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# Cauchy-Dirichlet Problem in Morrey Spaces for Parabolic Equations with Discontinuous Coefficients.

Dian K. Palagachev - Maria A. Ragusa - Lubomira G. Softova

**Sunto.** – *Siano* $Q_T$ *un cilindro in* $\mathbb{R}^{n+1}$ *ed* $x=(x', t) \in \mathbb{R}^n \times \mathbb{R}$. *Si studia il problema di Cauchy-Dirichlet per l'operatore uniformemente parabolico*

$$\begin{cases} u_t - \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u = f(x) & q.o. \ \ in \ \ Q_T, \\ u(x) = 0 & su \ \ \partial Q_T, \end{cases}$$

*nell'ambito degli spazi di Morrey* $W^{2,1}_{p,\lambda}(Q_T)$, $p \in (1, \infty)$, $\lambda \in (0, n+2)$, *supponendo che i coefficienti della parte principale appartengano alla classe delle funzioni con oscillazione media infinitesima. Si ottengono inoltre delle stime a priori nei suddetti spazi, e regolarità Hölderiana della soluzione e della sua derivata spaziale.*

**Summary.** – *Let* $Q_T$ *be a cylinder in* $\mathbb{R}^{n+1}$ *and* $x=(x', t) \in \mathbb{R}^n \times \mathbb{R}$. *It is studied the Cauchy-Dirichlet problem for the uniformly parabolic operator*

$$\begin{cases} u_t - \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u = f(x) & a.e. \ \ in \ \ Q_T, \\ u(x) = 0 & on \ \ \partial Q_T, \end{cases}$$

*in the Morrey spaces* $W^{2,1}_{p,\lambda}(Q_T)$, $p \in (1, \infty)$, $\lambda \in (0, n+2)$, *supposing the coefficients to belong to the class of functions with vanishing mean oscillation. There are obtained a priori estimates in Morrey spaces and Hölder regularity for the solution and its spatial derivatives.*

## 1. – Introduction.

The main goal of the present paper is to study qualitative properties in the framework of the parabolic Morrey spaces of the Cauchy-Dirichlet problem

$$(1.1) \qquad \begin{cases} \mathcal{P}u \equiv u_t - \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u = f(x) & \text{a.e. in } Q_T, \\ u(x) = 0 & \text{on } \partial Q_T \end{cases}$$

in the case of uniformly parabolic operator $\mathcal{P}$ with discontinuous coefficients. Here $\Omega \subset \mathbb{R}^n$ is a bounded and $C^{1,1}$-smooth domain, $n \geq 1$, and $Q_T$ stands for the cylinder $\Omega \times (0, T)$, $T > 0$. As usual, $S_T = \partial\Omega \times (0, T)$ means the lateral surface and $\partial Q_T = \Omega \cup S_T$ — the parabolic boundary of $Q_T$. Throughout the paper the standard summation convention on repeated upper and lower indices is adopted. For simplicity we denote the set of the parabolic variables by $x = (x', t) = (x_1, \dots, x_n, t) \in \mathbb{R}^{n+1}$ and $D_i u = \partial u/\partial x_i$, $D_{ij} u = \partial^2 u/\partial x_i \partial x_j$, $u_t = D_t u = \partial u/\partial t$, $D_{x'} u = (D_1 u, \dots, D_n u)$ means the spatial gradient of $u$, $D_{x'}^2 u = \{D_{ij} u\}_{ij=1}^n$. In our further considerations we shall use the notations $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ and $\mathbb{D}_+^{n+1} = \mathbb{R}_+^n \times \mathbb{R}_+ = \{x' \in \mathbb{R}^n : x_n > 0\} \times \{t > 0\}$.

The problem (1.1) is very well studied both in Hölder and Sobolev functional spaces when the coefficients $a^{ij}$ are Hölder or uniformly continuous functions in $Q_T$ (see [16]). Relevant $L^2$-theory of (1.1) was developed in [14] supposing $a^{ij}$'s to be discontinuous but owning suitable Sobolev regularity ($D_{x'} a^{ij} \in L^{n+2}$, $D_t a^{ij} \in L^{(n+2)/2}$).

Our principal assumption on the coefficients $a^{ij}$ is that they belong to the Sarason class of functions *VMO* with *vanishing mean oscillation* (cf. [20]). That class consists of functions $f$ which *mean oscillation* is not only *bounded*, i.e. $f \in BMO$ ([15]), but also converges uniformly to zero over balls shrinking to a point. The increasing interest to *VMO* in the last years is due mainly to the fact that it contains as a proper subspace the bounded uniformly continuous functions and this ensures the possibility to extend the $L^p$-theory of operators with *continuous* coefficients ([13], [16]) to the case of *discontinuous* ones ([8], [9], [3], [21]).

Differential operators with *VMO* principal coefficients have been considered for the first time by Chiarenza, Frasca and Longo in [8] and [9]. These authors succeeded to modify classical methods in deriving $L^p$-estimates for solutions of Dirichlet boundary problem for linear elliptic equations which allowed them to move from $a^{ij}(x) \in C^0(\overline{\Omega})$ into $a^{ij}(x) \in VMO$. Roughly speaking, their approach goes back to Calderón and Zygmund (see [4], [5]) and makes use of an explicit representation formula for the second derivatives $D^2 u$ in terms of singular integrals and commutators both with variable Calderón-Zygmund kernels.

In the articles [3] and [21], the parabolic Cauchy-Dirichlet and oblique derivative problems have been studied in the Sobolev spaces $W_p^{2,1}(Q_T)$, $p \in (1, \infty)$, under *VMO* hypothesis on the coefficients $a^{ij}$. These results along with other classical and modern techniques regarding both elliptic and parabolic equations with discontinuous data, including *VMO*, can be found in the monograph [17].

Here we are going to extend the considerations in [3] supposing the right-hand side of the equation (1.1) to belong to the parabolic Morrey spaces $L^{p,\lambda}(Q_T)$. Let us note that the space $L^{p,\lambda}$ is a subspace of $L^p_{\text{loc}}$ for every

$p \in (1, \infty)$ and $\lambda \in (0, n+2)$. This way, the existence results in Sobolev classes $W_p^{2,1}(Q_T)$ from [3] still hold if $f \in L^{p,\lambda}(Q_T)$. A natural question that arises is whether $\mathcal{P}u \in L^{p,\lambda}$ implies $u \in W^{2,1}_{p,\lambda}$.

We show that the solution of (1.1) belongs to $W^{2,1}_{p,\lambda}(Q_T)$ assuming the coefficients of the uniformly parabolic operator $\mathcal{P}$ to be *VMO* functions and $f \in L^{p,\lambda}$, $p \in (1, \infty)$, $\lambda \in (0, n+2)$. In our investigations we make use of the results obtained in [22], [23] and [19] in the framework of the Morrey spaces. These articles propose detailed study of singular integrals and commutators with kernel $k(x; y)$ depending on parameter $x$ and satisfying Calderón-Zygmund type conditions with respect to $y$. The mixed homogeneity of the kernel in $y$, which in [22] and [23] is of parabolic type and in [19] of general type, needs an appropriate metric as the one defined in [12].

Our goal here is to obtain $L^{p,\lambda}$ estimates for the nonsingular integrals which appear in the representation of the solution near the boundary. These estimates along with the estimates for the singular integrals lead to an a priori estimate of the solution of (1.1) in $W^{2,1}_{p,\lambda}(Q_T)$. Finally, Morrey's regularity of strong solution $u$ to (1.1) implies Hölder regularity both of $u$ and its gradient, which are finer than the already known in the case $\mathcal{P}u \in L^p$.

We refer the reader to [22] and [23] for similar results concerning oblique derivative problem for the parabolic operator $\mathcal{P}$, and to [11] and [18] for Morrey regularity results regarding boundary value problems for elliptic operators with *VMO* coefficients.

## 2. – Definitions and preliminaries.

Suppose $\mathcal{P}$ is a *uniformly parabolic operator,* i.e., there exists a constant $\Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \leq a^{ij}(x)\,\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad \text{a.a. } x \in Q_T,\ \forall \xi \in \mathbb{R}^n. \tag{2.1}$$

Besides that, requiring the coefficients matrix $\boldsymbol{a} = \{a^{ij}\}_{ij=1}^n$ to be symmetric, one gets immediately essential boundedness of $a^{ij}$'s.

Denote by $\mathcal{P}_0$ a linear parabolic operator with constant coefficients $a_0^{ij}$ which satisfy (2.1). The fundamental solution of the operator $\mathcal{P}_0$ with pole at the origin is given by the formula (cf. [16])

$$\Gamma^0(y) = \Gamma^0(y', \tau) = \begin{cases} \dfrac{1}{(4\pi\tau)^{n/2}\sqrt{\det \boldsymbol{a}_0}} \exp\left\{-\dfrac{A_0^{ij}y_iy_j}{4\tau}\right\} & \text{as } \tau > 0, \\ 0 & \text{as } \tau < 0, \end{cases}$$

where $\boldsymbol{a}_0 = \{a_0^{ij}\}$ is the matrix of the coefficients of $\mathcal{P}_0$ and $\boldsymbol{A}_0 = \{A_0^{ij}\} = \boldsymbol{a}_0^{-1}$.

In the problem under consideration, the coefficients of the operator $\mathcal{P}$ depend on $x$ and it reflects also on the fundamental solution. To express this dependence we define

$$\Gamma(x; y) = \begin{cases} \dfrac{1}{(4\pi\tau)^{n/2}\sqrt{\det \boldsymbol{a}(x)}} \exp\left\{-\dfrac{A^{ij}(x)\, y_i y_j}{4\tau}\right\} & \text{as } \tau > 0\,, \\ 0 & \text{as } \tau < 0\,, \end{cases} \tag{2.2}$$

with $\boldsymbol{a}(x) = \{a^{ij}(x)\}$ and $\boldsymbol{A}(x) = \{A^{ij}(x)\} = \boldsymbol{a}(x)^{-1}$. Set also $\Gamma_i = \partial\Gamma(x; y', \tau)/\partial y_i$, $\Gamma_{ij} = \partial^2\Gamma(x; y', \tau)/\partial y_i \partial y_j$ for $i, j = 1, \ldots, n$.

For the goal of our further considerations, besides the standard parabolic metric $\tilde{\varrho}(x) = \max\{|x'|, |t|^{1/2}\}$, $|x'| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$, $\tilde{d}(x, y) = \tilde{\varrho}(x-y)$, we are going to use the one introduced by Fabes and Riviére in [12]

$$\varrho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}}\,, \qquad d(x, y) = \varrho(x-y)\,. \tag{2.3}$$

The topology induced by $d$ is defined through open ellipsoids centered at zero and of radius $r$

$$\mathcal{E}_r(0) = \left\{x \in \mathbb{R}^{n+1}:\ \frac{|x'|^2}{r^2} + \frac{t^2}{r^4} < 1\right\}.$$

Obviously, the unit sphere with respect to that metric coincides with the unit Euclidean sphere in $\mathbb{R}^{n+1}$, i.e.

$$\partial\mathcal{E}_1(0) \equiv \Sigma_{n+1} = \left\{x \in \mathbb{R}^{n+1}:\ |x| = \left(\sum_{i=1}^{n} x_i^2 + t^2\right)^{1/2} = 1\right\}$$

and $\overline{x} = \dfrac{x}{\varrho(x)} \in \Sigma_{n+1}$. It is easy to see that for any ellipsoid $\mathcal{E}_r$, there exist cylinders $\underline{I}$ and $\overline{I}$ (these are balls with respect to the metric $\tilde{\varrho}$) with measures comparable to $r^{n+2}$ and such that $\underline{I} \subset \mathcal{E}_r \subset \overline{I}$. Obviously, that relation gives an equivalence of the metrics $\varrho$ and $\tilde{\varrho}$ and the induced by them topologies.

DEFINITION 2.1. – *A function $k(y)$: $\mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}$ is said to be a* constant parabolic Calderón-Zygmund (PCZ) kernel *if $k(y)$ is smooth on $\mathbb{R}^{n+1}\setminus\{0\}$; $k(ry', r^2\tau) = r^{-(n+2)}k(y', \tau)$ for each $r > 0$; $\int_{\varrho(y)=r} k(y)\, d\sigma_y = 0$ for each $r > 0$.*

*A function $k(x; y)$: $\mathbb{R}^{n+1} \times (\mathbb{R}^{n+1}\setminus\{0\}) \to \mathbb{R}$ is a* variable PCZ kernel, *if for any fixed $x \in \mathbb{R}^{n+1}$ $k(x; \cdot)$ is a parabolic PCZ kernel and*

$$\sup_{\varrho(y)=1}\left|\left(\frac{\partial}{\partial y}\right)^{\beta} k(x; y)\right| \leqslant C(\beta) \text{ for every multiindex } \beta, \text{ independently of } x.$$

For the sake of completeness we recall here the definitions and some properties of the spaces we are going to use.

DEFINITION 2.2. – *For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ define*

$$\gamma_f(R) = \sup_{I_r} \frac{1}{|I_r|} \int_{I_r} |f(y) - f_{I_r}|\, dy$$

*where $I_r$ ranges over all cylinders in $\mathbb{R}^{n+1}$ of radius $r$ and centered at some point $x$, i.e., $I_r(x) = \{y \in \mathbb{R}^{n+1} \colon |x' - y'| < r,\ |t - \tau| < r^2\}$ and $f_{I_r} = |I_r|^{-1} \int_{I_r} f(y)\, dy$.*

*Then, $f \in BMO$ (bounded mean oscillation, [15]) if $\|f\|_* = \sup_R \gamma_f(R) < +\infty$, while $f \in VMO$ (vanishing mean oscillation, [20]) if $\lim_{R \to 0} \gamma_f(R) = 0$ and the quantity $\gamma_f(R)$ is referred to as VMO-modulus of $f$.*

The spaces $BMO(Q_T)$ and $VMO(Q_T)$ of functions given on $Q_T$, can be defined in the same manner, taking $I_r \cap Q_T$ instead of $I_r$ above. As follows by result of Acquistapace (see [1, Proposition 1.3]), having a function $f$ defined in $Q_T$ and belonging to $BMO(Q_T)$, it is possible to extend it to the whole $\mathbb{R}^{n+1}$ preserving the $BMO$ seminorm of the extension. In particular, if $f \in VMO(Q_T)$ then the extended function $\tilde{f}$ belongs to $VMO(\mathbb{R}^{n+1})$ and $\gamma_{\tilde{f}}(R)$ is equivalent to $\gamma_f(R)$.

The problem (1.1) has been already studied in [3] in the framework of Sobolev spaces $W^{2,1}_p(Q_T)$, $p \in (1, \infty)$. Precisely, assuming (2.1) and $a^{ij} \in VMO(Q_T)$, it is proved that for any $f \in L^p(Q_T)$, $p \in (1, \infty)$, there exists a unique *strong solution*, i.e., a weakly differentiable function $u$ belonging to $L^p(Q_T)$ with all its derivatives $D_t^r D_{x'}^s u$, $0 \leq 2r + s \leq 2$, such that $u$ satisfies the equation in (1.1) almost everywhere in $Q_T$ and the boundary condition holds in the sense of trace on $\partial Q_T$.

Our goal here is to obtain finer regularity of that solution supposing $\mathcal{P}u$ belongs to the Morrey space $L^{p,\lambda}(Q_T)$, $p \in (1, \infty)$, $\lambda \in (0, n+2)$.

DEFINITION 2.3. – *A measurable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is said to belong to the parabolic Morrey space $L^{p,\lambda}(\mathbb{R}^{n+1})$ with $p \in (1, +\infty)$ and $\lambda \in (0, n+2)$, if the following norm is finite*

$$\|f\|_{p,\lambda} = \left( \sup_{r>0} \frac{1}{r^\lambda} \int_{I_r} |f(y)|^p\, dy \right)^{1/p},$$

*where $I_r$ is any cylinder of radius $r$. To define the space $L^{p,\lambda}(Q_T)$, we insist* *the norm*

$$\|f\|_{p,\lambda;\,Q_T} = \left( \sup_{r>0} \frac{1}{r^{\lambda}} \int_{Q_T \cap I_r} |f(y)|^p \, dy \right)^{1/p}$$

*to be finite.*

*We say that the function* $u(x)$ *lies in* $W^{2,1}_{p,\lambda}(Q_T)$, $1 < p < \infty$, $0 < \lambda < n+2$, *if it is weakly differentiable and belongs to* $L^{p,\lambda}(Q_T)$ *along with all its derivatives* $D_t^r D_{x'}^s u$, $0 \leqslant 2r + s \leqslant 2$. *Then the quantity*

$$\|u\|_{W^{2,1}_{p,\lambda}(Q_T)} = \|u\|_{p,\lambda;\,Q_T} + \|D^2_{x'} u\|_{p,\lambda;\,Q_T} + \|D_t u\|_{p,\lambda;\,Q_T}$$

*defines a norm under which* $W^{2,1}_{p,\lambda}(Q_T)$ *becomes a Banach space.*

For a given measurable function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ we define the *Hardy-Littlewood maximal operator*

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| \, dy \qquad \text{for a.a. } \ x \in \mathbb{R}^{n+1},$$

where the supremum is taken over all cylinders $I$ centered at the point $x$. A variant of it is the *sharp maximal operator*

$$f^{\#}(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y) - f_I| \, dy \ \text{ for a.a. } \ x \in \mathbb{R}^{n+1}.$$

The following lemmas give $L^{p,\lambda}$ estimates for $f$, $Mf$ and $f^{\#}$. Analogous bounds in the space $\mathbb{R}^n$ endowed with the Euclidean metric can be found in [7] and [11]. The $L^{p,\lambda}$ estimates below follow in the same manner, making use of the parabolic metrics $\tilde{\varrho}$ or $\varrho$ and corresponding to them diadic partition of the space $\mathbb{R}^{n+1} = 2I \cup \left( \bigcup_{k=1}^{\infty} 2^{k+1} I \setminus 2^k I \right)$ where $I$ is either a cylinder or an ellipsoid centered at some point $x \in \mathbb{R}^{n+1}$ and of radius $r$. We note that $2^k I$ means cylinder (ellipsoid) with the same center and of radius $2^k r$.

Lemma 2.1 (Maximal inequality). – *Let* $p \in (1, \infty)$, $\lambda \in (0, n+2)$ *and* $f \in L^{p,\lambda}(\mathbb{R}^{n+1})$. *Then there exists a constant* $C$ *independent of* $f$ *such that*

$$\|Mf\|_{p,\lambda} \leqslant C \|f\|_{p,\lambda}.$$

Lemma 2.2 (Sharp inequality). – *Let* $1 < p < \infty$, $0 < \lambda < n+2$, $f \in L^{p,\lambda}(\mathbb{R}^{n+1})$. *There exists a constant* $C$ *independent of* $f$ *such that*

$$\|f\|_{p,\lambda} \leqslant C \|f^{\#}\|_{p,\lambda}.$$

Analogous estimates are valid also in $\mathbb{D}_+^{n+1} = \mathbb{R}_+^n \times \mathbb{R}_+$, where the corresponding diadic partition of the space has the form $\mathbb{D}_+^{n+1} = 2I_+ \cup \left( \bigcup_{k=1}^{\infty} 2^{k+1} I_+ \setminus 2^k I_+ \right)$ with $I_+ = I \cap \{x_n > 0,\ t > 0\}$ and $I$ is a cylinder centered at $x \in \mathbb{D}_+^{n+1}$. Then

$$\|Mf\|_{p,\lambda;\,\mathbb{D}_+^{n+1}} \leq C\|f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}}, \qquad \|f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}} \leq C\|f^{\#}\|_{p,\lambda;\,\mathbb{D}_+^{n+1}}.$$

We shall exploit below the well known technique, based on an expansion into spherical harmonics of certain kernels (cf. [4], [5], [8], [3]). Recall that the restriction to the unit sphere $\Sigma_{n+1}$ of any homogeneous and harmonic polynomial $p(x)\colon \mathbb{R}^{n+1} \to \mathbb{R}$ of degree $m$ is called an $(n+1)$-*dimensional spherical harmonic of degree* $m$. Set $\mathcal{Y}_m$ for the space of all $(n+1)$-dimensional spherical harmonics of degree $m$. It is a finite-dimensional linear space and setting $g_m = \dim\ \mathcal{Y}_m$, we have

$$g_m = \binom{m+n}{n} - \binom{m+n-2}{n} \leq C(n)\, m^{n-1} \tag{2.4}$$

with the second binomial coefficient to be settled 0 when $m = 0, 1$, i.e., $g_0 = 1$, $g_1 = n+1$. Further, let $\{Y_{sm}(x)\}_{s=1}^{g_m}$ be an *orthonormal base* of $\mathcal{Y}_m$. Then $\{Y_{sm}(x)\}_{s=1,\,m=0}^{g_m,\,\infty}$ is a *complete orthonormal system* in $L^2(\Sigma_{n+1})$ and

$$\sup_{x \in \Sigma_{n+1}} \left| \left( \frac{\partial}{\partial x} \right)^{\beta} Y_{sm}(x) \right| \leq C(n)\, m^{|\beta| + (n-1)/2}, \qquad m = 1, 2, \ldots. \tag{2.5}$$

In particular, let $\phi \in C^{\infty}(\Sigma_{n+1})$ and $\sum_{s,m} b_{sm} Y_{sm}(x)$ be the Fourier series expansion of $\phi(x)$ with respect to $\{Y_{sm}\}$. Then

$$b_{sm} = \int_{\Sigma_{n+1}} \phi(x)\, Y_{sm}(x)\, d\sigma, \qquad |b_{sm}| \leq C(l)\, m^{-2l} \sup_{\substack{|\gamma| = 2l \\ x \in \Sigma_{n+1}}} \left| \left( \frac{\partial}{\partial x} \right)^{\gamma} \phi(x) \right| \tag{2.6}$$

for every integer $l > 1$ and $\sum_{s,m} \equiv \sum_{m=0}^{\infty} \sum_{s=1}^{g_m}$. Therefore, the expansion of $\phi$ into spherical harmonics converges uniformly to $\phi$ (see [4], [5] for details).

## 3. – Integral estimates in Morrey spaces.

This section is devoted to Morrey continuity of certain nonsingular integral operators near the lateral boundary $S_T$ of the cylinder $Q_T$. For what concerns the regularity of $\partial\Omega$ we will suppose that it is $C^{1,1}$-smooth. In other words, $\partial\Omega$ can be represented locally as a graph of function having Lipschitz continuous first derivatives. Indeed, by virtue of Rademacher's theorem, $C^{1,1} \equiv W^{2,\infty}$ and therefore all the diffeomorphisms which flatten locally $\partial\Omega$

(and thus $S_T$) will have $L^\infty$-smooth second-order generalized derivatives.

Suppose now that $S_T$ is locally flatten such that $Q_T \subset \mathbb{D}_+^{n+1} = \mathbb{R}_+^n \times \mathbb{R}_+$, and let the coefficients of the operator $\mathcal{P}$ be defined in $\mathbb{D}_+^{n+1}$. Construct a *generalized symmetry* $T$ in the next manner. Denote by $\boldsymbol{a}^n(y)$ the last row of the matrix $\boldsymbol{a} = \{a^{ij}\}$ and define

$$T(x', t; y', t) = x' - 2x_n \frac{\boldsymbol{a}^n(y', t)}{a^{nn}(y', t)}, \qquad T(x) = T(x', t; x', t),$$

for any $x', y' \in \mathbb{R}_+^n$ and any fixed $t \in \mathbb{R}_+$. Obviously $T$ maps $\mathbb{R}_+^n$ into $\mathbb{R}_-^n$ and if $k(x; \cdot)$ is a variable PCZ kernel then $k(x; T(x) - y)$ turns out to be a *nonsingular variable* kernel for any $x, y \in \mathbb{D}_+^{n+1}$.

Let $f \in L^{p,\lambda}(\mathbb{D}_+^{n+1})$ with $p \in (1, \infty)$, $\lambda \in (0, n+2)$ and $a \in BMO(\mathbb{D}_+^{n+1})$. Define the operators

$$\widetilde{\mathcal{K}} f(x) = = \int_{\mathbb{D}_+^{n+1}} k(x; T(x) - y)\, f(y)\, dy,$$

$$\widetilde{\mathcal{C}}[a, f](x) \int_{\mathbb{D}_+^{n+1}} k(x; T(x) - y)[a(y) - a(x)]\, f(y)\, dy.$$

We consider the series expansion of the nonsingular kernel $k(x; T(x) - y)$ on $\Sigma_{n+1}$ with respect to the base $\{Y_{sm}(x)\}_{s=1, m=0}^{g_m, \infty}$

$$k(x, T(x) - y) = \varrho(T(x) - y)^{-(n+2)} k(x, \overline{T(x) - y}) =$$

$$\varrho(T(x) - y)^{-(n+2)} \sum_{s,m} b_{sm}(x)\, Y_{sm}(\overline{T(x) - y}) = \sum_{s,m} b_{sm}(x)\, \mathcal{H}_{sm}(T(x) - y).$$

The kernels $\mathcal{H}_{sm}(\cdot) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ are constant parabolic Calderón-Zygmund kernels satisfying Hörmander type condition (see [3]). Thus $\mathcal{H}_{sm}(T(x) - y)$ for $x \in \mathbb{D}_+^{n+1}$ are nonsingular. Further, the expansion of $k(x, T(x) - y)$ leads also to series expansions of the integrals $\widetilde{\mathcal{K}} f$ and $\widetilde{\mathcal{C}}[a, f]$

$$\widetilde{\mathcal{K}} f(x) = \sum_{s,m} b_{sm}(x) \int_{\mathbb{D}_+^{n+1}} \mathcal{H}_{sm}(T(x) - y) f(y)\, dy = \sum_{s,m} b_{sm}(x)\, \widetilde{\mathcal{K}}_{sm} f(x),$$

$$\widetilde{\mathcal{C}}[a, f](x) = \sum_{s,m} b_{sm}(x) \int_{\mathbb{D}_+^{n+1}} \mathcal{H}_{sm}(T(x) - y)[a(y) - a(x)] f(y)\, dy \tag{3.1}$$

$$= \sum_{s,m} b_{sm}(x)\, \widetilde{\mathcal{C}}_{sm}[a, f](x).$$

Before proving the $L^{p,\lambda}$-boundedness of the above integrals we shall pay attention to some preliminary results. For any $x' \in \mathbb{R}_+^n$ and $t \in \mathbb{R}_+$ we define $\tilde{x}' =$

$(x_1, \dots, x_{n-1}, -x_n) \in \mathbb{R}^n_-$ and $\tilde{x} = (x_1, \dots, x_{n-1}, -x_n, t) \in \mathbb{D}^{n+1}_- = \mathbb{R}^n_+ \times \mathbb{R}_-$. Hence the following integral operators

$$\mathcal{T}f(x) = \int_{\mathbb{D}^{n+1}_+} \frac{f(y)}{\varrho(\tilde{x}-y)^{n+2}}\, dy,$$

$$\mathcal{S}(a,f)(x) = \int_{\mathbb{D}^{n+1}_+} \frac{|a(y)-a(x)|f(y)}{\varrho(\tilde{x}-y)^{n+2}}\, dy$$

are nonsingular.

THEOREM 3.1. – *Let* $f \in L^{p,\lambda}(\mathbb{D}^{n+1}_+)$ *with* $p \in (1, \infty)$, $\lambda \in (0, n+2)$ *and* $a \in BMO(\mathbb{D}^{n+1}_+)$. *Then*

$$\|\mathcal{T}f\|_{p,\lambda;\,\mathbb{D}^{n+1}_+} \leqslant C\|f\|_{p,\lambda;\,\mathbb{D}^{n+1}_+} \tag{3.2}$$

$$\|\mathcal{S}(a,f)\|_{p,\lambda;\,\mathbb{D}^{n+1}_+} \leqslant C\|a\|_* \|f\|_{p,\lambda;\,\mathbb{D}^{n+1}_+} \tag{3.3}$$

*and the constant* $C$ *depends on* $n, p, \lambda$ *but not on* $f$.

PROOF. – Let $I$ be a cylinder centered at $x_0 \in \mathbb{D}^{n+1}_+$ and of radius $r$. We set $I_+ = I \cap \mathbb{D}^{n+1}_+$ and $2^k I_+$ stands for $2^k I \cap \mathbb{D}^{n+1}_+$. Every function $f$ defined on $\mathbb{D}^{n+1}_+$ could be written as

$$f(x) = f(x)\,\chi_{2I_+}(x) + \sum_{k=1}^{\infty} f(x)\,\chi_{2^{k+1}I_+ \setminus 2^k I_+}(x) = \sum_{k=0}^{\infty} f_k(x)$$

with $\chi$ being the characteristic function of the respective set. As is shown in [3, Lemma 3.3], $\mathcal{T}$ is a continuous operator acting from $L^p$ into itself, whence

$$\begin{aligned}\int_{I_+} |\mathcal{T}f_0(y)|^p\, dy &\leqslant \|\mathcal{T}f_0\|^p_{p;\,\mathbb{D}^{n+1}_+} \leqslant C(p)\|f_0\|^p_{p;\,\mathbb{D}^{n+1}_+} \\ &= C(p)\int_{2I_+} |f(y)|^p\, dy \leqslant C(p)\, r^\lambda \|f\|^p_{p,\lambda;\,\mathbb{D}^{n+1}_+}.\end{aligned}$$

It is easy to see that for every $y \in 2^{k+1}I_+ \setminus 2^k I_+$ and $x \in I_+$, $k \geqslant 1$, one has

$$\varrho(\tilde{x}-y) \geqslant \varrho(x-y) \geqslant (2^k - 1)\, r \geqslant 2^{k-1} r.$$

Thus

$$
\begin{aligned}
|\mathcal{T}f_k(x)|^p &= \left( \int_{2^k r < \varrho(x_0 - y) < 2^{k+1} r} \frac{|f(y)|}{\varrho(\tilde{x} - y)^{n+2}}\, dy \right)^p \\
&\leqslant \left( \frac{1}{(2^{k-1} r)^{n+2}} \int_{\varrho(x_0 - y) < 2^{k+1} r} |f(y)|\, dy \right)^p \\
&\leqslant C \frac{1}{(2^{k-1} r)^{p(n+2)}} \left( \int_{\varrho(x_0 - y) < 2^{k+1} r} 1\, dy \right)^{p-1} \left( \int_{\varrho(x_0 - y) < 2^{k+1} r} |f(y)|^p\, dy \right) \\
&\leqslant C 2^{k(\lambda - (n+2))} r^{\lambda - (n+2)} \|f\|^p_{p, \lambda; \mathbb{D}^{n+1}_+}.
\end{aligned}
$$

Now we get

$$
\begin{aligned}
\int_{I_+} |\mathcal{T}f(y)|^p\, dy &= \sum_{k=0}^{\infty} \int_{I_+} |\mathcal{T}f_k(y)|^p\, dy \\
&\leqslant C r^\lambda \left( 1 + \sum_{k=1}^{\infty} 2^{k(\lambda - (n+2))} \right) \|f\|^p_{p, \lambda; \mathbb{D}^{n+1}_+} \leqslant C r^\lambda \|f\|^p_{p, \lambda; \mathbb{D}^{n+1}_+}
\end{aligned}
$$

and the constant depends on $n$, $p$ and $\lambda$. Moving $r^\lambda$ on the left-hand side and taking the supremum with respect to $r$ we get exactly (3.2).

To prove (3.3) we use the following inequality

$$
|\mathcal{S}(a, f)^{\#}(x)| \leqslant C\|a\|_* ((M((\mathcal{T}|f|)^q)(x))^{1/q} + (M(|f|^q)(x))^{1/q})
$$

proved in [2, Theorem 3.1]. Thus, for any $q \in (1, p)$ and $f \in L^{p,\lambda}(\mathbb{D}^{n+1}_+)$ we write

$$
\int_{I_+} |\mathcal{S}(a, f)^{\#}(y)|^p\, dy \leqslant
$$

$$
C\|a\|^p_* \left\{ \int_{I_+} |M(\mathcal{T}|f|)^q(y)|^{p/q}\, dy + \int_{I_+} |M(|f|^q)(y)|^{p/q}\, dy \right\} = C\|a\|^p_* (J_1 + J_2).
$$

Making use of Lemma 2.1 and (3.2), it is easy to see that

$$
\begin{aligned}
J_1 = \int_{I_+} |M(\mathcal{T}|f|)^q(y)|^{p/q}\, dy &\leqslant r^\lambda \|M(\mathcal{T}|f|)^q\|^{p/q}_{p/q, \lambda; \mathbb{D}^{n+1}_+} \\
&\leqslant r^\lambda \|(\mathcal{T}|f|)^q\|^{p/q}_{p/q, \lambda; \mathbb{D}^{n+1}_+} = r^\lambda \|\mathcal{T}|f|\|^p_{p, \lambda; \mathbb{D}^{n+1}_+} \\
&\leqslant C r^\lambda \|f\|^p_{p, \lambda; \mathbb{D}^{n+1}_+}.
\end{aligned}
$$

Analogous arguments hold also for the estimate of $J_2$.

The estimate (3.3) follows from the sharp inequality (Lemma 2.2) which completes the proof of Theorem 3.1. ∎

THEOREM 3.2. – *Let the functions $f$ and $a$ be as above and $\widetilde{\mathcal{K}}_{sm}f$ and $\widetilde{\mathcal{C}}_{sm}[a,f]$ be the integrals from the series expansions* (3.1). *Then there exist constants depending on $n$, $p$, $\lambda$ such that*

$$\|\widetilde{\mathcal{K}}_{sm} f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}} \leq Cm^{(n-1)/2}\|f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}} \tag{3.4}$$

$$\|\widetilde{\mathcal{C}}_{sm}[a,f]\|_{p,\lambda;\,\mathbb{D}_+^{n+1}} \leq Cm^{(n-1)/2}\|a\|_*\|f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}}. \tag{3.5}$$

PROOF. – From the boundedness of $Y_{sm}(x)$ (see (2.5)) and the relation between the distances (see [3, Lemma 3.2])

$$C_1\varrho(\tilde{x}-y) \leq \varrho(T(x)-y) \leq C_2\varrho(\tilde{x}-y)$$

we have

$$|\widetilde{\mathcal{K}}_{sm} f(x)| \leq \int_{\mathbb{D}_+^{n+1}} \frac{|Y_{sm}(\overline{T(x)-y})|}{\varrho(T(x)-y)^{n+2}}|f(y)|dy \leq Cm^{(n-1)/2}\int_{\mathbb{D}_+^{n+1}} \frac{|f(y)|}{\varrho(\tilde{x}-y)^{n+2}}dy.$$

The last integral is exactly $\mathcal{F}|f|$ so we can apply the estimate (3.2), which gives

$$\|\widetilde{\mathcal{K}}_{sm} f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}} \leq Cm^{(n-1)/2}\|f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}}.$$

Analogously we get (3.5), making use of (3.3). ∎

We are in position now to prove our main result concerning $L^{p,\lambda}(\mathbb{D}_+^{n+1})$ estimates for the nonsingular integral operators $\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{C}}$.

THEOREM 3.3. – *Let $f \in L^{p,\lambda}(\mathbb{D}_+^{n+1})$, $p \in (1,\infty)$, $\lambda \in (0, n+2)$ and $a \in BMO(\mathbb{D}_+^{n+1})$. There exists a constant $C(n,p,\lambda)$ such that*

$$\|\widetilde{\mathcal{K}} f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}} \leq C\|f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}} \tag{3.6}$$

$$\|\widetilde{\mathcal{C}}[a,f]\|_{p,\lambda;\,\mathbb{D}_+^{n+1}} \leq C\|a\|_*\|f\|_{p,\lambda;\,\mathbb{D}_+^{n+1}}.$$

PROOF. – The estimates (2.4), (2.6) and (3.4) ensure total convergence in $L^{p,\lambda}(\mathbb{D}_+^{n+1})$ of the series expansion (3.1) of $\widetilde{\mathcal{K}}f$

$$\|\widetilde{\mathcal{K}}f\|_{p,\lambda,\mathbb{D}_+^{n+1}} \leq \sum_{s,m} \|b_{sm}\|_\infty \|\widetilde{\mathcal{K}}_{sm}f\|_{p,\lambda,\mathbb{D}_+^{n+1}}$$
$$< C\|f\|_{p,\lambda,\mathbb{D}_+^{n+1}} \sum_{m=1}^{\infty} m^{-2l+(n-1)/2+n-1}$$

if the integer $l$ is preliminary chosen greater than $(3n-1)/4$. Analogous arguments hold also for the commutator. ■

## 4. – A priori estimates, strong solvability and Hölder continuity.

THEOREM 4.1. – *Suppose* $a^{ij} \in VMO(Q_T)$, (2.1), $\partial\Omega \in C^{1,1}$ *and let* $u \in W^{2,1}_{p,\lambda}(Q_T)$, $p \in (1, \infty)$, $\lambda \in (0, n+2)$, *be a strong solution to* (1.1). *Then*

$$\|u\|_{W^{2,1}_{p,\lambda}(Q_T)} \leq C\|f\|_{p,\lambda;Q_T} \tag{4.1}$$

*where the constant depends on* $n, p, \lambda, \Lambda, T, \partial\Omega$ *and the VMO-moduli of* $a^{ij}$.

PROOF. – *Step 1: Interior estimate.* The interior representation formula for the second spatial derivatives ([3, Theorem 1.4]) expresses $D_{ij}u$ in terms of singular integral operators and their commutators with kernels $\Gamma_{ij}(x; x-y)$ (the derivatives of the fundamental solution (2.2) with respect to the second variable). Further, $\Gamma_{ij}(x; x-y)$'s are variable PCZ kernels (cf. [12]) that are homogeneous of degree $-1$ with respect to $x'$ and of degree $-2$ with respect to $t$. Thus, the singular integrals and commutators under consideration are a particular case of more general class of singular operators with kernels $k(x; y)$ of *mixed homogeneity* studied in [19]. We refer the reader to [23, Theorem 3.1] for the continuity properties of these operators in Morrey spaces. As a consequence of [23, Eq. (5.5)] (see also [19, Theorem 2]), the following interior regularity of solutions to (1.1) follows

THEOREM 4.2. – *Let* $u \in W^{2,1}_p(Q_T)$ *be a strong solution to the uniformly parabolic equation* $D_t u - a^{ij}(x) D_{ij} u = f(x)$ *with* $a^{ij} \in VMO(Q_T)$ *and* $f \in L^{p,\lambda}(Q_T)$. *Then* $D^2_{x'} u, D_t u \in L^{p,\lambda}(Q'_T)$ *for any cylinder* $Q'_T = \Omega' \times (0, T)$, $\Omega' \subset\subset \Omega$, *and*

$$\|u\|_{W^{2,1}_{p,\lambda}(Q'_T)} \leq C\left(\|u\|_{p,\lambda;Q''_T} + \|f\|_{p,\lambda;Q_T}\right) \tag{4.2}$$

*where* $Q''_T = \Omega'' \times (0, T)$, $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ *and* $C$ *depends on known quantities and on* $\operatorname{dist}(\partial\Omega', \partial\Omega)$.

*Step 2: Boundary estimate.* Suppose $S_T$ is locally flatten near the point $x_0$ such that $Q_T \subset \mathbb{D}_+^{n+1}$ and consider a semicylinder $I_+$ centered at $x_0$ and of radius $r$. Recall the boundary representation formula for the second derivatives $D_{ij}u$ (see [3, Theorem 1.5])

$$D_{ij}u(x) = \mathcal{C}_{ij}[a^{hk}, D_{hk}u](x) + \mathcal{K}_{ij}f(x) + f(x) \int_{\Sigma_{n+1}} \Gamma_j(x; y)\, \nu_i\, d\sigma_y - I_{ij}(x), \tag{4.3}$$

with

$$\mathcal{C}_{ij}[a^{hk}, D_{hk}u](x) = P.V. \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x; x-y)[a^{hk}(y) - a^{hk}(x)]\, D_{hk}u(y)\, dy,$$

$$\mathcal{K}_{ij}f(x) = P.V. \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x; x-y)\, f(y)\, dy,$$

$$I_{ij}(x) = \widetilde{\mathcal{C}}_{ij}[a^{hk}, D_{hk}u](x) + \widetilde{\mathcal{K}}_{ij}f(x) \quad i, j = 1, \dots, n-1;$$

$$I_{in}(x) = I_{ni}(x) = \sum_{l=1}^{n} (D_n T(x))^l \left(\widetilde{\mathcal{C}}_{il}[a^{hk}, D_{hk}u](x) + \widetilde{\mathcal{K}}_{il}f(x)\right) \quad i = 1, \dots, n-1;$$

$$I_{nn}(x) = \sum_{l,r=1}^{n} (D_n T(x))^l (D_n T(x))^r \left(\widetilde{\mathcal{C}}_{lr}[a^{hk}, D_{hk}u](x) + \widetilde{\mathcal{K}}_{lr}f(x)\right)$$

where $(D_n T(x))^l$ stands for the $l$-th component of the vector $D_n T(x)$ and $\nu_i$ is the $i$-th component of the unit outward normal to $\Sigma_{n+1}$.

The first two integrals in (4.3) are singular and of the kind treated in [23, Theorem 3.1] and [19, Theorem 1], while the third one is bounded nonsingular integral. Thus

$$\|D_{ij}u\|_{p,\lambda; I_+} \leqslant C\left(\|a\|_* \|D_{x'}^2 u\|_{p,\lambda; I_+} + \|f\|_{p,\lambda; I_+}\right) + \|I_{ij}\|_{p,\lambda; I_+}, \tag{4.4}$$

where the constant depends on known quantities but not on $f$. To estimate the last norm above we use the results for nonsingular integrals established in Theorem 3.3. Thus

$$\|I_{ij}\|_{p,\lambda; I_+} \leqslant C\left(\|a\|_* \|D_{x'}^2 u\|_{p,\lambda; I_+} + \|f\|_{p,\lambda; I_+}\right)$$

where the constant depends on $n$, $p$, $\lambda$, $\Lambda$ and $\|a\|_* = \max_{1 \leqslant i,j \leqslant n} \|a^{ij}\|_*$. By means of the *VMO*-assumption on $a^{ij}$'s, we are able to choose $r > 0$ sufficiently small in order to move the term $\|D_{x'}^2 u\|_{p,\lambda; I_+}$ on the left-hand side of (4.4). Therefore,

$$\|D_{x'}^2 u\|_{p,\lambda; I_+} \leqslant C\|f\|_{p,\lambda; I_+}$$

and similar estimate holds true also for $\|D_t u\|_{p,\lambda;\,I_+}$ by virtue of $u_t = a^{ij}(x)\,D_{ij}u + f(x)$. Finally, expressing $u(x',t) = \int_0^t D_s u(x',s)\,ds$ and applying Jensen's integral inequality, we obtain

$$\|u\|_{W^{2,1}_{p,\lambda}(I_+)} \le C\|f\|_{p,\lambda;\,I_+}.$$

Covering $Q_T \setminus Q'$ with a finite number of subcylinders $I_+$ we get a $W^{2,1}_{p,\lambda}$-estimate of the solution near the lateral boundary $S_T$ which, combined with (4.2) completes the proof. ■

We are in a position now to derive existence of a unique strong solution to the Cauchy-Dirichlet problem (1.1).

THEOREM 4.3. – *Suppose* (2.1), $\partial\Omega \in C^{1,1}$ *and* $a^{ij} \in VMO(Q_T)$. *Then the problem* (1.1) *admits a unique strong solution* $u \in W^{2,1}_{p,\lambda}(Q_T)$ *with* $p \in (1, \infty)$, $\lambda \in (0, n+2)$, *for every* $f \in L^{p,\lambda}(Q_T)$.

PROOF. – The *unicity* assertion follows immediately from the a priori estimate (4.1).

To prove *existence* of solution to (1.1), the *continuity method* ([13, Theorem 5.2]) will be employed. Consider the Cauchy-Dirichlet problem for the heat equation

$$\begin{cases} \mathcal{H}u = u_t - \Delta u = f(x) & \text{a.e. in } Q_T \\ u = 0 & \text{on } \partial Q_T. \end{cases} \tag{4.5}$$

It is easy to see that for any $f \in L^{p,\lambda}(Q_T)$ the above problem is uniquely solvable in $W^{2,1}_{p,\lambda}(Q_T)$. In fact, the $L^p$-theory of linear parabolic operators (see [16]) asserts existence of a unique strong solution $u \in W^{2,1}_p(Q_T)$ of (4.5) because of $f \in L^p(Q_T)$. Further, in the interior and boundary representation formulas for that solution the commutators *disappear* since $\mathcal{H}$ is a *constant coefficients* operator. This means $u \in W^{2,1}_{p,\lambda}(Q_T)$ in view of Theorem 4.2 and Theorem 3.1.

To apply the method of continuity, we define the Banach space

$$\mathcal{M} = \left\{ u \in W^{2,1}_{p,\lambda}(Q_T) \colon u|_{\partial Q_T} = 0 \right\}, \qquad \|\cdot\|_{\mathcal{M}} = \|\cdot\|_{W^{2,1}_{p,\lambda}(Q_T)}$$

and for any $\varrho \in [0,1]$ consider the convex combination $\mathcal{P}_\varrho = \varrho\mathcal{P} + (1-\varrho)\,\mathcal{H}$. Obviously, $\mathcal{P}_0 = \mathcal{H}$, $\mathcal{P}_1 = \mathcal{P}$, $\mathcal{P}_\varrho \colon \mathcal{M} \to L^{p,\lambda}(Q_T)$, and the coefficients of $\mathcal{P}_\varrho$ satisfy (2.1). Furthermore, the a priori estimate (4.1) implies

$$\|u\|_{\mathcal{M}} \le C\|\mathcal{P}_\varrho u\|_{p,\lambda;\,Q_T}$$

with $C$ independent of $\varrho$.

Since $\mathcal{P}_0$ is a *surjective mapping,* the method of continuity asserts that $\mathcal{P}_1 = \mathcal{P}$ is *surjective* too. Bearing in mind the unicity assertion, we obtain that (1.1) possesses a unique solution $u \in W^{2,1}_{p,\lambda}(Q_T)$ for any $f \in L^{p,\lambda}(Q_T)$, $p \in (1, \infty)$, $\lambda \in (0, n+2)$. ∎

An immediate consequence of the last result is Hölder continuity of the strong solution $u$ to (1.1) or its spatial gradient $D_{x'} u$ for suitable values of $p$ and $\lambda$. To be more precise, define

$$[u]_{\alpha; Q_T} = \sup_{\substack{(x',t),(y',\tau) \in Q_T \\ (x',t) \neq (y',\tau)}} \frac{|u(x',t) - u(y',\tau)|}{\left(|x'-y'|^2 + |t-\tau|\right)^{\alpha/2}}, \qquad 0 < \alpha < 1$$

and set $C^{0,\alpha}(\overline{Q}_T)$ for the space of all functions $u: \overline{Q}_T \to \mathbb{R}$ of finite norm

$$\|u\|_{0,\alpha;Q_T} = \|u\|_{\infty;Q_T} + [u]_{\alpha;Q_T}.$$

COROLLARY 4.1. – *Suppose* $a^{ij} \in VMO(Q_T)$, $\partial\Omega \in C^{1,1}$, (2.1), $f \in L^{p,\lambda}(Q_T)$ *with* $p \in (1, \infty)$ *and* $\lambda \in (0, n+2)$ *and let* $u \in W^{2,1}_{p,\lambda}(Q_T)$ *be the unique strong solution of the problem* (1.1). *Then*

1. $u \in C^{0,\alpha}(\overline{Q}_T)$ *and* $\|u\|_{0,\alpha;Q_T} \leq C\|f\|_{p,\lambda;Q_T}$ *with* $\alpha = \dfrac{1}{n+1} + \dfrac{\lambda-(n+2)}{p}$ *if* $\lambda > \max\{0, n+2-p/(n+1)\}$,
2. $D_{x'} u \in C^{0,\alpha}(\overline{Q}_T)$ *and* $\|D_{x'} u\|_{0,\alpha;Q_T} \leq C\|f\|_{p,\lambda;Q_T}$ *with* $\alpha = 1 + \dfrac{\lambda-(n+2)}{p}$ *if* $\lambda > \max\{0, n+2-p\}$.

PROOF. – Hölder's regularity of the strong solution $u$ is a direct consequence of Theorem 4.3 and [10, Theorem 4.1].

Concerning the Hölder continuity of the spatial gradient $D_{x'} u$ it is a rather delicate matter because of the lack of derivatives $D_{tt} u$ and $D_{x't} u$. Anyway, a standard approach consisting of passage through the parabolic Poincaré inequality ([6, Lemma 2.2], [17, Chapter 3]) yields

$$\begin{aligned} \int_{Q_T \cap I} |D_{x'} u - (D_{x'} u)_{Q_T \cap I}|^p \, dx &\leq r^p \int_{Q_T \cap I} \left(|u_t|^p + |D^2_{x'} u|^p\right) dx \\ &\leq C r^{p+\lambda} \|u\|_{W^{2,1}_{p,\lambda}(Q_T)} \end{aligned}$$

for any cylinder $I \subset Q_T$ of radius $r$. Therefore, $D_{x'} u$ belongs to the Campanato space $\mathcal{L}^{p,p+\lambda}(Q_T)$ and it is known (see [10, Theorem 3.1], [17, Section 3.3.2]) that $\mathcal{L}^{p,p+\lambda}(Q_T)$ coincides with $C^{0,1+(\lambda-(n+2))/p}(\overline{Q}_T)$ for $\lambda \in (n+2-p, n+2)$. ∎

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