g(x) = \frac{1-\phi(x)}{\Phi_{2a,\mu}(ix)\, h'_{\mu}(S)(x)}, \quad x > A.$$

It is clear that g is a smooth function on $(0, \infty)$. Moreover, by taking into account that $h'_{\mu}(S)$ is a multiplier of H ([21, Proposition 4.2]) and [28, (5) and (8), 6.2], we can see that g is a multiplier of H. Indeed, by using the Leibniz rule we can see that, for every $k \in \boldsymbol{N}$,

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{k}\left(\frac{1-\phi(x)}{\Phi_{2a,\mu}(ix)\, h'_{\mu}(S)(x)}\right)\right|$$

has a polynomial growth at infinity. Hence the distribution $G = h'_{\mu}(g)$ is in $\mathcal{O}'_{\#}$ ([21, Proposition 4.2]).

Moreover,

$$\Phi_{2a,\mu}(\Delta_{\mu})(S\#G) = \delta - \Phi, \tag{2.13}$$

where $\Phi = h_{\mu}(\phi)$. Indeed, let $\varphi \in H$. We can write

$$\langle \Phi_{2a,\mu}(\Delta_{\mu})(S\#G), \varphi\rangle$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2a)^{2k}}{2^{2k} k!\,\Gamma(\mu+k+1)} \langle S\#\Delta_{\mu}^{k} G, \varphi\rangle$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2a)^{2k}}{2^{2k} k!\,\Gamma(\mu+k+1)} \langle h'_{\mu}(S)\, h'_{\mu}(\Delta_{\mu}^{k} G), h_{\mu}(\varphi)\rangle$$

$$= \sum_{k=0}^{\infty} \frac{(2a)^{2k}}{2^{2k} k! \, \Gamma(\mu+k+1)} \int_0^{\infty} x^{2k} g(x) \, h'_\mu(S)(x) \, h_\mu(\varphi)(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dx$$

$$= \sum_{k=0}^{\infty} \frac{(2a)^{2k}}{2^{2k} k! \, \Gamma(\mu+k+1)} \int_A^{\infty} x^{2k} \frac{1-\phi(x)}{\Phi_{2a,\mu}(ix)} h_\mu(\varphi)(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dx$$

$$= \int_0^{\infty} (1-\phi(x)) \, h_\mu(\varphi)(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dx$$

$$= h_\mu(h_\mu \varphi)(0) - \int_0^{\infty} h_\mu(\phi)(x) \, \varphi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dx$$

$$= \langle \delta, \varphi \rangle - \langle h_\mu(\phi), \varphi \rangle.$$

Then (2.13) is established. Note that (2.13) implies also that $\Phi_{2a,\mu}(\Delta_\mu)(S\#G)$ is in $\mathcal{O}'_\#$.

Also the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2a)^{2k}}{2^{2k} k! \, \Gamma(\mu+k+1)} \Delta_\mu^k (S\#G)$$

converges in the space $\mathcal{O}'_\#$. Indeed, let $\varphi \in H$. By proceeding as above we can see that

$$\left\langle h'_\mu \left(\sum_{k=0}^{n} \frac{(-1)^k (2a)^{2k}}{2^{2k} k! \, \Gamma(\mu+k+1)} \Delta_\mu^k (S\#G) \right), \varphi \right\rangle =$$

$$\sum_{k=0}^{n} \frac{(2a)^{2k}}{2^{2k} k! \, \Gamma(\mu+k+1)} \left\langle x^{2k} \frac{1-\phi(x)}{\Phi_{2a,\mu}(ix)}, \varphi(x) \right\rangle.$$

Hence, it is sufficient to show that the series

$$\sum_{k=0}^{\infty} \frac{(2ax)^{2k}}{2^{2k} k! \, \Gamma(\mu+k+1)} \frac{1-\phi(x)}{\Phi_{2a,\mu}(ix)}$$

converges in the topology of $\mathcal{O}$. Let $s \in \boldsymbol{N}$. By invoking [28, (5) and (8), 6.2] it obtains,

$$\begin{aligned}
&\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{s}\left(\sum_{k=0}^{n}\frac{(2ax)^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\,\frac{1-\phi(x)}{\Phi_{2a,\mu}(ix)}-(1-\phi(x))\right)\right| \\
&=\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{s}\left(\left(\sum_{k=0}^{n}\frac{(2ax)^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\,\frac{1}{\Phi_{2a,\mu}(ix)}-1\right)(1-\phi(x))\right)\right| \\
&\leqslant \sum_{j=0}^{s}\binom{s}{j}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{k-j}(1-\phi(x))\right|\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{j}\left(\sum_{k=0}^{n}\frac{(2ax)^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\,\frac{1}{\Phi_{2a,\mu}(ix)}-1\right)\right| \\
&\leqslant C\sum_{j=0}^{s}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{j}\left(\sum_{k=0}^{n}\frac{(2ax)^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\,\frac{1}{\Phi_{2a,\mu}(ix)}-1\right)\right| \\
&\leqslant C(1+x^2)^l,\ x\in(0,\infty)\ \text{and}\ n\in\boldsymbol{N},
\end{aligned}$$

for some $l \in \boldsymbol{N}$ that is not depending on $x \in (0,\ \infty)$ and $n \in \boldsymbol{N}$.

Let $\varepsilon > 0$ and $s \in \boldsymbol{N}$. If l is the nonnegative integer that is associated to s as above, there exists $x_0 > 0$ such that, for every $n \in \boldsymbol{N}$,

$$\sup_{x\geqslant x_0}\frac{1}{(1+x^2)^{l+1}}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{s}\left(\left(\sum_{k=0}^{n}\frac{(2ax)^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\,\frac{1}{\Phi_{2a,\mu}(ix)}-1\right)(1-\phi(x))\right)\right|<\varepsilon.$$

Moreover, we can find $n_0 \in \boldsymbol{N}$ for which

$$\sup_{0<x<x_0}\frac{1}{(1+x^2)^{l+1}}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{s}\left(\left(\sum_{k=0}^{n}\frac{(2ax)^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\,\frac{1}{\Phi_{2a,\mu}(ix)}-1\right)(1-\phi(x))\right)\right|<\varepsilon,$$

provided that $n \geqslant n_0$.

Hence, we conclude that, for every $n \geqslant n_0$,

$$\sup_{0<x<\infty}\frac{1}{(1+x^2)^{l+1}}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^{s}\left(\left(\sum_{k=0}^{n}\frac{(2ax)^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\,\frac{1}{\Phi_{2a,\mu}(ix)}-1\right)(1-\phi(x))\right)\right|<\varepsilon.$$

Thus, it is showed that

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(2ax)^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\,\frac{1-\phi(x)}{\Phi_{2a,\mu}(ix)}=1-\phi(x),$$

in the topology of $\mathcal{O}$.

Assume now that $T\#S = f$ where $T \in H'$ and $f \in \mathcal{E}H'$. According to (2.13) and by taking into account the series

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2a)^{2k}}{2^{2k}k!\,\Gamma(\mu+k+1)}\Delta_{\mu}^{k}(S\#G)$$

converges in $\mathcal{O}'_{\#}$ we can write

$$T = T\#(\Phi_{2a,\mu}(\Delta_\mu)(S\#G)) + T\#\Phi$$
$$= \Phi_{2a,\mu}(\Delta_\mu)((T\#S)\#G) + T\#\Phi$$
$$= \Phi_{2a,\mu}(\Delta_\mu)(f\#G) + T\#\Phi. \quad (2.14)$$

By Propositions 2.4 and 2.5, $\Phi_{2a,\mu}(\Delta_\mu)(f\#G)$ is in $\mathcal{E}H'$.
Moreover, $T\#\Phi \in \mathcal{E}H'$. Indeed, by [4, (3.1)], we have

$$(T\#\Phi)(x) = \langle T, \tau_x \Phi\rangle$$
$$= \langle h'_\mu(T)(t), 2^\mu \Gamma(\mu+1)(xt)^{-\mu} J_\mu(xt)\, \phi(t)\rangle, \quad x \in (0, \infty).$$

For every $x \in \boldsymbol{C}$, the series

$$(xt)^{-\mu} J_\mu(xt)\phi(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (xt)^{2k}}{2^{2k+\mu} k!\, \Gamma(\mu+k+1)} \phi(t)$$

converges in $\mathcal{B}$. Then it deduces that

$$(T\#\Phi)(x) = \Gamma(\mu+1) \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!\, \Gamma(\mu+k+1)} \langle (h'_\mu T)(t), t^{2k} \phi(t)\rangle, \quad x \in (0, \infty).$$

Hence $T\#\Phi$ can be extended as an even and entire function.
By virtue of [6, Lemma 2.2], we get, for every $z, z_1, z_2, \ldots, z_n \in \boldsymbol{C}$ and $n \in \boldsymbol{N}$,

$$\tau_{z_1}\tau_{z_2}\ldots\tau_{z_n}(T\#\Phi)(z) =$$
$$\langle h'_\mu(T)(t), (2^\mu\Gamma(\mu+1))^{n+1} (z_1 t)^{-\mu} J_\mu(z_1 t)(z_2 t)^{-\mu} J_\mu(z_2 t)\ldots$$
$$(z_n t)^{-\mu} J_\mu(z_n t)(zt)^{-\mu} J_\mu(zt)\, \phi(t)\rangle.$$

Since $h'_\mu T \in \mathcal{B}'$ and $\phi \in \mathcal{B}$, there exist $r \in \boldsymbol{N}$ and $C > 0$ for which

$$|\tau_{z_1}\tau_{z_2}\ldots\tau_{z_n}(T\#\Phi)(z)| \leqslant$$
$$C \max_{0\leqslant k\leqslant r} \sup_{0<t<A+1} \left| \left(\frac{1}{t}\frac{d}{dt}\right)^k ((z_1 t)^{-\mu} J_\mu(z_1 t)(z_2 t)^{-\mu} J_\mu(z_2 t)\ldots \right.$$
$$\left. (z_n t)^{-\mu} J_\mu(z_n t)(zt)^{-\mu} J_\mu(zt)\, \phi(t)) \right|,$$

for each $z, z_1, z_2, \ldots, z_n \in \boldsymbol{C}$.

Therefore, according to [28, (7), 5.1] and [13, (5.3.a)], for each n, $l \in \boldsymbol{N}$, one has, for every $z, z_1, z_2, \ldots, z_n \in I_l$,

$$|\tau_{z_1} \tau_{z_2} \ldots \tau_{z_n} (T \# \Phi)(z)| \leq C((1+|z|)(1+|z_1|) \ldots (1+|z_n|))^{2r}.$$

Thus we prove that S is entire elliptic in H'. ■

We now give a distribution $S \in \mathcal{O}'_\#$ that is entire elliptic but it is not hypoelliptic in H'.

Let $a > 0$. We define the function $\phi(x) = 1/\Phi_{a,\mu}(ix)$, $x > 0$. By taking into account [28, (5) and (8), 6.2] we can see that $\phi \in H$. Then, according to [1, Satz 5], $S = h_\mu(\phi) \in H$. By [21, Proposition 4.2] it follows that $S \in \mathcal{O}'_\#$. Moreover, for every $k \in \boldsymbol{N}$, $y^k h'_\mu(S)(y) = y^k \phi(y) \to 0$, as $y \to \infty$. By invoking [9, Proposition 3.3] we conclude that S is not hypoelliptic in H'.

Moreover, S is entire elliptic in H'. Indeed, according to [28, (5), 6.2], there exists $C > 0$ such that

$$|h'_\mu(S)(y)| = \phi(y) \geq Ce^{-y},$$

when y is large enough. Hence, from Proposition 2.8 one deduces that S is entire elliptic in H'.

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