Alberto Farina

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One-Dimensional Symmetry for Solutions of Quasilinear Equations in $\mathbb{R}^2$.

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Sunto. – *In questo lavoro si considerano le equazioni quasilineari della forma* $\operatorname{div}(a(|\nabla u|)\nabla u) + f(u) = 0$ *in* $\mathbb{R}^2$ *e si studiano le proprietà delle soluzioni u il cui gradiente è limitato e non si annulla mai. Sotto un'ipotesi naturale, riguardante la crescita della fase del gradiente di u (si noti che la funzione* $\arg(\nabla u)$ *è ben definita in quanto* $|\nabla u| > 0$ *in* $\mathbb{R}^2$*), si dimostra che u è a simmetria unidimensionale, ovvero* $u = u(\nu \cdot x)$*, dove* ν *è un vettore unitario di* $\mathbb{R}^2$*. Come conseguenza di questo risultato si ottiene che ogni soluzione u avente una derivata positiva è a simmetria unidimensionale. Questo risultato fornisce la dimostrazione di una congettura di E. De Giorgi nel più ampio contesto delle equazioni quasilineari. In particolare, nel caso delle equazioni semilineari, si ottiene una nuova e semplice dimostrazione della (classica) congettura di De Giorgi.*

Summary. – *In this paper we consider two-dimensional quasilinear equations of the form* $\operatorname{div}(a(|\nabla u|)\nabla u) + f(u) = 0$ *and study the properties of the solutions u with bounded and non-vanishing gradient. Under a weak assumption involving the growth of the argument of* ∇u *(notice that* $\arg(\nabla u)$ *is a well-defined real function since* $|\nabla u| > 0$ *on* $\mathbb{R}^2$*) we prove that u is one-dimensional, i.e.,* $u = u(\nu \cdot x)$ *for some unit vector* ν*. As a consequence of our result we obtain that any solution u having one positive derivative is one-dimensional. This result provides a proof of a conjecture of E. De Giorgi in dimension 2 in the more general context of the quasilinear equations. In particular we obtain a new and simple proof of the classical De Giorgi's conjecture.*

1. – Introduction and main results.

This paper is concerned with the study of the one-dimensional symmetry for C^1 solutions of the quasilinear equation

$$\operatorname{div}(a(|\nabla u|)\nabla u) + f(u) = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^2) \tag{1.1}$$

satisfying

$$|\nabla u| > 0 \quad \text{in} \quad \mathbb{R}^2. \tag{1.2}$$

Typical representatives of quasilinear operators appearing in (1.1) are given by the p-Laplacian operator $(1 < p < +\infty)$ or by the prescribed mean curvature operator.

The class of nonlinear problems described by the equation (1.1) naturally arises in mathematical physics (reaction-diffusion problems, non-Newtonian fluids, porous media, plasma and nuclear physics, cosmology, etc.) as well as in geometry (theory of non parametric surfaces, theory of quasiregular and quasiconformal mappings, etc). Due to the role played by these models it is important to understand the symmetry properties of the solutions of (1.1). It is the purpose of this paper to study the one-dimensional symmetry properties of solutions u under the assumption (1.2).

On the other hand this type of symmetry problem is also related to a conjecture formulated by E. De Giorgi in 1978 (see [4], open question (3), page 175) and, more generally, to the following result:

THEOREM 1.1 ([6], [1]). – *Assume $F \in C^2(\mathbb{R})$. Let $u \in C^2(\mathbb{R}^N)$ be a bounded solution of*

$$\Delta u - F'(u) = 0 \quad \text{in} \quad \mathbb{R}^N \tag{1.3}$$

such that

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.4}$$

If $N = 2$ or $N = 3$, then u is one-dimensional, i.e., there are $a \in \mathbb{R}^N$ and $g \in C^2(\mathbb{R})$ such that

$$u(x) = g(a \cdot x) \qquad \forall\, x \in \mathbb{R}^N. \tag{1.5}$$

The case $N = 2$ was proved by N. Ghoussoub and C. Gui in [6] while the case $N = 3$ has been recently established by G. Alberti, L. Ambrosio and X. Cabré in [1]. In particular, Theorem 1.1. gives a positive answer, in the case of $\mathbb{R}^2$ and $\mathbb{R}^3$, to the above mentioned conjecture of E. De Giorgi (about this subject see also [2]).

In the present work we consider only the two-dimensional case. The purpose of this paper is to extend the one-dimensional symmetry result of Theorem 1.1 in different directions:

(i) by only assuming $|\nabla u| > 0$ in $\mathbb{R}^2$ instead of the monotonicity assumption (1.4).

(ii) by only assuming $|\nabla u| \in L^\infty(\mathbb{R}^2)$ (in particular we do not assume anything about the boundedness of u, see Remark 1.4.).

(iii) by treating quasilinear equations of the form (1.1), with $f \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$ and $a \in C^{1,1}_{\mathrm{loc}}(0, +\infty)$.

The function a satisfies the structural assumptions:

$$(S_1) \qquad \lim_{t\to 0} ta(t) = 0, \quad (ta(t))' > 0 \quad \forall t > 0$$

and

$$(S_2) \qquad \exists T' > 0, \quad \exists C = C(T') > 0 : (ta(t))' \leqslant Ct^{-2} \quad \forall t \in (0, T'].$$

These assumptions are satisfied by every function of the form:

$$a(t) = t^{\gamma}(\eta + t^2)^{\beta} \tag{1.6}$$

with, $\eta > 0$, $\gamma > -1$ and $\beta \geqslant -\frac{\gamma+1}{2}$. For a suitable choice of the parameters in (1.6), we recover the p-Laplacian operator, with $1 < p < +\infty$, the prescribed mean curvature operator as well as some more general operators satisfying non-standard growth conditions.

Ou main results are:

THEOREM 1.2. – *Let $f \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$ and suppose that the structural assumptions $(S_1)-(S_2)$ are satisfied. Let u be a $C^1(\mathbb{R}^2)$ solution of*

$$\begin{cases} \operatorname{div}(a(|\nabla u|)\nabla u) + f(u) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2) \\ |\nabla u| > 0 & \text{in } \mathbb{R}^2 \end{cases} \tag{1.7}$$

such that $\nabla u \in L^{\infty}(\mathbb{R}^2)$. Assume that there exists $\delta < 1$ such that:

$$|\arg(\nabla u)(x)| = O(\ln^{\delta}|x|), \quad \text{as } |x| \to +\infty, \tag{1.8}$$

then u is one-dimensional, i.e., there are $\nu \in \boldsymbol{S}^1$ and a function $g \in C^2(\mathbb{R})$ such that

$$u(x) = g(\nu\cdot x) \qquad \forall x \in \mathbb{R}^2,$$

$$|g'(x)| > 0 \qquad \forall x \in \mathbb{R}^2.$$

When we suppose that u satisfies:

$$\frac{\partial u}{\partial x_1} > 0 \quad \text{in} \quad \mathbb{R}^2$$

we have that the argument of the gradient of u can be written as:

$$\arg(\nabla u) = \arctan\left(\frac{u_2}{u_1}\right) \quad \text{in } \mathbb{R}^2,$$

where $u_j := \frac{\partial u}{\partial x_j}$, for $j = 1, 2$. Since in this case (1.8) is satisfied, a direct application of Theorem 1.2. gives the following extension of Theorem 1.1.

COROLLARY 1.3. – *Let $f \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$ and suppose that the structural assumptions $(S_1)-(S_2)$ are satisfied. Let $u \in C^1(\mathbb{R}^2)$ be a solution of*

$$\begin{cases} \operatorname{div}(a(|\nabla u|)\nabla u) + f(u) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2) \\ \dfrac{\partial u}{\partial x_2} > 0 & \text{in } \mathbb{R}^2 \end{cases} \tag{1.9}$$

such that $\nabla u \in L^\infty(\mathbb{R}^2)$. Then, u is one-dimensional, i.e., there are $\nu \in \boldsymbol{S}^1$ and a function $g \in C^2(\mathbb{R})$ such that

$$u(x) = g(\nu \cdot x) \qquad \forall x \in \mathbb{R}^2.$$

To obtain our symmetry results we consider the linear system satisfied by the first derivatives of u and rewrite it by considering ∇u as complex number. This procedure leads to a linear system of two equations with respect to new variables ϱ and θ, which are respectively the norm and the argument of the gradient vector of u. At this point we remark that, one of the equations writes as $\operatorname{div}(B(x)\nabla\theta) = 0$, where $B = (b_{hk})$ is a symmetric real matrix, whose entries are continuous and bounded functions on $\mathbb{R}^2$, such that:

$$\sum_{h,k=1}^{2} b_{hk}(x)\,\xi_h\xi_k > 0 \qquad \forall x \in \mathbb{R}^2, \qquad \forall \xi \in \mathbb{R}^2 \setminus \{0\}.$$

To conclude we invoke a classical Liouville-type theorem for elliptic (not necessarily uniformly elliptic) linear operators in divergence form defined over $\mathbb{R}^2$, proved by D. Gilbarg and J. Serrin in 1956 ([7]).

REMARK 1.4. – Observe that any C^1 solution of (1.1)-(1.2) actually belongs to C^2. This is an immediate consequence of the regularity results of Tolksdorf ([9]) and Ladyzhenskaya-Uraltseva ([8]) since u satisfies the assumption (1.2), f is locally Lipschitz-continuous and $a \in C^{1,1}_{\mathrm{loc}}(0, +\infty)$ satisfies the structural assumptions (S_1).

Standard elliptic estimates imply that any bounded C^1 solution of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$ has bounded gradient. On the other hand, the function $u(x) = x_1$ is an unbounded one-dimensional harmonic function with bounded and non-vanishing gradient. Thus our result generalizes Theorem 1.1. and it provides a proof of De Giorgi's conjecture in dimension 2 in the more general context of the quasilinear equations. In particular we obtain a new and simple proof of the classical De Giorgi's conjecture.

The latter remark also applies to solutions of a large class of quasilinear equations of the form (1.1). By making use of regularity results for solutions of quasilinear equations [9], it is easy to see that the property: $u \in L^\infty \Rightarrow \nabla u \in L^\infty$ is still true for the p-Laplacian operator or, more generally, for any operator satisfying some standard growth structural conditions.

REMARK 1.5. – In Theorem 1.2 and Corollary 1.3. the assumption $\nabla u \in L^\infty$ is necessary. Indeed, the function $u(x, y) = -\frac{x^2}{2} + y$ satisfies

$$\begin{cases} \Delta u + 1 = 0 & \text{in } \mathbb{R}^2 \\ \dfrac{\partial u}{\partial y} = 1 > 0 & \text{in } \mathbb{R}^2 \\ |\nabla u| = \sqrt{1 + x^2} \notin L^\infty(\mathbb{R}^2) \end{cases} \tag{1.10}$$

and it is not one-dimensional.

Before completing this section we want to mention the papers [3] and [5], where the one-dimensional symmetry, for solutions of quasilinear equations in any dimension $N \geqslant 2$, is obtained under different assumptions and with different methods. The results of the beautiful paper [3] are based on P-functions, $C^{1,\alpha}$ a priori estimates and the strong maximum principle. The results of [5] are obtained by variational methods.

2. – Assumptions and proofs.

We consider quasilinear operators of the form (1.1) where the function a belongs to $C^{1,1}_{\mathrm{loc}}(0, +\infty)$ and satisfies the structural assumptions (S_1) and (S_2). It follows that:

$$a(t) > 0 \quad \forall t > 0\,, \tag{2.1}$$

$$\forall T > 0 \quad \exists C = C(T) > 0:$$

$$a(t) \leqslant Ct^{-2}, \quad (ta(t))' \leqslant Ct^{-2} \quad \forall t \in (0, T]. \tag{2.2}$$

Let $p = (p_1, p_2) \neq 0$. Then we have

$$\frac{\partial (p_h a(|p|))}{\partial p_k} = \frac{a'(|p|)}{|p|} p_h p_k + a(|p|)\,\delta_{hk}, \qquad h, k = 1, 2\,. \tag{2.3}$$

Let the right-hand side of (2.3) be denoted by $\alpha_{hk}(p)$. The following lemma will be useful in the sequel.

LEMMA 2.1. – *The matrix $(\alpha_{hk}(p))$ is symmetric and positive-definite for all $p \in \mathbb{R}^2 \setminus \{0\}$. Furthermore, the entries $\alpha_{hk}(p)$ belong to $C^{0,1}_{\mathrm{loc}}(0, +\infty)$ and satisfy:*

$$\forall T > 0 \quad \exists K = K(T) > 0:$$

$$|\alpha_{hk}(p)| \leqslant K|p|^{-2}, \quad \forall p \neq 0 : |p| \leqslant T. \tag{2.4}$$

PROOF. – The entries α_{hk} are in $C^{0,1}_{\mathrm{loc}}(0, +\infty)$ by the regularity assumption on a, furthermore, the matrix $(\alpha_{hk}(p))$ is symmetric by (2.3). A direct calculation shows that the eigenvalues of $(\alpha_{hk}(p))$ are:

$$a(|p|), \qquad |p|a'(|p|)+a(|p|). \tag{2.5}$$

Note that $(ta(t))' = ta'(t) + a(t)$ so that both eigenvalues in (2.5) are positive by (S_1) and (2.1). This yields the first desired conclusion.

Since, for $p \neq 0$,

$$\alpha_{hk}(p) = \frac{a'(|p|)}{|p|} p_h p_k + a(|p|)\,\delta_{hk} = \\ (|p|a'(|p|)+a(|p|))\frac{p_h p_k}{|p|^2} + a(|p|)\left(\delta_{hk} - \frac{p_h p_k}{|p|^2}\right) \tag{2.6}$$

we have that (2.4) follows immediately from (S_2) and (2.2) and the fact that the eigenvalues are positive. ■

The next lemma is crucial to obtain our results. Before stating this lemma we notice that, since $|\nabla u| > 0$ on $\mathbb{R}^2$ and $\nabla u \in C^1$, we have that $\dfrac{\nabla u}{|\nabla u|} \in C^1(\mathbb{R}^2, \boldsymbol{S}^1)$. Hence, there exists a $C^1(\mathbb{R}^2)$ real function θ such that

$$\nabla u(x) = |\nabla u(x)|\,e^{i\theta(x)} := \varrho(x)\,e^{i\theta(x)} \quad \text{in} \quad \mathbb{R}^2. \tag{2.7}$$

LEMMA 2.2. – *Let u be a $C^1(\mathbb{R}^2)$ solution of*

$$\begin{cases} \operatorname{div}(a(|\nabla u|)\,\nabla u) + f(u) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2) \\ |\nabla u| > 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{2.8}$$

Then

$$\begin{cases} \operatorname{div}(\varrho^2 A\nabla\theta) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2) \\ \operatorname{div}(A\nabla\varrho) = \varrho(-f'(u) + (A\nabla\theta)\,\nabla\theta) & \text{in } \mathcal{D}'(\mathbb{R}^2) \end{cases} \tag{2.9}$$

where $A = (a_{hk})$ is the real matrix whose entries are $C^{0,1}_{\mathrm{loc}}$ functions given by

$$a_{hk} := \frac{a'(|\nabla u|)}{|\nabla u|} u_h u_k + a(|\nabla u|)\,\delta_{hk} = \alpha_{hk}(|\nabla u|). \tag{2.10}$$

PROOF. – Differentiating the equation in (2.8) yields, for $s = 1, 2$,

$$\operatorname{div}([a(|\nabla u|)\,\nabla u]_s) + f'(u)\,u_s = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{2.11}$$

The vector field $a(|\nabla u|)\nabla u$ belongs to C^1, then a direct calculation gives

$$\operatorname{div}(A(x)\,\nabla u_s) + f'(u)\,u_s = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{2.12}$$

where A is the matrix whose entries are given by (2.10).

Define the complex function $z = u_1 + iu_2$ then, $z \in C^1$ and satisfies the following complex Schroedinger equation:

$$\operatorname{div}(A\nabla z) + f'(u)\,z = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{2.13}$$

Inserting (2.7) into (2.13) we have

$$-f'(u)\varrho e^{i\theta} = -f'(u)z = \operatorname{div}(A\nabla z) = \operatorname{div}(e^{i\theta}A\nabla\varrho) + i\operatorname{div}(\varrho e^{i\theta}A\nabla\theta)) = \tag{2.14}$$

$$e^{i\theta}\operatorname{div}(A\nabla\varrho) + ie^{i\theta}(A\nabla\varrho)\,\nabla\theta +$$

$$i\varrho e^{i\theta}\operatorname{div}(A\nabla\theta) + ie^{i\theta}(A\nabla\theta)\,\nabla\varrho - e^{i\theta}\varrho(A\nabla\theta)\,\nabla\theta \ \text{ in } \ \mathcal{D}'(\mathbb{R}^2)$$

hence

$$-f'(u)\,\varrho = \operatorname{div}(A\nabla\varrho) - \varrho(A\nabla\theta)\,\nabla\theta + 2i(A\nabla\varrho)\,\nabla\theta + i\varrho\operatorname{div}(A\nabla\theta) \ \text{ in } \ \mathcal{D}'(\mathbb{R}^2)$$

where in the last identity we used the symmetry of A.

Separating the imaginary and the real parts we obtain

$$\begin{cases} \varrho\operatorname{div}(A\nabla\theta) + 2(A\nabla\varrho)\,\nabla\theta = 0 & \text{in } \ \mathcal{D}'(\mathbb{R}^2) \\ \operatorname{div}(A\nabla\varrho) - \varrho(A\nabla\theta)\,\nabla\theta + \varrho f'(u) = 0 & \text{in } \ \mathcal{D}'(\mathbb{R}^2)\,. \end{cases} \tag{2.15}$$

In particular, the second equation in (2.9) is established. To prove the first one we see that

$$\operatorname{div}(\varrho^2 A\nabla\theta) = \varrho^2\operatorname{div}(A\nabla\theta) + 2\varrho(A\nabla\theta)\,\nabla\varrho =$$

$$\varrho(\varrho\operatorname{div}(A\nabla\theta) + 2(A\nabla\varrho)\,\nabla\theta) \quad \text{in } \ \mathcal{D}'(\mathbb{R}^2)$$

thus, the claim follows from the first equation in (2.15). This fact concludes the proof. ∎

To obtain our symmetry result we need the following Liouville theorem for non-uniformly linear elliptic equations in divergence form. This result was proved in [7] (see pp. 333-334 and also p. 330).

THEOREM 2.3. – *Let $B = (b_{ij})$ be a symmetric real matrix, whose entries are bounded and mesurable functions on $\mathbb{R}^2$, such that:*

$$\sum_{i,j=1}^{2} b_{ij}(x)\,\xi_i\xi_j > 0 \qquad \forall x \in \mathbb{R}^2, \ \ \forall \xi \in \mathbb{R}^2 \setminus \{0\}\,. \tag{2.16}$$

Then every function $v \in C^1(\mathbb{R}^2)$ satisfying

$$\begin{cases} \operatorname{div}(B(x)\,\nabla v) = 0 \quad \text{in } \ \mathcal{D}'(\mathbb{R}^2), \\ \exists\,\delta < 1 \ : \ |v(x)| = O(\ln^\delta |x|), \quad \text{as } \ |x| \to +\infty\,, \end{cases} \tag{2.17}$$

is a constant function.

Now, we are in position to prove Theorem 1.2.

PROOF OF THEOREM 1.2. – By the assumptions we have that $\theta = \arg(\nabla u)$ is a C^1 solution of the first equation in (2.9). By applying Lemma 2.1, we obtain that the real matrix $B = \varrho^2 A = |\nabla u|^2 A$ is symmetric, has continuous and bounded entries and satisfies (2.16).

By (1.8) and Theorem 2.3. we conclude that θ is constant on $\mathbb{R}^2$. So $\nabla u(x) = |\nabla u(x)| e^{i\theta_0}$ in $\mathbb{R}^2$, for a real constant θ_0. Setting $\tau = (-\sin(\theta_0), \cos(\theta_0))$ we have $\nabla u \cdot \tau = 0$ everywhere. The latter implies the desired conclusion with $\nu = (\cos(\theta_0), \sin(\theta_0))$. ■

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LAMFA, CNRS UMR 6140, Université de Picardie Jules Verne
Faculté de Mathématiques et d'Informatique, 33, rue Saint-Leu 80039 Amiens, France
E-mail: alberto.farina@u-picardie.fr

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