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Schwartz Kernels on the Heisenberg Group.

ALESSANDRO VENERUSO

Sunto. – *Sia $\boldsymbol{H}_n$ il gruppo di Heisenberg di dimensione $2n+1$. Siano $\mathcal{L}_1, \ldots, \mathcal{L}_n$ i sub-Laplaciani parziali su $\boldsymbol{H}_n$ e T l'elemento centrale dell'algebra di Lie di $\boldsymbol{H}_n$. In questo lavoro dimostriamo che, data una funzione m appartenente allo spazio di Schwartz $\mathcal{S}(\boldsymbol{R}^{n+1})$, il nucleo dell'operatore $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$ è una funzione in $\mathcal{S}(\boldsymbol{H}_n)$. Inoltre dimostriamo che, date altre due funzioni $h \in \mathcal{S}(\boldsymbol{R}^n)$ e $g \in \mathcal{S}(\boldsymbol{R}^2)$, i nuclei degli operatori $h(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ e $g(\mathcal{L}, -iT)$ stanno in $\mathcal{S}(\boldsymbol{H}_n)$. Qui $\mathcal{L} = \mathcal{L}_1 + \ldots + \mathcal{L}_n$ è il sub-Laplaciano su $\boldsymbol{H}_n$.*

Summary. – *Let $\boldsymbol{H}_n$ be the Heisenberg group of dimension $2n+1$. Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be the partial sub-Laplacians on $\boldsymbol{H}_n$ and T the central element of the Lie algebra of $\boldsymbol{H}_n$. We prove that the kernel of the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$ is in the Schwartz space $\mathcal{S}(\boldsymbol{H}_n)$ if $m \in \mathcal{S}(\boldsymbol{R}^{n+1})$. We prove also that the kernel of the operator $h(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ is in $\mathcal{S}(\boldsymbol{H}_n)$ if $h \in \mathcal{S}(\boldsymbol{R}^n)$ and that the kernel of the operator $g(\mathcal{L}, -iT)$ is in $\mathcal{S}(\boldsymbol{H}_n)$ if $g \in \mathcal{S}(\boldsymbol{R}^2)$. Here $\mathcal{L} = \mathcal{L}_1 + \ldots + \mathcal{L}_n$ is the Kohn-Laplacian on $\boldsymbol{H}_n$.*

1. – Introduction.

Let $\mathcal{L}$ be the Kohn-Laplacian on a stratified group G and let m be the restriction on $[0, +\infty)$ of a function in the Schwartz space $\mathcal{S}(\boldsymbol{R})$. Then it is well known that the kernel of the operator $m(\mathcal{L})$, i.e. the unique tempered distribution M such that $m(\mathcal{L})f = f * M$ for every $f \in \mathcal{S}(G)$, is in $\mathcal{S}(G)$ (see [5, 7]).

Let G be the Heisenberg group $\boldsymbol{H}_n$ of dimension $2n+1$. We denote by $\mathcal{L}_1, \ldots, \mathcal{L}_n$ the partial sub-Laplacians and by T the central element of the Lie algebra of $\boldsymbol{H}_n$. The Kohn-Laplacian on $\boldsymbol{H}_n$ is $\mathcal{L} = \mathcal{L}_1 + \ldots + \mathcal{L}_n$. The operators $\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT$ form a commutative family of self-adjoint operators, so they admit a joint spectral resolution and it is possible to define the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$ when m is a bounded Borel function on the joint spectrum Σ of $\{\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT\}$. It has been proved by Benson, Jenkins and Ratcliff [1, Corollary 6.3] that the kernel of the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$ is in $\mathcal{S}(\boldsymbol{H}_n)$ if $m \in C_c^\infty(\boldsymbol{R}^{n+1})$ (here we identify m with its restriction on Σ) and the kernel of the operator $g(\mathcal{L}, -iT)$ is in $\mathcal{S}(\boldsymbol{H}_n)$ if $g \in C_c^\infty(\boldsymbol{R}^2)$.

In this paper we prove the following stronger result (for the definitions of the norms in $\mathcal{S}(\boldsymbol{H}_n)$ and in $\mathcal{S}(\boldsymbol{R}^d)$ see Sections 2 and 3):

THEOREM 1.1.

(a) *Let H denote the kernel of the operator $h(\mathcal{L}_1, \ldots, \mathcal{L}_n)$. Then $h \mapsto H$ is a bounded linear map from $\mathcal{S}(\boldsymbol{R}^n)$ to $\mathcal{S}(\boldsymbol{H}_n)$.*

(b) *Let M denote the kernel of the operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$. Then $m \mapsto M$ is a bounded linear map from $\mathcal{S}(\boldsymbol{R}^{n+1})$ to $\mathcal{S}(\boldsymbol{H}_n)$.*

(c) *Let G denote the kernel of the operator $g(\mathcal{L}, -iT)$. Then $g \mapsto G$ is a bounded linear map from $\mathcal{S}(\boldsymbol{R}^2)$ to $\mathcal{S}(\boldsymbol{H}_n)$.*

2. – Notation and preliminaries.

In this paper $\boldsymbol{N}$ denotes the set of nonnegative integers, $\boldsymbol{Z}_+$ the set of positive integers and $\boldsymbol{R}^*$ the set of non-zero real numbers. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \boldsymbol{N}^d$, we put $|\alpha| = \sum_{j=1}^{d} \alpha_j$. We shall denote by C a constant which will not be necessarily the same at each occurrence.

Fix $n \in \boldsymbol{Z}_+$. The $2n+1$-dimensional Heisenberg group $\boldsymbol{H}_n$ is the nilpotent Lie group whose underlying manifold is $\boldsymbol{C}^n \times \boldsymbol{R}$, with multiplication given by

$$(z, t)(z', t') = (z + z', t + t' + 2\ \mathrm{Im}\langle z, z'\rangle)$$

where $\langle z, z'\rangle = \sum_{j=1}^{n} z_j \overline{z_j'}$. The Lie algebra of $\boldsymbol{H}_n$ is generated by the left-invariant vector fields $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T$, where

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial t};$$

$$\overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - iz_j \frac{\partial}{\partial t};$$

$$T = \frac{\partial}{\partial t}.$$

The commutators are

$$[Z_j, \overline{Z}_k] = -2i\delta_{j,k} T; \tag{2.1}$$

$$[Z_j, Z_k] = [\overline{Z}_j, \overline{Z}_k] = [Z_j, T] = [\overline{Z}_j, T] = 0. \tag{2.2}$$

$\boldsymbol{H}_n$ is a stratified group endowed with a family of dilations $\{\delta_r : r > 0\}$ defined by

$$\delta_r(z, t) = (rz, r^2 t).$$

The homogeneous dimension of $\boldsymbol{H}_n$ is therefore $Q = 2n + 2$. We fix on $\boldsymbol{H}_n$ the following subadditive homogeneous norm (see [3]):

$$|(z,t)|_{\boldsymbol{H}_n} = (|z|^4 + t^2)^{1/4}$$

where $|z| = \left(\sum_{j=1}^{n} |z_j|^2\right)^{1/2}$. We observe that

$$|(z,t)|_{\boldsymbol{H}_n} \simeq \sum_{j=1}^{n} |z_j| + |t|^{1/2}. \tag{2.3}$$

The following lemma will be useful later:

LEMMA 2.1. – *Fix $u, v \in \boldsymbol{H}_n$ and $a \geqslant 1$. Then*

$$a + |u|_{\boldsymbol{H}_n} \leqslant (a + |v|_{\boldsymbol{H}_n})(1 + |uv^{-1}|_{\boldsymbol{H}_n}).$$

PROOF.

$$\begin{aligned} a + |u|_{\boldsymbol{H}_n} &= a + |uv^{-1}v|_{\boldsymbol{H}_n} \\ &\leqslant a + |uv^{-1}|_{\boldsymbol{H}_n} + |v|_{\boldsymbol{H}_n} \\ &\leqslant a + a|uv^{-1}|_{\boldsymbol{H}_n} + |v|_{\boldsymbol{H}_n} + |v|_{\boldsymbol{H}_n}|uv^{-1}|_{\boldsymbol{H}_n} \\ &= (a + |v|_{\boldsymbol{H}_n})(1 + |uv^{-1}|_{\boldsymbol{H}_n}). \quad \blacksquare \end{aligned}$$

The bi-invariant Haar measure on $\boldsymbol{H}_n$ coincides with the Lebesgue measure on $\boldsymbol{R}^{2n+1}$. The convolution $f * g$ of two functions $f, g \in L^1(\boldsymbol{H}_n)$ is defined by

$$\begin{aligned} (f*g)(z,t) &= \int_{\boldsymbol{H}_n} f((z,t)(\zeta,\tau)^{-1})\, g(\zeta,\tau)\, d\zeta\, d\tau \\ &= \int_{\boldsymbol{H}_n} f(z-\zeta, t-\tau-2\ \mathrm{Im}\langle z,\zeta\rangle)\, g(\zeta,\tau)\, d\zeta\, d\tau. \end{aligned} \tag{2.4}$$

As usual, we denote by $\mathcal{S}(\boldsymbol{H}_n)$ the Schwartz space of rapidly decreasing smooth functions on $\boldsymbol{H}_n$ and by $\mathcal{S}'(\boldsymbol{H}_n)$ the dual space of $\mathcal{S}(\boldsymbol{H}_n)$, i.e. the space of tempered distributions on $\boldsymbol{H}_n$. The topology of the Fréchet space $\mathcal{S}(\boldsymbol{H}_n)$ is given by the family of norms $\|\cdot\|_{(N,\boldsymbol{H}_n)}$ $(N \in \boldsymbol{N})$ defined by

$$\|f\|_{(N,\boldsymbol{H}_n)} = \sup_{\substack{|I| \leqslant N \\ x \in \boldsymbol{H}_n}} (1 + |x|_{\boldsymbol{H}_n})^{(N+1)(Q+1)} |X^I f(x)| \tag{2.5}$$

where $I = (i_1, \ldots, i_n, j_1, \ldots, j_n, l) \in \boldsymbol{N}^{2n+1}$ and

$$X^I = Z_1^{i_1} \ldots Z_n^{i_n} \overline{Z}_1^{j_1} \ldots \overline{Z}_n^{j_n} T^l. \tag{2.6}$$

If $\{f_k\}_{k \in \boldsymbol{N}}$ is a sequence of functions in $\mathcal{S}(\boldsymbol{H}_n)$, the series $\sum_{k=0}^{+\infty} f_k$ converges absolutely in $\mathcal{S}(\boldsymbol{H}_n)$ if and only if

$$\sum_{k=0}^{+\infty} \|f_k\|_{(N,\, \boldsymbol{H}_n)} < +\infty$$

for every $N \in \boldsymbol{N}$. If $f \in \mathcal{S}(\boldsymbol{H}_n)$ and $u \in \mathcal{S}'(\boldsymbol{H}_n)$, the convolution $f * u$ is the tempered distribution defined by

$$\langle f * u,\, \varphi\rangle = \langle u, \tilde{f} * \varphi\rangle$$

for any $\varphi \in \mathcal{S}(\boldsymbol{H}_n)$, where the function $\tilde{f} \in \mathcal{S}(\boldsymbol{H}_n)$ is defined by

$$\tilde{f}(x) = f(x^{-1}).$$

The partial sub-Laplacians $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on $\boldsymbol{H}_n$ are defined by

$$\mathcal{L}_j = -\frac{1}{2}(Z_j \overline{Z}_j + \overline{Z}_j Z_j).$$

The Kohn-Laplacian on $\boldsymbol{H}_n$ is $\mathcal{L} = \sum_{j=1}^{n} \mathcal{L}_j$. The operators $\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT$ form a family of commuting self-adjoint operators. Their joint spectrum (see [2]) is the subset $\Sigma = \Sigma_1 \cup \Sigma_2$ of $\boldsymbol{R}^{n+1}$, where

$$\Sigma_1 = \left\{((2k_1+1)|\lambda|, \ldots, (2k_n+1)|\lambda|, \lambda) : k_1, \ldots, k_n \in \boldsymbol{N},\ \lambda \in \boldsymbol{R}^*\right\}$$

and

$$\Sigma_2 = \{(\mu_1, \ldots, \mu_n, 0) : \mu_1, \ldots, \mu_n \in [0, +\infty)\}.$$

For any bounded Borel function m on Σ, the multiplier operator $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT)$ is bounded on $L^2(\boldsymbol{H}_n)$ by the spectral theorem. Such operator commutes with left translations, so by [6, Theorem 3.2] it admits a kernel $M \in \mathcal{S}'(\boldsymbol{H}_n)$ which satisfies

$$m(\mathcal{L}_1, \ldots, \mathcal{L}_n, -iT) f = f * M$$

for any $f \in \mathcal{S}(\boldsymbol{H}_n)$.

3. – Schwartz functions on $\boldsymbol{R}^d$ and tensor products.

Fix $d \in \boldsymbol{Z}_+$. Following [4] and by analogy with the definition of the norms (2.5) on $\mathcal{S}(\boldsymbol{H}_n)$, we define the following family of norms on $\mathcal{S}(\boldsymbol{R}^d)$, which gives the usual topology of the Fréchet space $\mathcal{S}(\boldsymbol{R}^d)$:

$$\|f\|_{(N,\, \boldsymbol{R}^d)} = \sup_{\substack{|\alpha| \leq N \\ x \in \boldsymbol{R}^d}} (1 + |x|)^{(N+1)(d+1)} |D^\alpha f(x)| \tag{3.7}$$

where $N \in \boldsymbol{N}$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \boldsymbol{N}^d$ and $D^\alpha = \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}, \dots, \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \right)$. The notion of absolute convergence of a series in $\mathcal{S}(\boldsymbol{R}^d)$ is the same as in $\mathcal{S}(\boldsymbol{H}_n)$.

Fix $m, n \in \boldsymbol{Z}_+$. If $f \in \mathcal{S}(\boldsymbol{R}^m)$ and $g \in \mathcal{S}(\boldsymbol{R}^n)$, their tensor product is the function $f \otimes g \in \mathcal{S}(\boldsymbol{R}^{m+n})$ defined by the formula

$$(f \otimes g)(x_1, \dots, x_{m+n}) = f(x_1, \dots, x_m)\, g(x_{m+1}, \dots, x_{m+n}).$$

By straight-forward calculations involving the norms (3.7), it is easy to verify that for every $N \in \boldsymbol{N}$ the following inequality holds:

$$\|f\|_{(N, \boldsymbol{R}^m)} \|g\|_{(N, \boldsymbol{R}^n)} \le \|f \otimes g\|_{(2N+1, \boldsymbol{R}^{m+n})}. \tag{3.8}$$

Moreover, combining Theorems 45.1 and 51.6 in [8], we have the following

THEOREM 3.1. – *For every $h \in \mathcal{S}(\boldsymbol{R}^{m+n})$ there exist $f_k \in \mathcal{S}(\boldsymbol{R}^m)$ and $g_k \in \mathcal{S}(\boldsymbol{R}^n)$ $(k \in \boldsymbol{N})$ such that the series $\sum_{k=0}^{+\infty} (f_k \otimes g_k)$ converges absolutely to h in $\mathcal{S}(\boldsymbol{R}^{m+n})$.*

4. – Proof of Theorem 1.1.

In order to avoid confusion, since we have to deal with Heisenberg groups of different dimensions, in this section $\mathcal{L}_j^{\boldsymbol{H}_n}$, $\mathcal{L}^{\boldsymbol{H}_n}$ and $*_{\boldsymbol{H}_n}$ will denote the j-th sub-Laplacian, the Kohn-Laplacian and convolution on $\boldsymbol{H}_n$, respectively. Moreover, $*_{\boldsymbol{R}}$ will denote convolution on $\boldsymbol{R}$ and $\mathcal{F}$ the Fourier transform on $\boldsymbol{R}$ defined by

$$\mathcal{F}f(\xi) = \int_{\boldsymbol{R}} f(x)\, e^{-ix\xi}\, dx$$

for every $f \in L^1(\boldsymbol{R})$ and $\xi \in \boldsymbol{R}$.

Fix $f \in \mathcal{S}(\boldsymbol{H}_n)$, $j \in \{1, \dots, n\}$ and $(z, t) \in \boldsymbol{H}_n$. It is immediate to verify that

$$(\mathcal{L}_j^{\boldsymbol{H}_n} f)(z, t) = (\mathcal{L}^{\boldsymbol{H}_1} f(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_n, \cdot))(z_j, t).$$

So, if γ is a bounded Borel function on $[0, +\infty)$ and $\Gamma \in \mathcal{S}'(\boldsymbol{H}_1)$ is the kernel of the operator $\gamma(\mathcal{L}^{\boldsymbol{H}_1})$, we have

$$(\gamma(\mathcal{L}_j^{\boldsymbol{H}_n})\, f)(z, t) = (f(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_n, \cdot) *_{\boldsymbol{H}_1} \Gamma)(z_j, t).$$

Moreover, for $n \ge 2$, if β is a bounded Borel function on $\boldsymbol{R}^{n-1}$ and $B \in \mathcal{S}'(\boldsymbol{H}_{n-1})$ is the kernel of the operator $\beta(\mathcal{L}_1^{\boldsymbol{H}_{n-1}}, \dots, \mathcal{L}_{n-1}^{\boldsymbol{H}_{n-1}})$, we have

$$(\beta(\mathcal{L}_1^{\boldsymbol{H}_n}, \dots, \mathcal{L}_{n-1}^{\boldsymbol{H}_n})\, f)(z, t) = (f(\cdot, \dots, \cdot, z_n, \cdot) *_{\boldsymbol{H}_{n-1}} B)(z_1, \dots, z_{n-1}, t). \tag{4.10}$$

We prove part (a) of Theorem 1.1 by induction on n. We know that for $n = 1$ it is verified (see [5, Theorem 2.4]), so we take $n \geqslant 2$ and suppose that the statement holds for any integer up to $n-1$. Fix $h \in \mathcal{S}(\boldsymbol{R}^n)$. By Theorem 3.1 there exist $\varphi_k \in \mathcal{S}(\boldsymbol{R}^{n-1})$ and $\psi_k \in \mathcal{S}(\boldsymbol{R})$ $(k \in \boldsymbol{N})$ such that the series $\sum_{k=0}^{+\infty} (\varphi_k \otimes \psi_k)$ converges absolutely to h in $\mathcal{S}(\boldsymbol{R}^n)$. We denote by Φ_k, Ψ_k and H_k the kernels of the operators $\varphi_k(\mathcal{L}_1^{\boldsymbol{H}_{n-1}}, \ldots, \mathcal{L}_{n-1}^{\boldsymbol{H}_{n-1}})$, $\psi_k(\mathcal{L}^{\boldsymbol{H}_1})$ and $(\varphi_k \otimes \psi_k)(\mathcal{L}_1^{\boldsymbol{H}_n}, \ldots, \mathcal{L}_n^{\boldsymbol{H}_n})$, respectively. By the inductive hypothesis $\Phi_k \in \mathcal{S}(\boldsymbol{H}_{n-1})$ and $\Psi_k \in \mathcal{S}(\boldsymbol{H}_1)$. Fix $f \in \mathcal{S}(\boldsymbol{H}_n)$ and $(z, t) \in \boldsymbol{H}_n$. By (4.10) and (2.4) we have

$$
\begin{aligned}
&(\varphi_k(\mathcal{L}_1^{\boldsymbol{H}_n}, \ldots, \mathcal{L}_{n-1}^{\boldsymbol{H}_n}) f)(z, t) \\
&= (f(\cdot, \ldots, \cdot, z_n, \cdot) *_{\boldsymbol{H}_{n-1}} \Phi_k)(z_1, \ldots, z_{n-1}, t) \\
&= \int_{\boldsymbol{H}_{n-1}} f\Big(z_1 - \zeta_1, \ldots, z_{n-1} - \zeta_{n-1}, z_n, t - \tau - 2 \operatorname{Im}\Big(\sum_{j=1}^{n-1} z_j \overline{\zeta}_j\Big)\Big) \\
&\quad \cdot \Phi_k(\zeta_1, \ldots, \zeta_{n-1}, \tau)\, d\zeta_1 \ldots d\zeta_{n-1}\, d\tau\,.
\end{aligned}
$$

Then, by applying also (4.9), we obtain

$$
\begin{aligned}
(f *_{\boldsymbol{H}_n} H_k)(z, t) &= (\psi_k(\mathcal{L}_n^{\boldsymbol{H}_n})\, \varphi_k(\mathcal{L}_1^{\boldsymbol{H}_n}, \ldots, \mathcal{L}_{n-1}^{\boldsymbol{H}_n}) f)(z, t) \\
&= ((\varphi_k(\mathcal{L}_1^{\boldsymbol{H}_n}, \ldots, \mathcal{L}_{n-1}^{\boldsymbol{H}_n}) f)(z_1, \ldots, z_{n-1}, \cdot, \cdot) *_{\boldsymbol{H}_1} \Psi_k)(z_n, t) \\
&= \int_{\boldsymbol{H}_1} \Bigg(\int_{\boldsymbol{H}_{n-1}} f\Big(z_1 - \zeta_1, \ldots, z_n - \zeta_n, t - \vartheta - \tau - 2 \operatorname{Im}\Big(\sum_{j=1}^{n} z_j \overline{\zeta}_j\Big)\Big) \\
&\quad \cdot \Phi_k(\zeta_1, \ldots, \zeta_{n-1}, \tau)\, d\zeta_1 \ldots d\zeta_{n-1}\, d\tau \Bigg) \Psi_k(\zeta_n, \vartheta)\, d\zeta_n\, d\vartheta\,.
\end{aligned}
$$

The change of variable $\sigma = \tau + \vartheta$ in the inner integral leads to

$$
\begin{aligned}
(f *_{\boldsymbol{H}_n} H_k)(z,t) &= \int_{\boldsymbol{H}_1} \Bigg(\int_{\boldsymbol{H}_{n-1}} f\Big(z_1 - \zeta_1, \ldots, z_n - \zeta_n, t - \sigma - 2 \operatorname{Im}\Big(\sum_{j=1}^{n} z_j \overline{\zeta}_j\Big)\Big) \\
&\quad \cdot \Phi_k(\zeta_1, \ldots, \zeta_{n-1}, \sigma - \vartheta)\, d\zeta_1 \ldots d\zeta_{n-1}\, d\sigma \Bigg) \Psi_k(\zeta_n, \vartheta)\, d\zeta_n\, d\vartheta \\
&= \int_{\boldsymbol{H}_n} f(z - \zeta, t - \sigma - 2 \operatorname{Im}\langle z, \zeta\rangle) \\
&\quad \cdot \Bigg(\int_{\boldsymbol{R}} \Phi_k(\zeta_1, \ldots, \zeta_{n-1}, \sigma - \vartheta)\, \Psi_k(\zeta_n, \vartheta)\, d\vartheta \Bigg) d\zeta\, d\sigma\,.
\end{aligned}
$$

Therefore, the kernel H_k is the Schwartz function on $\boldsymbol{H}_n$ defined by

$$H_k(z, t) = \int_{\boldsymbol{R}} \Phi_k(z_1, \ldots, z_{n-1}, t-\tau)\, \Psi_k(z_n, \tau)\, d\tau\,.$$

Fix $I = (i_1, \ldots, i_n, j_1, \ldots, j_n, l) \in \boldsymbol{N}^{2n+1}$. By (2.6), (2.1) and (2.2) we have that $X^I = VU$ where $U = Z_1^{i_1} \ldots Z_{n-1}^{i_{n-1}} \overline{Z}_1^{j_1} \ldots \overline{Z}_{n-1}^{j_{n-1}}$ and $V = Z_n^{i_n} \overline{Z}_n^{j_n} T^l$. For every $k \in \boldsymbol{N}$ and $(z, t) \in \boldsymbol{H}_n$ we have

$$\begin{aligned} UH_k(z, t) &= \int_{\boldsymbol{R}} U\Phi_k(z_1, \ldots, z_{n-1}, t-\tau)\, \Psi_k(z_n, \tau)\, d\tau \\ &= \int_{\boldsymbol{R}} U\Phi_k(z_1, \ldots, z_{n-1}, \tau)\, \Psi_k(z_n, t-\tau)\, d\tau \end{aligned}$$

and hence

$$\begin{aligned} X^I H_k(z, t) &= \int_{\boldsymbol{R}} U\Phi_k(z_1, \ldots, z_{n-1}, \tau)\, V\Psi_k(z_n, t-\tau)\, d\tau \\ &= \int_{\boldsymbol{R}} U\Phi_k(z_1, \ldots, z_{n-1}, t-\tau)\, V\Psi_k(z_n, \tau)\, d\tau\,. \end{aligned} \tag{4.11}$$

Fix $N_0 \in \boldsymbol{N}$. By the inductive hypothesis there exist $C > 0$ and $L \in \boldsymbol{N}$, which do not depend on k, such that

$$|U\Phi_k(u)| \leq C(1 + |u|_{\boldsymbol{H}_{n-1}})^{-(N_0+3)} \|\varphi_k\|_{(L, \boldsymbol{R}^{n-1})} \tag{4.12}$$

for every $u \in \boldsymbol{H}_{n-1}$ and

$$|V\Psi_k(v)| \leq C(1 + |v|_{\boldsymbol{H}_1})^{-(N_0+3)} \|\psi_k\|_{(L, \boldsymbol{R})} \tag{4.13}$$

for every $v \in \boldsymbol{H}_1$. Fix $x = (z, t) \in \boldsymbol{H}_n$, put $u = (z_1, \ldots, z_{n-1}, t) \in \boldsymbol{H}_{n-1}$ and define the function $P : \boldsymbol{R} \ni \tau \mapsto (0, \ldots, 0, \tau) \in \boldsymbol{H}_{n-1}$. Note that

$$(z_1, \ldots, z_{n-1}, t-\tau) = u \cdot P(\tau)^{-1}$$

for every $\tau \in \boldsymbol{R}$. Moreover, by (2.3) we observe that

$$|(z_n, \tau)|_{\boldsymbol{H}_1} \simeq |z_n| + |\tau|^{1/2} = |z_n| + |P(\tau)|_{\boldsymbol{H}_{n-1}}$$

for every $\tau \in \boldsymbol{R}$. Then, by applying (4.11), (4.12) and (4.13), we have

$$|X^I H_k(x)| \leq C \|\varphi_k\|_{(L, \boldsymbol{R}^{n-1})} \|\psi_k\|_{(L, \boldsymbol{R})} \int_{\boldsymbol{R}} (1 + |u \cdot P(\tau)^{-1}|_{\boldsymbol{H}_{n-1}})^{-(N_0+3)} \cdot$$

$$(1 + |z_n| + |P(\tau)|_{\boldsymbol{H}_{n-1}})^{-(N_0+3)}\, d\tau\,.$$

But Lemma 2.1, applied in $\boldsymbol{H}_{n-1}$ with $v = P(\tau)$ and $a = 1 + |z_n|$, yields

$$(1 + |z_n| + |P(\tau)|_{\boldsymbol{H}_{n-1}})^{-N_0} \leqslant (1 + |z_n| + |u|_{\boldsymbol{H}_{n-1}})^{-N_0}(1 + |u \cdot P(\tau)^{-1}|_{\boldsymbol{H}_{n-1}})^{N_0}$$

$$\simeq (1 + |x|_{\boldsymbol{H}_n})^{-N_0}(1 + |u \cdot P(\tau)^{-1}|_{\boldsymbol{H}_{n-1}})^{N_0}$$

for every $\tau \in \boldsymbol{R}$. By this inequality and (3.8) we have

$$|X^I H_k(x)| \leqslant C \|\varphi_k \otimes \psi_k\|_{(2L+1, \boldsymbol{R}^n)}(1 + |x|_{\boldsymbol{H}_n})^{-N_0} \cdot$$

$$\int_{\boldsymbol{R}} (1 + |u \cdot P(\tau)^{-1}|_{\boldsymbol{H}_{n-1}})^{-3}(1 + |z_n| + |\tau|^{1/2})^{-3} \, d\tau \, .$$

The preceding integral is bounded by the constant $\int_{\boldsymbol{R}} (1 + |\tau|^{1/2})^{-3} d\tau$. Since I and N_0 are arbitrary, for every $N \in \boldsymbol{N}$ there exist $C > 0$ and $N' \in \boldsymbol{N}$ such that

$$\|H_k\|_{(N, \boldsymbol{H}_n)} \leqslant C \|\varphi_k \otimes \psi_k\|_{(N', \boldsymbol{R}^n)}$$

for every $k \in \boldsymbol{N}$. Therefore, since the series $\sum_{k=0}^{+\infty} (\varphi_k \otimes \psi_k)$ converges absolutely to h in $\mathcal{S}(\boldsymbol{R}^n)$, the series $\sum_{k=0}^{+\infty} H_k$ converges absolutely in $\mathcal{S}(\boldsymbol{H}_n)$ to some function F. Then, for a fixed $f \in \mathcal{S}(\boldsymbol{H}_n)$, the series $\sum_{k=0}^{+\infty} (f *_{\boldsymbol{H}_n} H_k)$ converges to $f *_{\boldsymbol{H}_n} F$ in $\mathcal{S}(\boldsymbol{H}_n)$, since convolution is continuous from $\mathcal{S}(\boldsymbol{H}_n) \times \mathcal{S}(\boldsymbol{H}_n)$ to $\mathcal{S}(\boldsymbol{H}_n)$ (see e.g. [4, Proposition 1.47]). On the other hand, the series $\sum_{k=0}^{+\infty} (f *_{\boldsymbol{H}_n} H_k)$ converges to $f *_{\boldsymbol{H}_n} H$ in $L^2(\boldsymbol{H}_n)$ by the spectral theorem. Since f is an arbitrary function in $\mathcal{S}(\boldsymbol{H}_n)$, we conclude that $H = F \in \mathcal{S}(\boldsymbol{H}_n)$. The boundedness of the operator $h \mapsto H$ is an easy consequence of the closed graph theorem.

Now we prove part (b). Fix $m \in \mathcal{S}(\boldsymbol{R}^{n+1})$. By Theorem 3.1 there exist $h_k \in \mathcal{S}(\boldsymbol{R}^n)$ and $\gamma_k \in \mathcal{S}(\boldsymbol{R})$ $(k \in \boldsymbol{N})$ such that the series $\sum_{k=0}^{+\infty} (h_k \otimes \gamma_k)$ converges absolutely to m in $\mathcal{S}(\boldsymbol{R}^{n+1})$. We denote by H_k and M_k the kernels of the operators $h_k(\mathcal{L}_1^{\boldsymbol{H}_n}, \ldots, \mathcal{L}_n^{\boldsymbol{H}_n})$ and $(h_k \otimes \gamma_k)(\mathcal{L}_1^{\boldsymbol{H}_n}, \ldots, \mathcal{L}_n^{\boldsymbol{H}_n}, -iT)$, respectively. Note that $H_k \in \mathcal{S}(\boldsymbol{H}_n)$ by part (a). Moreover, we denote by Γ_k the kernel of the operator $\gamma_k\left(-i\frac{d}{dt}\right)$ which acts on $L^2(\boldsymbol{R})$. Fix $f \in \mathcal{S}(\boldsymbol{H}_n)$ and $(z, t) \in \boldsymbol{H}_n$. We observe that $\Gamma_k = \mathcal{F}^{-1}\gamma_k \in \mathcal{S}(\boldsymbol{R})$ and

$$(\gamma_k(-iT) f)(z, t) = (f(z, \cdot) *_{\boldsymbol{R}} \Gamma_k)(t) \, .$$

Then

$$
\begin{aligned}
&(f *_{\boldsymbol{H}_n} M_k)(z, t) \\
&= ((f *_{\boldsymbol{H}_n} H_k)(z, \cdot) *_{\boldsymbol{R}} \Gamma_k)(t) \\
&= \int_{\boldsymbol{R}} (f *_{\boldsymbol{H}_n} H_k)(z, t-\tau)\, \Gamma_k(\tau)\, d\tau \\
&= \int_{\boldsymbol{R}} \left(\int_{\boldsymbol{H}_n} f(z-\zeta, t-\tau-\vartheta-2\, \mathrm{Im}\langle z, \zeta\rangle)\, H_k(\zeta, \vartheta)\, d\zeta\, d\vartheta \right) \Gamma_k(\tau)\, d\tau \\
&= \int_{\boldsymbol{R}} \left(\int_{\boldsymbol{H}_n} f(z-\zeta, t-\sigma-2\, \mathrm{Im}\langle z, \zeta\rangle)\, H_k(\zeta, \sigma-\tau)\, d\zeta\, d\sigma \right) \Gamma_k(\tau)\, d\tau \\
&= \int_{\boldsymbol{H}_n} f(z-\zeta, t-\sigma-2\, \mathrm{Im}\langle z, \zeta\rangle) \left(\int_{\boldsymbol{R}} H_k(\zeta, \sigma-\tau)\, \Gamma_k(\tau)\, d\tau \right) d\zeta\, d\sigma .
\end{aligned}
$$

Therefore, the kernel M_k is the Schwartz function on $\boldsymbol{H}_n$ defined by

$$M_k(z, t) = \int_{\boldsymbol{R}} H_k(z, t-\tau)\, \Gamma_k(\tau)\, d\tau .$$

Fix $I \in \boldsymbol{N}^{2n+1}$ and $N_0 \in \boldsymbol{N}$. By part (a) and by the continuity of the operator $\mathcal{F}\colon \mathcal{S}(\boldsymbol{R}) \to \mathcal{S}(\boldsymbol{R})$, there exist $C > 0$ and $L \in \boldsymbol{N}$, which do not depend on k, such that

$$|X^I H_k(x)| \leqslant C(1 + |x|_{\boldsymbol{H}_n})^{-N_0} \|h_k\|_{(L, \boldsymbol{R}^n)}$$

for every $x \in \boldsymbol{H}_n$ and

$$|\Gamma_k(\tau)| \leqslant C(1 + |\tau|^{1/2})^{-(N_0+3)} \|\gamma_k\|_{(L, \boldsymbol{R})}$$

for every $\tau \in \boldsymbol{R}$. Fix $x = (z, t) \in \boldsymbol{H}_n$. Then

$$
\begin{aligned}
|X^I M_k(x)| &= \left| \int_{\boldsymbol{R}} X^I H_k(z, t-\tau)\, \Gamma_k(\tau)\, d\tau \right| \\
&\leqslant C \|h_k\|_{(L, \boldsymbol{R}^n)} \|\gamma_k\|_{(L, \boldsymbol{R})} \int_{\boldsymbol{R}} (1 + |(z, t-\tau)|_{\boldsymbol{H}_n})^{-N_0} (1 + |\tau|^{1/2})^{-(N_0+3)}\, d\tau \\
&= C \|h_k\|_{(L, \boldsymbol{R}^n)} \|\gamma_k\|_{(L, \boldsymbol{R})} \int_{\boldsymbol{R}} (1 + |(z, t)(0, \tau)^{-1}|_{\boldsymbol{H}_n})^{-N_0} \\
&\qquad \cdot (1 + |(0, \tau)|_{\boldsymbol{H}_n})^{-N_0} (1 + |\tau|^{1/2})^{-3}\, d\tau .
\end{aligned}
$$

But Lemma 2.1, applied with $u = (z, t)$, $v = (0, \tau)$ and $a = 1$, yields

$$(1 + |(z, t)(0, \tau)^{-1}|_{\boldsymbol{H}_n})^{-N_0} (1 + |(0, \tau)|_{\boldsymbol{H}_n})^{-N_0} \leqslant (1 + |(z, t)|_{\boldsymbol{H}_n})^{-N_0}$$

for every $\tau \in \boldsymbol{R}$. By this inequality and (3.8) we have

$$|X^I M_k(x)| \leqslant C \|h_k \otimes \gamma_k\|_{(2L+1, \boldsymbol{R}^{n+1})} (1 + |x|_{\boldsymbol{H}_n})^{-N_0}.$$

Since I and N_0 are arbitrary, for every $N \in \boldsymbol{N}$ there exist $C > 0$ and $N' \in \boldsymbol{N}$ such that

$$\|M_k\|_{(N,\,\boldsymbol{H}_n)} \leqslant C\|h_k \otimes \gamma_k\|_{(N',\,\boldsymbol{R}^{n+1})}$$

for every $k \in \boldsymbol{N}$. From now on, we only have to apply the argument in the final part of the proof of part (a).

Part (c) is a corollary of part (b). We only need to observe that

$$g(\mathcal{L},\ -iT) = m(\mathcal{L}_1,\ \dots,\ \mathcal{L}_n,\ -iT)$$

where m is the function defined by

$$m(x_1,\ \dots,\ x_{n+1}) = g(x_1 + \dots + x_n,\ x_{n+1})\,.$$

It is easy to verify that if $g \in \mathcal{S}(\boldsymbol{R}^2)$ then $m \in \mathcal{S}(\boldsymbol{R}^{n+1})$ and

$$\|m\|_{(N,\,\boldsymbol{R}^{n+1})} \leqslant \|g\|_{(n(N+1),\,\boldsymbol{R}^2)}$$

for every $N \in \boldsymbol{N}$.

REFERENCES

[1] C. Benson - J. Jenkins - G. Ratcliff, *The spherical transform of a Schwartz function on the Heisenberg group*, J. Funct. Anal., **154** (1998), 379-423.

[2] C. Benson - J. Jenkins - G. Ratcliff - T. Worku, *Spectra for Gelfand pairs associated with the Heisenberg group*, Colloq. Math., **71** (1996), 305-328.

[3] J. Cygan, *Subadditivity of homogeneous norms on certain nilpotent Lie groups*, Proc. Amer. Math. Soc., **83** (1981), 69-70.

[4] G. B. Folland - E. M. Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton, 1982.

[5] A. Hulanicki, *A functional calculus for Rockland operators on nilpotent Lie groups*, Studia Math., **78** (1984), 253-266.

[6] A. Korányi - S. Vági - G. V. Welland, *Remarks on the Cauchy integral and the conjugate function in generalized half-planes*, J. Math. Mech., **19** (1970), 1069-1081.

[7] G. Mauceri, *Maximal operators and Riesz means on stratified groups*, Symposia Math., **29** (1987), 47-62.

[8] F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.

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