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Entire elliptic Hankel convolution equations


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Entire Elliptic Hankel Convolution Equations (*).

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Summary. – In this paper we characterize the entire elliptic Hankel convolutors on tempered distributions in terms of the growth of their Hankel transforms.

1. – Introduction and preliminaries.

The Hankel transformation is usually defined by ([18])

\[ h_\mu(f)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(x) x^{2\mu + 1} dx, \quad y > 0. \]

Here \( J_\mu \) denotes the Bessel function of the first kind and order \( \mu \). Throughout this paper we will assume that \( \mu > -\frac{1}{2} \).

The Hankel transformation \( h_\mu \) has been studied in spaces of distributions of slow growth by G. Altenburg [1]. Altenburg's investigation was inspired in the studies of A. H. Zemanian ([26] and [28]) about the variant \( \mathcal{C}_\mu \) of the Hankel transformation defined through

\[ \mathcal{C}_\mu(f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x) dx, \quad y > 0. \]

It is clear that \( h_\mu \) and \( \mathcal{C}_\mu \) are closely connected.

G. Altenburg [1] introduced the space \( H \) constituted by all those complex valued and smooth functions \( \phi \) on \( (0, \infty) \) such that, for every \( m, n \in \mathbb{N} \),

\[ \gamma_{m,n}(\phi) = \sup_{x \in (0, \infty)} (1 + x^2)^m \left| \left( \frac{1}{x} \frac{d}{dx} \right)^n \phi(x) \right| < \infty. \]

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On $H$ it considers the topology associated with the family $\{g_{m, n}\}_{m, n \in \mathbb{N}}$ of seminorms. Thus $H$ is a Fréchet space and $h_\mu$ is an automorphism of $H$ ([1, Satz 5]). According to [12, p. 85] the space $H$ coincides with the space $S_{\text{even}}$ constituted by all the even functions in the Schwartz space $S$. From [3, Theorem 2.3] it is immediately deduced that a function $f$ defined on $[0, \infty)$ is a pointwise multiplier of $H$, write $f \in \mathcal{O}$, if, and only if, $f$ is smooth on $[0, \infty)$ and, for every $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ for which $(1 + x^2)^{-\alpha} \left( \frac{1}{x} \frac{d}{dx} \right)^k f(x)$ is bounded on $(0, \infty)$.

The dual space of $H$, is, as usual, represented by $H'$. If $f$ is a measurable function on $(0, \infty)$ such that $(1 + x^2)^{-\alpha} f(x)$ is a bounded function on $(0, \infty)$, for some $n \in \mathbb{N}$, then $f$ generates an element of $H'$, that we continue calling $f$, by

$$\langle f, \phi \rangle = \int_0^\infty f(x) \phi(x) \frac{x^{2\mu + 1}}{2^\mu \Gamma(\mu + 1)} dx, \quad \phi \in H.$$ 

The Hankel transformation $h_\mu'$ is defined on $H'$ as the transpose of $h_\mu$-transformation of $H$. That is, if $T \in H'$ the Hankel transformation $h_\mu' T$ is the element of $H'$ given through

$$\langle h_\mu' T, \phi \rangle = \langle T, h_\mu \phi \rangle, \quad \phi \in H.$$ 

Thus $h_\mu'$ is an automorphism of $H'$ when on $H'$ it considers the weak * or the strong topologies.

Also in [1] G. Altenburg considered, for every $a > 0$ the space $\mathcal{B}_a$ constituted by all those functions $\phi$ in $H$ such that $\phi(x) = 0$, $x \geq a$. $\mathcal{B}_a$ is endowed with the topology induced on it by $H$. The Hankel transform $h_\mu(\mathcal{B}_a)$ of $\mathcal{B}_a$ can be characterized by invoking [27, Theorem 1]. The union space $\mathcal{B} = \bigcup_{a > 0} \mathcal{B}_a$ is equipped with the inductive topology. The dual spaces of $\mathcal{B}_a$, $a > 0$, and $\mathcal{B}$ are denoted, as usual, by $\mathcal{B}'_a$, $a > 0$, and $\mathcal{B}'$, respectively.

In [24] K. Trimèche introduced, for every $a > 0$, the space $\mathcal{O}_* a$ constituted by all those smooth and even functions $\phi$ on $\mathbb{R}$ such that $\phi(x) = 0$, $|x| \geq a$. Also he considered the union space $\mathcal{O}_* = \bigcup_{a > 0} \mathcal{O}_* a$. According to [12, p. 85], the spaces $\mathcal{B}_a$, $a > 0$, and $\mathcal{B}$, coincides with the spaces $\mathcal{O}_* a$, $a > 0$, and $\mathcal{O}_*$, respectively.

F. M. Cholewinski [10], D. T. Haimo [17] and I. I. Hirschman [19] investigated the convolution operation of the Hankel transformation $h_\mu$ on Lebesgue spaces. We say that a measurable function $f$ is in $L_{1, \mu}$ when

$$\int_0^\infty |f(x)| x^{2\mu + 1} dx < \infty.$$
If \(f, g \in L_{1, \mu}\) the Hankel convolution \(\mathcal{C}_\mu f \# \mu g\) of \(f\) and \(g\) is defined by

\[
(\mathcal{C}_\mu f \# \mu g)(x) = \int_0^\infty f(y) (x, y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dy, \quad a.e. \quad x \in (0, \infty),
\]

where the Hankel translated \(\mathcal{C}_\mu f \# \mu g\), \(x \in (0, \infty)\), is given through

\[
(\mathcal{C}_\mu f \# \mu g)(y) = \int_0^\infty g(z) D_\mu(x, y, z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dz, \quad a.e. \quad y \in (0, \infty),
\]

and being

\[
D_\mu(x, y, z) = (2^\mu \Gamma(\mu+1))^2 \int_0^\infty (xt)^{-\mu} J_{\mu}(xt)(yt)^{-\mu} J_{\mu}(yt)(zt)^{-\mu} J_{\mu}(zt) t^{2\mu+1} dt,
\]

\(x, y, z \in (0, \infty)\).

Here \(a.e.\) is understood respect to the Lebesgue measure on \((0, \infty)\).

The Hankel transformation \(h_{\mu}\) and the Hankel convolution \(\mathcal{C}_\mu\) are related by ([19, Theorem 2.d])

\[
h_{\mu}(\mathcal{C}_\mu f \# \mu g) = h_{\mu}(f) h_{\mu}(g), \quad f, g \in L_{1, \mu}.
\]

Since we think no confusion will appear, in the sequel we will write \(\#\), \(\tau_x\), \(x \in (0, \infty)\), and \(D\) instead of \(\mathcal{C}_\mu\), \(\mathcal{C}_\mu \mathcal{C}_\mu\), \(x \in (0, \infty)\), and \(D_\mu\), respectively.

As it was mentioned the transformations \(\mathcal{C}_\mu\) and \(h_{\mu}\) are closely connected. After a straightforward manipulation it can be deduced from \(\#\) a form for the convolution operation \(*\) for the Hankel transformation \(\mathcal{C}_\mu\).

The investigation of the \(*\) convolution on the distribution spaces was began by J. de Sousa-Pinto [23]. He considered the 0-order transformation \(\mathcal{C}_0\) and compact support distributions on \((0, \infty)\). More recently in a series of papers J. J. Betancor and I. Marrero ([4], [5], [6], [7] and [21]) have extended the studies of J. de Sousa-Pinto. They defined the \(*\) convolution of the Hankel transformation \(\mathcal{C}_0\) on Zemanian distribution spaces of slow growth ([21]) and rapid growth ([4]). J. J. Betancor and L. Rodriguez-Mesa ([9]) studied the hypoellipticity of Hankel \(*\) convolution on Zemanian distribution spaces.

The main aspects of the distributional theory developed by the \(*\) convolution can be transplanted to the \# convolution. Our objective in this paper is to analyze the entire ellipticity of the \# convolution operators on the spaces \(H\) and \(H'\).

For every \(x \in (0, \infty)\), the Hankel translated \(\tau_x\) defines a continuous linear mapping from \(H\) into itself ([21, Proposition 2.1]). For every \(T \in H'\) and \(\phi \in H\)
the Hankel convolution $T\#\phi$ of $T$ and $\phi$ is defined by

$$(T\#\phi)(x) = \langle T, \tau_x \phi \rangle, \quad x \in (0, \infty).$$

By [21, Proposition 3.5], $T\#\phi$ is a multiplier of $H$, for each $T \in H'$ and $\phi \in H$. In general $T\#\phi$ is not in $H$ when $T \in H'$ and $\phi \in H$. Indeed, if we define the functional $T$ on $H$ by

$$\langle T, \phi \rangle = \int_0^\infty \phi(x) \frac{x^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \, dx, \quad \phi \in H,$$

then $T \in H'$ and, for every $\phi \in H$,

$$(T\#\phi)(x) = \int_0^\infty (\tau_x \phi)(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \, dy = \int_0^\infty \phi(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \, dy, \quad x \in (0, \infty).$$

Hence $T\#\phi \notin H$ when $\int_0^\infty \phi(y) \frac{y^{2\mu+1}}{2^\mu \Gamma(\mu+1)} \, dy \neq 0$. According to [21, Proposition 4.2] we can characterize the subspace constituted by all those $T \in H'$ such that $T\#\phi \in H$, for every $\phi \in H$. Let $m \in \mathbb{Z}$. We say that a complex valued and smooth function $\phi$ on $(0, \infty)$ is in $O_{\mu, m, \#}$ if and only if, for every $k \in \mathbb{N}$,

$$w_{m, \mu}^k(\phi) = \sup_{x \in (0, \infty)} (1 + x^2)^m |\Delta_\mu^k \phi(x)| < \infty,$$

where $\Delta_\mu$ denotes the Bessel operator $x^{-2\mu-1} D x^{2\mu+1} D$. $O_{\mu, m, \#}$ is a Fréchet space when it is endowed with the topology associated with the system $\{w_{m, \mu}^k\}_{k \in \mathbb{N}}$ of seminorms. It is clear that $H$ is contained in $O_{\mu, m, \#}$. We denote by $\mathcal{O}_{\mu, m, \#}$ the closure of $H$ in $O_{\mu, m, \#}$. By $\mathcal{O}_{\mu, \#}$ we represent the inductive limit space $\bigcup_{m \in \mathbb{Z}} \mathcal{O}_{\mu, m, \#}$. The dual space $\mathcal{O}_{\mu, \#}'$ of $\mathcal{O}_{\mu, \#}$ can be characterized as the subspace of $H'$ of $\#$-convolution operators on $H$ ([5, Proposition 2.5]). Moreover, by defining on $\mathcal{O}_{\mu, \#}'$ the topology associated with the family $\{\eta_{m, k, \phi}\}_{m \in \mathbb{N}, \phi \in H}$ of seminorms, where, for each $m, k \in \mathbb{N}$ and $\phi \in H$,

$$\eta_{m, k, \phi}(T) = w_{m, \mu}^k(T\#\phi), \quad T \in \mathcal{O}_{\mu, \#}' ,$$

and by considering on $\mathcal{O}$ the topology induced by the simple topology of the space $\mathcal{C}(H)$ of the linear and continuous mappings from $H$ into itself, the Hankel transformation $b_{\mu}$ is an isomorphism from $\mathcal{O}_{\mu, \#}'$ onto $\mathcal{O}$.

The Hankel convolution $T\#S$ of $T \in H'$ and $S \in \mathcal{O}_{\mu, \#}'$ is defined by

$$\langle T\#S, \phi \rangle = \langle T, S\#\phi \rangle, \quad \phi \in H.$$

Thus $T\#S \in H'$, for each $T \in H'$ and $S \in \mathcal{O}_{\mu, \#}'$.

In [9] J. J. Betancor and L. Rodríguez-Mesa investigated the hypoellipticity of the $*$-Hankel convolution equations on Zemanian spaces. Results as in
[9] can be obtained for the \#-Hankel convolutions. A distribution $S \in \mathcal{O}^\mu_{\#}$ is said to be hypoelliptic in $H'$ when the following property holds: $T \in \mathcal{O}^\mu_{\#}$ provided that $T \in H'$ and $T \# S \in \mathcal{O}^\mu_{\#}$. From [9, Proposition 3.3] it infers that $S \in \mathcal{O}^\mu_{\#}$ is hypoelliptic in $H'$ when, and only when, there exist $b, B > 0$ such that

$$|h^\mu_\mu(S)(y)| \geq y^{-b}, \quad y \geq B.$$ 

Motivated by the celebrated paper of L. Ehrenpreis [14] and the investigations of Z. Zielezny [29], we study in this paper the entire elliptic Hankel convolution equations on $H'$.

By $H_e$ we represent the space of even and entire functions. It is equipped, as usual, with the topology of the uniform convergence of the bounded sets of $C$.

We will say that $f \in H_e$ is in $\mathcal{E}_H$ if, and only if, for every $l, n \in \mathbb{N}$, there exist $C > 0$ and $k \in \mathbb{N}$ for which

$$|\tau_{z_1} \tau_{z_2} \cdots \tau_{z_n}(f)(z)| \leq C((1 + |z|)(1 + |z_1|) \cdots (1 + |z_n|))^k, \quad z, z_1, z_2, \ldots, z_n \in I_l,$$

where $I_l = \{w \in C : |\text{Im} w| \leq l\}$.

Here the complex Hankel translation operator $\tau_z$, $z \in C$, must be understood as in [11]. If $f \in H_e$ and $f(z) = \sum_{k=0}^\infty a_k z^{2k}$, $z \in C$, then

$$(\tau_w f)(z) = \sum_{n=0}^\infty a_n \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(n + \mu + 1) \Gamma(\mu + 1)}{\Gamma(n - k + \mu + 1) \Gamma(k + \mu + 1)} z^{2(n-k)} w^{2k}, \quad z, w \in C.$$ 

Thus, the Hankel translation operator is extended to the complex plane.

A distribution $S \in \mathcal{O}^\mu_{\#}$ will say to be entire elliptic in $H'$ when the following property holds: $T \in \mathcal{E}_H$ provided that $T \in H'$ and $T \# S \in \mathcal{E}_H$.

We will start Section 2 proving that the space $\mathcal{O}^\mu_{\#}$ of Hankel convolution operators of $H$ is really not depending on $\mu$. Also, in Section 2 we obtain a characterization for the entire elliptic elements of $\mathcal{O}^\mu_{\#}$ in terms of the growth of their Hankel transforms. We will prove that $S \in \mathcal{O}^\mu_{\#}$ is entire elliptic on $H'$ if, and only if, there exist $a, A > 0$ such that

$$|h^\mu_\mu(S)(y)| \geq e^{-ay}, \quad y \geq A.$$ 

Throughtout this paper by $C$ we always represent a suitable positive constant that can change from a line to the other one.
2. – Entire elliptic Hankel convolution equations in $H'$. 

We firstly prove that the space $O_{\mu, #}$ of Hankel convolution operators is really not depending on $\mu$.

Let $m \in \mathbb{Z}$, $m \leq 0$. We denote by $O_{m, #}$ the space constituted by all those smooth functions $\phi$ on $(0, \infty)$ for which there exists an even and smooth function $\psi$ such that $\psi(x) = \phi(x)$, $x \in (0, \infty)$, and that

$$\gamma_m^k(\phi) = \sup_{x \in (0, \infty)} (1 + x^2)^m |D^k \phi(x)| < \infty,$$

for every $k \in \mathbb{N}$. $O_{m, #}$ is endowed with the topology associated with the family $\{\gamma_m^k\}_{k \in \mathbb{N}}$ of seminorms. Thus, $O_{m, #}$ is a Fréchet space. By $O_{m, #}$ we understood the closure of $O_{*, #}$ in $O_{m, #}$. It is clear that $O_{m, #}$ is a Fréchet space. Moreover, $O_{m, #}$ contains continuously $O_{m+1, #}$. The union space $\bigcup_{m \in \mathbb{Z}, m \leq 0} O_{m, #}$ is denoted by $O_{#}$ and it is contained in the space $O$ of the pointwise multipliers of $H$.

Note that, for every $m \in \mathbb{Z}$, $m \leq 0$, a function $\phi \in O_{m, #}$ if, and only if, $\phi$ can be extended to an even function $\psi$ that is in the space $S_m$ studied in [20] and [22]. Hence an even and smooth function $\phi$ on $\mathbb{R}$ is in $O_{m, #}$ when, and only when, for every $k \in \mathbb{N}$, $\lim_{x \to \infty} (1 + x^2)^m D^k \phi(x) = 0$.

**Proposition 2.1.** – Let $m \in \mathbb{Z}$, $m \leq 0$. The spaces $O_{\mu, m, #}$ and $O_{m, #}$ coincide topologically and algebraically.

**Proof.** – Assume that $\phi \in O_{\mu, m, #}$. There exists a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ in $O_{*, #}$ such that $\phi_n \to \phi$, as $n \to \infty$, in $O_{\mu, m, #}$.

Let $k \in \mathbb{N}$. We choose a function $\alpha \in O_{*, 2k}$, such that $\alpha(x) = 1$, $x \in (-k, k)$. Then, since $\{\phi_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $O_{\mu, m, #}$, $\{\phi_n \alpha\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $O_{*, 2k}$. Hence, there exists $\psi \in O_{*, 2k}$ for which $\phi_n \alpha \to \psi$, as $n \to \infty$, in $O_{*, 2k}$. Since the convergence in $O_{\mu, m, #}$ implies the pointwise convergence on $(0, \infty)$, we conclude that $\phi$ admits an even and smooth extension to $\mathbb{R}$.

We can write

$$\left( \frac{1}{x^\mu} \right) \phi(x) = x^{-2\mu - 2} \int_0^x \Delta^\mu \phi(t) t^{2\mu + 1} dt, \quad x \in (0, \infty).$$

Hence, it obtains

$$\sup_{x \in (0, \infty)} (1 + x^2)^m \left| \left( \frac{1}{x^\mu} \right) \phi(x) \right| \leq C \sup_{x \in (0, \infty)} (1 + x^2)^m |\Delta^\mu \phi(x)|.$$
Moreover, since
\[ \Delta_{\mu} \phi(x) = D^2 \phi(x) + \frac{2\mu + 1}{x} D\phi(x), \quad x \in (0, \infty), \]
we have that
\[ (2.1) \quad \sup_{x \in (0, \infty)} (1 + x^2)^m |D^2 \phi(x)| \leq C \sup_{x \in (0, \infty)} (1 + x^2)^m |\Delta_{\mu} \phi(x)|. \]

On the other hand, a straightforward manipulation allows to get
\[ (2.2) \quad \int_{x}^{x+1} (x + 1 - t) D^2 \phi(t) \, dt = -D\phi(x) + \phi(x + 1) - \phi(x), \quad x \in (0, \infty). \]

Hence, we deduce from (2.1) and (2.2) that
\[ (2.3) \quad \sup_{x \in (0, \infty)} (1 + x^2)^m |D\phi(x)| \leq C \left( \sup_{x \in (0, \infty)} (1 + x^2)^m |D^2 \phi(x)| + \sup_{x \in (0, \infty)} (1 + x^2)^m |\phi(x)| \right). \]

Also we have that
\[ (2.4) \quad D\Delta_{\mu} \phi(x) = D^3 \phi(x) + (2\mu + 1) x \left( \frac{1}{x} D \right)^2 \phi(x), \quad x \in (0, \infty). \]

The family \( \{ w_{m, \mu}^k \}_{m, k \in N} \) generates the topology of \( H \). Then, we can find \( k \in N \) such that
\[ \sup_{x \in (0, 1)} \left| \left( \frac{1}{x} D \right)^2 \phi(x) \right| \leq \sup_{x \in (0, 1)} \left| \left( \frac{1}{x} D \right)^2 (\phi(x) \alpha(x)) \right| \leq C \sup_{x \in (0, 2)} |D_{\mu}^k (\phi(x) \alpha(x))|, \]
where \( \alpha \in O_{s, 2} \) and \( \alpha(x) = 1, \ |x| \leq 1. \)

Hence from (2.1), (2.3) and (2.4), since \( \sup_{x \in (0, \infty)} (1 + x^2)^m |D\Delta_{\mu} \phi(x)| < \infty \), it is deduced that
\[ \sup_{x \in (0, \infty)} (1 + x^2)^m |D^3 \phi(x)| < \infty. \]

By repeating the above procedure we can prove that \( \phi \in O_{m, \#}. \)

Moreover, since \( \phi_n \to \phi \), as \( n \to \infty \), in \( O_{\mu, m, \#} \), the above arguments allows us to conclude that \( (1 + x^2)^m |D_k \phi(x)| \to 0 \), as \( x \to \infty \), for every \( k \in N \). Thus we show that \( \phi \in O_{m, \#}. \)
Suppose now that $\phi \in \mathcal{O}_{m, \#}$. Let $k \in \mathbb{N}$. It is not hard to see that
\begin{equation}
|\Delta_{\mu}^k \phi(x)| \leq C \sum_{j=0}^{2k} |D^j \phi(x)|, \quad x \geq 1.
\end{equation}

Moreover, by choosing a function $\alpha \in \mathcal{O}_{*, 2}$, since $\{w_{l, \mu}\}_{l, \mu \in \mathbb{N}}$ generates the topology of $H$, we can find $l \in \mathbb{N}$ such that
\begin{equation}
\sup_{x \in (0, 1)} (1 + x^2)^m |D_{\mu}^k \phi(x)| \leq \sup_{x \in (0, 1)} |D_{\mu}^k (\phi(x) \alpha(x))| \\
\leq C \sum_{j=0}^{l} \sup_{x \in (0, 2)} |D^j (\phi(x) \alpha(x))| \\
\leq C \sum_{j=0}^{l} \sup_{x \in (0, \infty)} (1 + x^2)^m |D^j \phi(x)|.
\end{equation}

By combining (2.5) and (2.6) we obtain that $\phi \in O_{\mu, m, \#}$. Also, we can see that if $\{\phi_{n}\}_{n \in \mathbb{N}} \subset \mathcal{O}_{\#}$ and $\phi_{n} \rightarrow \phi$, as $n \rightarrow \infty$, in $O_{m, \#}$, then $\phi_{n} \rightarrow \phi$, as $n \rightarrow \infty$, in $O_{\mu, m, \#}$. Hence we deduce that $\phi \in \mathcal{O}_{\mu, m, \#}$.

Thus we proved that $O_{\mu, m, \#} = \mathcal{O}_{m, \#}$. Moreover (2.5) and (2.6) imply that the topology generated by $\{\gamma_{m, k}\}_{k \in \mathbb{N}}$ is stronger than the one induced by $\{w_{m, \mu}\}_{k \in \mathbb{N}}$. Then the open mapping theorem allows to conclude that the topologies defined by $\{\gamma_{m, k}\}_{k \in \mathbb{N}}$ and $\{w_{m, \mu}\}_{k \in \mathbb{N}}$ coincide.

Thus the proof is finished.}

From Proposition 2.1 we infer that $\mathcal{O}_{\#} = \mathcal{O}_{\mu, \#}$. Hence the space of Hankel convolution operators $\mathcal{O}_{\mu, \#}$, $\mu > -\frac{1}{2}$, coincides with the dual space $\mathcal{O}_{\#}^\prime$ of $\mathcal{O}_{\#}$.

Although, according to Proposition 2.1, the space of Hankel convolution operators is not depending on $\mu$, the representation given in [21, Proposition 4.2] that involves the Bessel operator $\Delta_{\mu}$ is very useful.

Our next objective is to obtain a characterization of the entire elliptic elements of $\mathcal{O}_{\#}$ involving the Hankel transformation.

Firstly some properties of the elements of $\mathcal{E}H$ are established.

**Proposition 2.2.** – Let $f \in \mathcal{E}H$. Then, for every $l \in \mathbb{N}$, there exists $C > 0$ and $r \in \mathbb{N}$, such that, for each $0 < R < l$,
\begin{equation}
|\Delta_{\mu}^k f(z)| \leq C \left( \frac{2}{R} \right)^{2k} k! \Gamma(\mu + k + 1)(1 + |z|)^r (1 + R)^r, \quad z \in I_l \text{ and } k \in \mathbb{N}.
\end{equation}
PROOF. – Since $f$ is an even and entire function, according to [11], we can write
\[ (\tau_z f)(w) = \sum_{k=0}^{\infty} \frac{w^{2k}}{2^k k! \Gamma(\mu + k + 1)} (\Delta_{\mu}^k f)(z), \quad w, z \in \mathbb{C}. \]

Hence, for every $k \in \mathbb{N}$, $R > 0$ and $z \in \mathbb{C}$, it has
\[ (\Delta_{\mu}^k f)(z) = \frac{2^k k! \Gamma(\mu + k + 1)}{2\pi i} \int_{C_R} \frac{(\tau_z f)(w)}{w^{2k+1}} \, dw. \]

Here $C_R$ denotes the circle having as a parametric representation to $w(t) = Re^{it}$, $T \in [0, 2\pi)$. Then, for every $l \in \mathbb{N}$ and $0 < R < l$, there exists $C > 0$ and $r \in \mathbb{N}$, for which
\[ |\Delta_{\mu}^k f(z) | \leq C \left( \frac{2}{R} \right)^{2k} k! \Gamma(\mu + k + 1)(1 + |z|)^r(1 + R)^r, \quad z \in I_l \text{ and } k \in \mathbb{N}. \]

A consequence of Proposition 2.2 is the following one.

COROLLARY 2.3. – Let $f \in \mathcal{E} \mathcal{H}'$. Then $f \in \mathcal{O}_\#$.

PROOF. – To see that $f \in \mathcal{O}_\#$ it is sufficient to use Proposition 2.2 and to argue as in the proof of Proposition 2.1. □

By proceeding as in [16, Proposition 5.2] (see also [2, Proposition 3.5]) we can prove that if $L$ is a continuous linear mapping from $H_e$ into itself that commutes with Hankel translations, that is, $\tau_z L = L \tau_z$, for every $z \in \mathbb{C}$, then there exists an even and entire function $\Phi$ of exponential type such that, for every $f \in H_e$,
\[ Lf(z) = \sum_{k=0}^{\infty} a_k \Delta_{\mu}^k f(z), \quad z \in \mathbb{C}, \]
where $\Phi(w) = \sum_{k=0}^{\infty} a_k w^{2k}$, $w \in \mathbb{C}$.

In the sequel, if $\Phi$ is an even and entire function admitting the representation $\Phi(w) = \sum_{k=0}^{\infty} a_k w^{2k}$, $w \in \mathbb{C}$, we will understand by $\Phi(\Delta_{\mu})$ the operator defined by
\[ \Phi(\Delta_{\mu}) f = \sum_{k=0}^{\infty} a_k \Delta_{\mu}^k f, \quad f \in D_{\Phi}. \]

Here the domain $D_{\Phi}$ of $\Phi(\Delta_{\mu})$ is constituted by all those even and entire functions $f$ such that the series $\sum_{k=0}^{\infty} a_k \Delta_{\mu}^k f(z)$ converges for every $z \in \mathbb{C}$. In particu-
lar, if $r > 0$ and

$$\Phi_{r, \mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (rz)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)}, \quad z \in \mathbb{C},$$

from Proposition 2.2 we deduce that $\mathcal{E}H'$ is contained in $D_{\Phi_{r, \mu}}$. Note that the function $\Phi_{r, \mu}$, $r > 0$, is closely connected with the Bessel function $J_\mu$ of the first kind and order $\mu$ (see [25]).

**Proposition 2.4.** Let $f \in \mathcal{E}H'$. Then $\Delta_\mu f \in \mathcal{E}H'$. Moreover $\Phi_{r, \mu}(\Delta_\mu f)$ is in $\mathcal{E}H'$, for every $r > 0$.

**Proof.** Assume that $z_1, z_2, \ldots, z_n \in \mathbb{C}$ with $n \in \mathbb{N}$. By taking into account that the operators $\Delta_\mu$ and $\tau_z$, $z \in \mathbb{C}$, commute on $H_e$, (2.7) leads to

$$\tau_{z_1} \tau_{z_2} \cdots \tau_{z_n}(\Delta_\mu f)(z) = \frac{2 \Gamma(\mu + 2)}{\pi i} \int_{C_1} \frac{\tau_{z_1} \cdots \tau_{z_n} \tau_z f(w)}{w^{3}} dw, \quad z \in \mathbb{C}.$$  

Here $C_1$ denotes the circle with parametric representation $w = e^{it}$, $t \in [0, 2\pi)$.

Since $f \in \mathcal{E}H'$, $\Delta_\mu f$ is an even and entire function and, by (2.8), for every $n, l \in \mathbb{N}$ there exist $C > 0$ and $r \in \mathbb{N}$ such that

$$|\tau_{z_1} \tau_{z_2} \cdots \tau_{z_n}(\Delta_\mu f)(z)| \leq C((1 + |z_1|)(1 + |z_2|) \cdots (1 + |z_n|)(1 + |z|))^r, \quad z_1, z_2, \ldots, z_n, z \in I_l.$$  

Hence $\Delta_\mu f \in \mathcal{E}H'$.

Let now $r > 0$. As it was mentioned $\mathcal{E}H'$ is contained in $D_{\Phi_{r, \mu}}$. Moreover, by Proposition 2.2, the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \Delta_\mu^k f(z)$$

is convergent in $H_e$. Hence, according to (2.7), we can write

$$\tau_{z_1} \tau_{z_2} \cdots \tau_{z_n}(\Phi_{r, \mu}(\Delta_\mu f))(z) = \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \Delta_\mu^k f(z)$$

for every $z, z_1, \ldots, z_n \in \mathbb{C}$, where $C_{2r}$ represents the circle with parametric representation $w = 2re^{it}$, $t \in [0, 2\pi)$. Then, since $f \in \mathcal{E}H'$, we conclude that $\Phi_{r, \mu}(\Delta_\mu f) \in \mathcal{E}H'$.  

We now establish that the Hankel convolution maps $\mathcal{O}' \times \mathcal{E}H'$ into $\mathcal{E}H'$.  

PROPOSITION 2.5. – Let $S \in \mathcal{O}_\#$ and $f \in \mathcal{E} H'$. Then $S \# f \in \mathcal{E} H'$.

PROOF. – According to [21, Proposition 4.2], for every $m \in \mathbb{N}$ there exist $k \in \mathbb{N}$ and continuous functions $f_j$ on $(0, \infty)$ such that $(1 + x^2)^m + 1 x^{2n + 1} f_j(x)$ is bounded on $(0, \infty)$, $j = 0, 1, \ldots, k$, and

$$
(S, \phi) = \sum_{j=0}^{k} \int_{0}^{\infty} f_j(x) \Delta^j_{\mu} \phi(x) x^{2n + 1} dx, \quad \phi \in \mathcal{O}_{-m, \#}.
$$

Let $l \in \mathbb{N}$. Since $f \in \mathcal{E} H'$, by Proposition 2.2, there exist $C > 0$ and $r \in \mathbb{N}$ for which

$$
|\Delta^j_{\mu}(\tau_x f)(x)| \leq C((1 + x)(1 + |x|))^r,
$$

when $x \in (0, \infty)$, $j \in \mathbb{N}$ and $z \in I_l$. Here $C$ can be depending on $j$ but $r$ is not depending on $j$.

We choose $m \in \mathbb{N}$ such that $f \in \mathcal{O}_{-m, \#}$ and that $2m + 1 > r$. Then

$$
(S \# f)(z) = \sum_{j=0}^{k} \int_{0}^{\infty} f_j(x) \tau_x(\Delta^j_{\mu} f)(x) x^{2n + 1} dx, \quad z \in (0, \infty).
$$

Moreover, since for every $j = 0, 1, \ldots, k$ the function $\tau_x(\Delta^j_{\mu} f)(x)$ is continuous on the set $\{(x, z) : x \in (0, \infty), z \in C\}$, $S \# f$ can be continuously extended to $C$ as an even function.

Let $j \in \mathbb{N}$, $0 \leq j \leq k$. We can write

$$
\frac{d}{dz} \tau_x(\Delta^j_{\mu} f)(x) = z^{-2n - 1} \int_{0}^{z} \omega^{2n + 1} \Delta_{\mu, \omega} \tau_x(\Delta^j_{\mu} f)(x) d\omega, \quad z \in C \setminus \{0\}.
$$

The last integral is extended on the segment from 0 to $z$.

Then if $l \in \mathbb{N}$, for a certain $r \in \mathbb{N}$ it has

$$
\left| \frac{d}{dz} \tau_x(\Delta^j_{\mu} f)(x) \right| \leq |z|^{-2n - 1} \int_{0}^{z} |\omega|^{2n + 1} |\tau_x(\Delta^j_{\mu} f)(x)| |d\omega|
$$

$$
\leq C(1 + |x|)^{r + 1}(1 + x)^r, \quad x \in (0, \infty) \text{ and } z \in I_l \setminus \{0\}.
$$

Hence, $S \# f$ is a holomorphic function on $I_l \setminus \{0\}$ and

$$
\frac{d}{dz} (S \# f)(z) = \sum_{j=0}^{k} \int_{0}^{\infty} f_j(x) \frac{d}{dz} \tau_x(\Delta^j_{\mu} f)(x) x^{2n + 1} dx, \quad z \in I_l \setminus \{0\}.
$$

Since $S \# f$ is continuous on $C$, Riemann theorem implies that $S \# f$ is holomorphic on $I_l$. Arbitrariness of $l$ allows to conclude that $S \# f$ is an entire function.

Also, for every $w \in C$, the function $\tau_w(S \# f)$ is even and entire.
By choosing a suitable representation (according to [21, Proposition 4.2]) for $S$ and by proceeding as above we can see that, for every $l, n \in \mathbb{N}$, there exist $C > 0$ and $s \in \mathbb{N}$, for which

\[
|\tau_{z_1} \tau_{z_2} \cdots \tau_{z_n}(S \# f)(z)| \leq C((1 + |z|)(1 + |z_1|) \cdots (1 + |z_n|))^s, \quad z, z_1, z_2, \ldots, z_n \in I_l.
\]

Thus we conclude that $S \# f \in \mathcal{E} H'$.

Next result will be very useful in the sequel. Similar results can be encountered in [9, Proposition 3.2] and [29, Lemma 1]

**Proposition 2.6.** – Assume that $\{\xi_j\}_{j \in \mathbb{N}}$ is a sequence of positive real numbers being $\xi_0 > 1$ and $\xi_{j+1} - \xi_j > 1$, for every $j \in \mathbb{N}$, and that $\{a_j\}_{j \in \mathbb{N}}$ is a sequence of complex numbers for which there exists a positive real number $\gamma$ verifying that $|a_j| = O(e^{-\gamma \xi_j})$, as $j \to \infty$. Then the series

\[
\sum_{j=0}^{\infty} a_j \tau_{\xi_j} \delta
\]

converges in the weak * topology of $H'$, where $\delta$ denotes, as usual, the Dirac functional. Moreover, $h_{\mu}' \left( \sum_{j=0}^{\infty} a_j \tau_{\xi_j} \delta \right)$ is in $\mathcal{E} H'$ if, and only if, for every $\eta > 0$, $|a_j| = O(e^{-\eta \xi_j})$, as $j \to \infty$.

**Proof.** – Let $\phi \in H$. For every $n, m \in \mathbb{N}, n > m$, we can write

\[
\left| \sum_{j=m}^{n} a_j \langle \tau_{\xi_j} \delta, \phi \rangle \right| \leq \sum_{j=m}^{n} |a_j| |\phi(\xi_j)| \leq C \sum_{j=m}^{n} e^{-\gamma j}.
\]

Hence, the series $\sum_{j=0}^{\infty} a_j \langle \tau_{\xi_j} \delta, \phi \rangle$ converges in $C$. Thus we proved that the series $\sum_{j=0}^{\infty} a_j \tau_{\xi_j} \delta$ converges in the weak * topology of $H'$.

According to [6, Lemma 2.1] we have that

\[
h_{\mu}' \left( \sum_{j=0}^{\infty} a_j \tau_{\xi_j} \delta \right) = 2^\mu \Gamma(\mu + 1) \sum_{j=0}^{\infty} a_j \langle ., \xi_j \rangle^{-\mu} J_{\mu}(., \xi_j),
\]

where the convergence of the last series is understood in the weak * topology of $H'$. Moreover, by taking into account [13, (5.3.a)] the last series defines a holomorphic function in the interior of the strip $I_\gamma$. Indeed, for every $n, m \in \mathbb{N}$, being $n > m$, it has

\[
\left| \sum_{j=m}^{n} a_j (z \xi_j)^{-\mu} J_{\mu}(z \xi_j) \right| \leq C \sum_{j=m}^{n} e^{-(\gamma - |\text{Im} z|) \xi_j}, \quad |\text{Im} z| < \gamma.
\]
We now define
\[ F(z) = \sum_{j=0}^{\infty} a_j (z^{\xi_j})^{-\mu} J_\mu(z^{\xi_j}), \quad |\text{Im} \, z| < \gamma. \]

Suppose that \(|a_j| = O(e^{-\eta j})\), as \(j \to \infty\), for each \(\eta > 0\). Then, by proceeding as above, we can see that \(F\) is an even and entire function that is bounded in \(I_l\), for each \(l \in \mathbb{N}\). Since the series defining \(F\) converges in \(H_e\), by \([19, 2, (1)]\), we get

\[ \tau_{z_1} \tau_{z_2} \ldots \tau_{z_n} (F)(z) = \]

\[ (2^\mu \Gamma(\mu + 1))^n \sum_{j=0}^{\infty} a_j (z^{\xi_j})^{-\mu} J_\mu(z_1^{\xi_j})(z_1^{\xi_j})^{-\mu} J_\mu(z_1^{\xi_j}) \ldots (z_n^{\xi_j})^{-\mu} J_\mu(z_n^{\xi_j}), \]

for every \(z, z_1, z_2, \ldots, z_n \in \mathcal{C}\). By invoking again \([13, (5.3.a)]\) we can see that \(F \in \mathcal{E}^H\).

Assume now that \(F \in \mathcal{E}^H\). Let \(r > 0\). By Proposition 2.4, \(\Phi_{r, \mu}(\Delta_\mu) F \in \mathcal{E}^H\). Moreover, for every \(l \in \mathbb{N}\) there exists \(m \in \mathbb{N}\) such that

\[ (1 + |z|)^{-m} \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \Delta_\mu^k F(z) \]

converges uniformly in \(I_l\).

According to \([4, (3.1)]\), we can write, for every \(\phi \in H\),

\[ 2^\mu \Gamma(\mu + 1) \int_0^\infty (xy)^{-\mu} J_\mu(xy) \Phi_{r, \mu}(\Delta_\mu) F(x) \phi(x) x^{2\mu + 1} dx \]

\[ = \int_0^\infty \Phi_{r, \mu}(\Delta_\mu) F(x) h_\mu(\tau_y(h_\mu \phi))(x) x^{2\mu + 1} dx \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \int_0^\infty \Delta_\mu^k F(x) h_\mu(\tau_y(h_\mu \phi))(x) x^{2\mu + 1} dx \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \int_0^\infty F(x) \Delta_\mu^k h_\mu(\tau_y(h_\mu \phi))(x) x^{2\mu + 1} dx \]

\[ = \sum_{k=0}^{\infty} \frac{r^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \langle h_\mu'(F)(x), x^{2k} \tau_y(h_\mu \phi)(x) \rangle \]

\[ = \sum_{k=0}^{\infty} \frac{r^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \sum_{j=0}^{\infty} a_j z_j^{2k} \tau_y(h_\mu \phi)(\xi_j), \quad y \in (0, \infty). \]

By invoking Proposition 2.4 and Corollary 2.3, \(\Phi_{r, \mu}(\Delta_\mu) F\) is a multiplier of
From [1, Satz 5] it follows that, for every \( f \in H \) and \( l \in N \),

\[
y^l \int_0^x (xy)^{-\mu} J_{\mu}(xy) \Phi_{r,\mu}(\Lambda_{\mu}) F(x) \phi(x) x^{2\mu+1} dx \to 0, \quad \text{as } y \to \infty.
\]

We now choose a function \( \phi \in H \) such that \( h_\mu(\phi)(x) \geq 0, \ x \in (0, \infty) \), \( h_\mu(\phi)(x) = 0, \ x \notin (0, 1) \), and \( h_\mu(\phi)(x) > \frac{1}{2}, \ x \in \left(0, \frac{1}{2}\right) \). Note that such a function can be easily found.

If \( x, y \in (0, \infty) \) and \( x - y > 1 \), by using [15, 8.11, (31)] (see also [19, p. 308, (2)]) then

\[
\tau_y(h_\mu \phi)(x) = \int_{x-y}^{x+y} D(x, y, z) h_\mu(\phi)(z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu + 1)} dz = 0.
\]

On the other hand, according to again [15, 8.11, (31)], we can write

\[
\tau_x(h_\mu \phi)(x) = \int_0^{2x} D(x, x, z) h_\mu(\phi)(z) \frac{z^{2\mu+1}}{2^\mu \Gamma(\mu + 1)} dz
\]

\[
= C \int_0^1 x^{-4\mu} z^{2\mu}(4x^2 - z^2)^{\mu - 1/2} h_\mu(\phi)(z) \ dz
\]

\[
= C \int_0^{1/2x} u^{2\mu}(1 - u^2)^{\mu - 1/2} h_\mu(\phi)(2xu) \ du
\]

\[
\geq C \int_{1/8x}^{1/4x} u^{2\mu}(1 - u^2)^{\mu - 1/2} h_\mu(\phi)(2xu) \ du
\]

\[
\geq C \int_{1/8x}^{1/4x} u^{2\mu}(1 - u^2)^{\mu - 1/2} \ du
\]

\[
\geq C x^{-2\mu - 1}, \quad x \geq \frac{1}{2}.
\]
From (2.10) we deduce that
\[ 2^\mu \Gamma(\mu + 1) \int_0^\infty (x\xi) - \mu J_\mu(x\xi) \Phi_{r,\mu}(\Delta_\mu) F(x) \phi(x) x^{2\mu + 1} dx \]
\[ = \sum_{k=0}^\infty \frac{r^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \xi_j^{2k} a_l \tau_h(\phi)(\xi_j) \]
\[ = \Phi_{r,\mu}(i\xi_j) a_l \tau_h(\phi)(\xi_j), \quad l \in \mathbb{N}. \]

Hence, (2.9) and (2.11) imply that
\[ a_l \Phi_{r,\mu}(i\xi_j) \to 0, \quad \text{as } l \to \infty. \]

By taking into account \( \Phi_{r,\mu}(iz) = 2^\mu (rz)^{-\mu} I_\mu(rz), \quad z \in \mathbb{C} \) and \( r > 0 \), where \( I_\mu \) denotes the modified Bessel function of the first kind and order \( \mu \), from [26, (5), 6.2] (see also [25, p. 203, (2) and (3)]) it infers that
\[ \Phi_{r,\mu}(ir\xi_j) \geq C(r\xi_j)^{-\mu - 1/2} e^{-r\xi_j}, \quad l \in \mathbb{N}. \]

Hence, it is conclude that \( \|a_l\| = O(e^{-r\xi_j}), \) as \( l \to \infty \), for every \( r > 0 \).

Thus the proof is finished. \( \blacksquare \)

The last proposition allows us to obtain necessary conditions in order that a distribution \( T \in \mathcal{O}_\mu \) is entire elliptic in \( H' \).

**Proposition 2.7.** – Let \( S \in \mathcal{O}_\mu \). If \( S \) is entire elliptic in \( H' \) then, there exist positive constants \( a \) and \( A \) such that
\[ |h_\mu'(S)(y)| \geq e^{-ay}, \quad y > A. \] (2.12)

**Proof.** – Suppose that we can not find \( a, A > 0 \) for which (2.12) holds. Then there exists a sequence \( \{\xi_j\}_{j \in \mathbb{N} \subset (0, \infty)} \) such that \( \xi_0 > 1, \xi_j - \xi_{j-1} > 1, \) for every \( j \in \mathbb{N}\setminus\{0\} \), and \( |h_\mu'(S)(\xi_j)| < e^{-j\xi_j} \), for each \( j \in \mathbb{N} \).

We define the distribution
\[ T = 2^\mu \Gamma(\mu + 1) \sum_{j=0}^\infty (r\xi_j)^{-\mu} J_\mu(\xi_j). \]

It is not hard to see that the series defining \( T \) converges in \( H' \). Moreover, Proposition 2.6 implies that \( T \not\in \mathcal{E}H' \). On the other hand, by the interchange formula for the distributional Hankel transformation ([21, Proposition 4.5]), we have
\[ h_\mu'(T\#S) = h_\mu'(T) h_\mu'(S) \]
\[ = \sum_{j=0}^\infty h_\mu'(S)(\xi_j) \tau_{\xi_j} \delta. \]
Hence,

\[ T\#S = 2^\mu \Gamma(\mu + 1) \sum_{j=0}^{\infty} h_\mu'(S)(\xi_j)(.\xi_j)^{-\mu} J_\mu(.\xi_j), \]

and by taking into account Proposition 2.6, \( T\#S \in \mathcal{E} \).

Thus we conclude that \( S \) is not entire elliptic on \( H' \).

In the next proposition we prove that the condition (2.12) implies the entire ellipticity of the element \( S \) of \( \mathcal{O}_\# \).

**Proposition 2.8.** – Let \( S \in \mathcal{O}_\# \). If there exist \( a, A > 0 \) such that (2.12) holds for \( S \), then \( S \) is entire elliptic on \( H' \).

**Proof.** – We first take a function \( f \in H \) such that \( f(x) = 1, x \leq A \), and \( f(x) = 0, x > A + 1 \). We define the function \( g \) by

\[ g(x) = 0, \quad 0 < x \leq A, \quad \text{and} \quad g(x) = \frac{1 - \phi(x)}{\Phi_{2a,\mu}(ix) h_\mu'(S)(x)}, \quad x > A. \]

It is clear that \( g \) is a smooth function on \((0, \infty)\). Moreover, by taking into account that \( h_\mu'(S) \) is a multiplier of \( H \) ([21, Proposition 4.2]) and [28, (5) and (8), 6.2], we can see that \( g \) is a multiplier of \( H \). Indeed, by using the Leibniz rule we can see that, for every \( k \in \mathbb{N} \),

\[ \left\| \left( \frac{1}{x} \frac{d}{dx} \right)^k \left( \frac{1 - \phi(x)}{\Phi_{2a,\mu}(ix) h_\mu'(S)(x)} \right) \right\| \]

has a polynomial growth at infinity. Hence the distribution \( G = h_\mu'(g) \) is in \( \mathcal{O}_\# \) ([21, Proposition 4.2]).

Moreover,

(2.13) \[ \Phi_{2a,\mu}(\Delta_\mu)(S\#G) = \delta - \Phi, \]

where \( \Phi = h_\mu(\phi) \). Indeed, let \( \varphi \in H \). We can write

\[ \langle \Phi_{2a,\mu}(\Delta_\mu)(S\#G), \varphi \rangle \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k (2a)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \langle S\#\Delta^k_\mu G, \varphi \rangle \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k (2a)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \langle h_\mu'(S) h_\mu'(\Delta^k_\mu G), h_\mu(\varphi) \rangle \]
\[\left(2a\right)^{2k}k!\Gamma(\mu + k + 1)\int_0^\infty x^{2k}g(x)h_\mu'(S)(x)h_\mu(\varphi)(x)\frac{x^{2\mu + 1}}{2^\mu\Gamma(\mu + 1)}dx\]

\[= \sum_{k=0}^{\infty} \frac{\left(2a\right)^{2k}}{2^{2k}k!\Gamma(\mu + k + 1)}\int_0^\infty x^{2k}\frac{1 - \phi(x)}{\Phi_{2a,\mu}(ix)}h_\mu(\varphi)(x)\frac{x^{2\mu + 1}}{2^\mu\Gamma(\mu + 1)}dx\]

\[= \int_0^\infty (1 - \phi(x))h_\mu(\varphi)(x)\frac{x^{2\mu + 1}}{2^\mu\Gamma(\mu + 1)}dx\]

\[= h_\mu(\varphi)(0) - \int_0^\infty h_\mu(\varphi)(x)\varphi(x)\frac{x^{2\mu + 1}}{2^\mu\Gamma(\mu + 1)}dx\]

\[= \langle \delta, \varphi \rangle - \langle h_\mu(\varphi), \varphi \rangle.\]

Then (2.13) is established. Note that (2.13) implies also that \(\Phi_{2a,\mu}(\Delta k)(S\#G)\) is in \(\mathcal{O}_\#\).

Also the series

\[\sum_{k=0}^{\infty} \frac{(-1)^k\left(2a\right)^{2k}}{2^{2k}k!\Gamma(\mu + k + 1)}\Delta_k(S\#G)\]

converges in the space \(\mathcal{O}_\#\). Indeed, let \(\varphi \in H\). By proceeding as above we can see that

\[\left\langle h_\mu' \left(\sum_{k=0}^{n} \frac{(-1)^k\left(2a\right)^{2k}}{2^{2k}k!\Gamma(\mu + k + 1)}\Delta_k(S\#G)\right), \varphi \right\rangle =\]

\[\sum_{k=0}^{n} \frac{\left(2a\right)^{2k}}{2^{2k}k!\Gamma(\mu + k + 1)}\int_0^\infty x^{2k}\frac{1 - \phi(x)}{\Phi_{2a,\mu}(ix)}\times\varphi(x)dx.\]

Hence, it is sufficient to show that the series

\[\sum_{k=0}^{\infty} \frac{\left(2ax\right)^{2k}}{2^{2k}k!\Gamma(\mu + k + 1)}\frac{1 - \phi(x)}{\Phi_{2a,\mu}(ix)}\]
converges in the topology of \( \mathcal{O} \). Let \( s \in \mathbb{N} \). By invoking [28, (5) and (8), 6.2] it obtains,

\[
\left| \left( \frac{1}{x} \frac{d}{dx} \right)^s \left( \sum_{k=0}^{n} \frac{(2ax)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \frac{1 - \phi(x)}{\Phi_{2\alpha, \mu}(ix)} - (1 - \phi(x)) \right) \right| = \left| \left( \frac{1}{x} \frac{d}{dx} \right)^s \left( \sum_{k=0}^{n} \frac{(2ax)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \frac{1 - (1 - \phi(x))}{\Phi_{2\alpha, \mu}(ix)} - (1 - \phi(x)) \right) \right| \\
\leq \sum_{j=0}^{s} \left( \left( \frac{1}{x} \frac{d}{dx} \right)^{k-j} (1 - \phi(x)) \left| \left( \frac{1}{x} \frac{d}{dx} \right)^{j} \left( \sum_{k=0}^{n} \frac{(2ax)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \frac{1}{\Phi_{2\alpha, \mu}(ix)} - 1 \right) \right| \right) \\
\leq C \sum_{j=0}^{s} \left( \left( \frac{1}{x} \frac{d}{dx} \right)^{j} \left( \sum_{k=0}^{n} \frac{(2ax)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \frac{1}{\Phi_{2\alpha, \mu}(ix)} - 1 \right) \right) \\
\leq C(1 + x^2)^l, \ x \in (0, \infty) \quad \text{and} \quad n \in \mathbb{N},
\]

for some \( l \in \mathbb{N} \) that is not depending on \( x \in (0, \infty) \) and \( n \in \mathbb{N} \).

Let \( \varepsilon > 0 \) and \( s \in \mathbb{N} \). If \( l \) is the nonnegative integer that is associated to \( s \) as above, there exists \( x_0 > 0 \) such that, for every \( n \in \mathbb{N} \),

\[
\sup_{x \geq x_0} 1 + x^{2l+1} \left| \left( \frac{1}{x} \frac{d}{dx} \right)^{s} \left( \sum_{k=0}^{n} \frac{(2ax)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \frac{1}{\Phi_{2\alpha, \mu}(ix)} - 1 \right) (1 - \phi(x)) \right| < \varepsilon.
\]

Moreover, we can find \( n_0 \in \mathbb{N} \) for which

\[
\sup_{0 < x < x_0} \frac{1}{1 + x^{2l+1}} \left| \left( \frac{1}{x} \frac{d}{dx} \right)^{s} \left( \sum_{k=0}^{n} \frac{(2ax)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \frac{1}{\Phi_{2\alpha, \mu}(ix)} - 1 \right) (1 - \phi(x)) \right| < \varepsilon,
\]

provided that \( n \geq n_0 \).

Hence, we conclude that, for every \( n \geq n_0 \),

\[
\sup_{0 < x < \infty} \frac{1}{1 + x^{2l+1}} \left| \left( \frac{1}{x} \frac{d}{dx} \right)^{s} \left( \sum_{k=0}^{n} \frac{(2ax)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \frac{1}{\Phi_{2\alpha, \mu}(ix)} - 1 \right) (1 - \phi(x)) \right| < \varepsilon.
\]

Thus, it is showed that

\[
\lim_{n \to \infty} \sum_{k=0}^{n} \frac{(2ax)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \frac{1 - \phi(x)}{\Phi_{2\alpha, \mu}(ix)} = 1 - \phi(x),
\]

in the topology of \( \mathcal{O} \).

Assume now that \( T\# S = f \) where \( T \in \mathcal{H}' \) and \( f \in \mathcal{S}' \). According to (2.13) and by taking into account the series

\[
\sum_{k=0}^{\infty} \frac{(-1)^k (2a)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} A_k^\mu(S\# G)
\]
converges in $O_\#$ we can write
\begin{equation}
T = T\#_0(\Phi_{2\alpha,\mu}(\Delta_\mu)(S\#G)) + T\#_0
\end{equation}
\begin{equation}
= \Phi_{2\alpha,\mu}(\Delta_\mu)(T\#S\#G) + T\#_0
\end{equation}
\begin{equation}
= \Phi_{2\alpha,\mu}(\Delta_\mu)(f\#G) + T\#_0.
\end{equation}
By Propositions 2.4 and 2.5, $\Phi_{2\alpha,\mu}(\Delta_\mu)(f\#G)$ is in $\mathcal{S}'_\mu$.
Moreover, $T\#_0 \Phi \in \mathcal{S}'_\mu$. Indeed, by [4, (3.1)], we have
\begin{equation}
(T\#_0 \Phi)(x) = \langle T, \tau_x \Phi \rangle
\end{equation}
\begin{equation}
= \langle h_\mu'(T)(t), 2^\mu \Gamma(\mu + 1)(xt)^{-\mu} J_\mu(x t) \phi(t) \rangle, \quad x \in (0, \infty).
\end{equation}
For every $x \in \mathcal{C}$, the series
\begin{equation}
(x t)^{-\mu} J_\mu(x t) \phi(t) = \sum_{k = 0}^{\infty} \frac{(-1)^k (x t)^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \phi(t)
\end{equation}
converges in $\mathcal{B}$. Then it deduces that
\begin{equation}
(T\#_0 \Phi)(x) = \Gamma(\mu + 1) \sum_{k = 0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k! \Gamma(\mu + k + 1)} \langle (h_\mu'(T)(t), t^{2k} \phi(t) \rangle, \quad x \in (0, \infty).
\end{equation}
Hence $T\#_0 \Phi$ can be extended as an even and entire function.
By virtue of [6, Lemma 2.2], we get, for every $z, z_1, z_2, \ldots, z_n \in \mathcal{C}$ and $n \in \mathcal{N},$
\begin{equation}
\tau_{z_1} \tau_{z_2} \ldots \tau_{z_n}(T\#_0 \Phi)(z) =
\end{equation}
\begin{equation}
\langle h_\mu'(T)(t), (2^\mu \Gamma(\mu + 1))^{n+1} (z_1 t)^{-\mu} J_\mu(z_1 t)(z_2 t)^{-\mu} J_\mu(z_2 t)\ldots
\end{equation}
\begin{equation}
(z_n t)^{-\mu} J_\mu(z_n t)(zt)^{-\mu} J_\mu(zt) \phi(t) \rangle.
\end{equation}
Since $h_\mu'(T) \in \mathcal{B}'$ and $\phi \in \mathcal{B}$, there exist $r \in \mathcal{N}$ and $C > 0$ for which
\begin{equation}
|\tau_{z_1} \tau_{z_2} \ldots \tau_{z_n}(T\#_0 \Phi)(z)| \leq
\end{equation}
\begin{equation}
C \max_{0 \leq k \leq r} \left( \sup_{0 < t < A + 1} \left| \left( \frac{1}{t} \frac{d}{dt} \right)^k ((z_1 t)^{-\mu} J_\mu(z_1 t)(z_2 t)^{-\mu} J_\mu(z_2 t)\ldots
\end{equation}
\begin{equation}
(z_n t)^{-\mu} J_\mu(z_n t)(zt)^{-\mu} J_\mu(zt) \phi(t) \rangle \right),
\end{equation}
for each $z, z_1, z_2, \ldots, z_n \in \mathcal{C}$. 

\
Therefore, according to [28, (7), 5.1] and [13, (5.3.a)], for each \( n, l \in \mathbb{N} \), one has, for every \( z, z_1, z_2, \ldots, z_n \in I_l \),

\[
|\tau_z \tau_{z_1} \cdots \tau_{z_n} (T \# \Phi)(z)| \leq C(1 + |z|)(1 + |z_1|)\ldots(1 + |z_n|)^{2r}.
\]

Thus we prove that \( S \) is entire elliptic in \( H' \). \( \blacksquare \)

We now give a distribution \( S \in \mathcal{O}'_g \) that is entire elliptic but it is not hypoelliptic in \( H' \).

Let \( a > 0 \). We define the function \( \phi(x) = 1/\Phi_{a,\mu}(ix), \ x > 0 \). By taking into account [28, (5) and (8), 6.2] we can see that \( \phi \in H \). Then, according to [1, Satz 5], \( S = h_\mu'(\phi) \in H \). By [21, Proposition 4.2] it follows that \( S \in \mathcal{O}'_g \). Moreover, for every \( k \in \mathbb{N}, y^k h_\mu'(S)(y) = y^k \phi(y) \to 0 \), as \( y \to \infty \). By invoking [9, Proposition 3.3] we conclude that \( S \) is not hypoelliptic in \( H' \).

Moreover, \( S \) is entire elliptic in \( H' \). Indeed, according to [28, (5), 6.2], there exists \( C > 0 \) such that

\[
|h_\mu'(S)(y)| = \phi(y) \geq Ce^{-y},
\]

when \( y \) is large enough. Hence, from Proposition 2.8 one deduces that \( S \) is entire elliptic in \( H' \).

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