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Regular Permutation Sets and Loops.

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Sunto. – *Utilizzando insiemi regolari di permutazioni e due operazioni opportunamente definite, si ottengono nuove caratterizzazioni di importanti classi di cappi.*

Summary. – *Two suitable composition laws are defined in a regular permutation set in order to find new characterizations of some important classes of loops.*

1. – Introduction and preliminaries.

The notation and terminology employed in this paper are standard: in particular if T is a mapping of a set Ω into itself or some other set and if $x \in \Omega$, then $x\mathrm{T}$ shall denote the image of x under T.

Let $\mathrm{Sym}(\Omega)$ be the group of all bijections of a set Ω: as usually a subset $\Phi \subseteq \mathrm{Sym}(\Omega)$ is called a *regular permutation set (r.p.s.) on* Ω if

– $\mathrm{Id}_\Omega \in \Phi$

– if $a, b \in \Omega$ then $\exists !\ P \in \Phi$ such that $aP = b$.

In section 2 two suitable composition laws are defined in a r.p.s. Φ on a set Ω. If these composition laws are denoted by $\otimes$ and $\perp$ respectively, then we can prove that

– $(\Phi, \otimes)$ and $(\Phi, \perp)$ are loops, which, in general, are neither isomorphic nor anti-isomorphic.

– $(\Phi, \otimes)$ is a right Bol loop if and only if $(\Phi, \perp)$ is a left Bol loop; moreover, in this case, they are anti-isomorphic. (For the historical relevance of Bol loops see [17, p. 113], for their main properties see [18], for further information see [16].)

– $(\Phi, \otimes)$ is a Moufang loop if and only if $(\Phi, \perp)$ is a Moufang

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loop; moreover, in this case, Id_{Φ} is an isomorphism between them. (Moufang loops are the most studied loops, for their relevance see [17, p. 88].)

– $(\Phi, \otimes)$ is a Bruck loop if and only if $(\Phi, \perp)$ is a Bruck loop (In [14] it was shown that Bruck loops and K-loops are the same; for their history see [12], for the connection among the theory of K-loops, the hyperbolic geometry and the theory of relativity see [5], [8], [9], [10].)

– $(\Phi, \perp)$ is an A-loop if and only if $(\Phi, \otimes)$ is a dual A-loop. (Dual A-loops are defined in this paper, while A-loops are well known, for their study see [2].)

These results can be more interesting if we remember (see [3], [4], [6] and theorem 3) that every loop $(\Gamma, \cdot)$ is isomorphic to a loop $(\Phi, \otimes)$, where Φ is a suitable r.p.s. on Γ.

2. – Permutation Loops.

Let Φ be a r.p.s. on a set Ω. If $1 \in \Omega$ is fixed, we define two composition laws in Φ:

(1) $$P \otimes Q = T \Leftrightarrow 1PQ = 1T$$

(2) $$P \perp Q = T \Leftrightarrow 1(PQ)^{-1} = 1T^{-1}$$

where PQ is the composition of P and Q in $\mathrm{Sym}(\Omega)$ and T^{-1} is the inverse of T in $\mathrm{Sym}(\Omega)$. The element 1 will be called *special element.*

THEOREM 1. *If Φ is a r.p.s. on Ω then*

– $(\Phi, \otimes)$ *and* $(\Phi, \perp)$ *are loops and* Id_{Ω} *is the neutral element in each of them*

– $P \in \Phi$ *has the inverse in* $(\Phi, \otimes)$ *if and only if it has the inverse in* $(\Phi, \perp)$

– Q *is the inverse of* P *in* $(\Phi, \otimes)$ *and in* $(\Phi, \perp)$ *if and only if* $1PQ = 1QP = 1$

– $(\Phi, \otimes)$ *and* $(\Phi, \perp)$ *are groups if and only if* $PQ \in \Phi$ *for all* $P, Q \in \Phi$.

PROOF. Trivial. ■

Let $\chi, \psi : \Omega \to \Phi$ be the bijections defined respectively by

(3) $$\forall a \in \Omega, \qquad a\chi = S \Leftrightarrow aS = 1; \qquad a\psi = R \Leftrightarrow 1R = a\,.$$

If we write $S_a = a\chi$, then $\Phi = \{S_a\}_{a \in \Omega}$ is a labelled set and we have

$$S_a \perp S_b = S_c \Leftrightarrow cS_a = b\ . \tag{4}$$

Likewise if $R_a = a\psi$, then $\Phi = \{R_a\}_{a \in \Omega}$ is a labelled set and we have

$$R_a \otimes R_b = R_c \Leftrightarrow aR_b = c\ . \tag{5}$$

Obviously $R_1 = S_1 = \mathrm{Id}_\Omega$

DEFINITION 1. *A loop* $(\Phi, \otimes)$ *is an upright permutation loop if*

- $\Phi = \{R_a\}_{a \in \Omega}$ *is a labelled regular permutation set on* Ω
- $aR_a^{-1} = bR_b^{-1}$ *for all* $a, b \in \Omega$
- $R_a \otimes R_b = R_c \Leftrightarrow aR_b = c$.

DEFINITION 2. *A loop* $(\Phi, \perp)$ *is a capsized permutation loop if*

- $\Phi = \{S_a\}_{a \in \Omega}$ *is a labelled regular permutation set on* Ω
- $aS_a = bS_b$ *for all* $a, b \in \Omega$
- $S_a \perp S_b = S_c \Leftrightarrow cS_a = b$.

THEOREM 2. – *By the choice of the special element in* Ω, *each r.p.s.* Φ *on* Ω *can be turned in an upright permutation loop* $(\Phi, \otimes)$ *or in a capsized permutation loop* $(\Phi, \perp)$.

PROOF. Trivial. ■

In general, a r.p.s. Φ can be turned in capsized (or in upright) permutation loops which are not isomorphic (a result on this subject was found by Burn: see the corollary of theorem 6). Moreover, in general, $(\Phi, \otimes)$ and $(\Phi, \perp)$ are neither isomorphic nor anti-isomorphic.

EXAMPLE 1. If Ω is finite, we write each $P \in \Phi$ as the product of disjoint cycles (see [21]). Let us consider $\Omega = \{1, 2, \dots, 5\}$ and $\Phi = \{\mathrm{Id}_\Omega,$ (123)(45), (24)(135), (14)(253), (152)(34)$\}$. If 1 is the special element, in $(\Phi, \otimes)$ only one element has order 2, if 4 is the special element in $(\Phi, \otimes)$ each element does. Let us choose $1 \in \Omega$ as special element and $\Phi_1 = \{\mathrm{Id}_\Omega, (1, 2, 3, 4, 5)$, $((1, 3)(2, 5, 4)$, $(1, 4, 3, 5, 2)$, $(1, 5, 3, 2, 4)$: $(\Phi_1, \otimes)$ and $((\Phi_1, \perp)$ are neither isomorphic nor anti-isomorphic because in $(\Phi_1, \otimes)$ the only element of order 2 is a square, while in $(\Phi_1, \perp)$ the only element of order 2 doesn't.

Let $(\Omega, \cdot)$ be a loop and let 1 be its identity element; as it is known, each $a \in \Omega$ determines a bijection $\Omega \mapsto \Omega$ called *right translation* and defined by $x \mapsto x \cdot a$. Obviously the set of all right translations is a r.p.s. on Ω

and, since $1 \cdot a = a$, in accordance with our notation, the right translation determined by a can be denoted by R_a.

THEOREM 3. *If $\Phi = \{R_a\}_{a \in \Omega}$ is the set of all right translations of a loop $(\Omega, \cdot)$ and 1 is the neutral element of Ω, let us choose 1 as special element; then*

a) $(\Phi, \otimes)$ is an upright permutation loop isomorphic to $(\Omega, \cdot)$

b) if $\forall b \in \Omega$ we define $S_b = R_a \Leftrightarrow bR_a = 1$, then $\Phi = \{S_b\}_{b \in \Omega}$ and $(\Phi, \perp)$ is a capsized permutation loop.

c) if in $(\Omega, \cdot)$ each element b has the inverse b^i and

$$(a \cdot b) \cdot b^i = a \quad \text{for all} \quad a, b \in \Omega \tag{6}$$

then the capsized permutation loop $(\Phi, \perp)$ is anti-isomorphic to $(\Omega, \cdot)$.

PROOF. a) and b) are trivial (for a) see also [3] [4] and [6].)

c) Obviously $S_a \colon x \to x \cdot a^i$; we have: $S_a \perp S_b = S_c \Leftrightarrow cS_a = b \Leftrightarrow c \cdot a^i = b \Leftrightarrow (c \cdot a^i) \cdot a = c = b \cdot a$. This means that the map $\zeta : \Omega \to \Phi$ defined by $a\zeta = S_a$ is an anti-isomorphism. ∎

Let Φ be the set of all right translations of the loop $(\Omega, \cdot)$. Then the loops $(\Phi, \otimes)$ and $(\Phi, \perp)$ considered in theorem 3 are the *permutation loops associated to* $(\Omega, \cdot)$.

If $(\Phi, \otimes)$ is an upright permutation loop with $\Phi = \{R_i\}_{i \in \Omega}$ and $\forall a, b \in \Omega$ we define $a \cdot b = c \Leftrightarrow R_a \otimes R_b = R_c$, then $(\Omega, \cdot)$ is a loop isomorphic to $(\Phi, \otimes)$. So we can suppose that each loop is an upright permutation loop.

REMARK. Let $P \in \Phi$ have the inverse P^i in the loop $(\Phi, \otimes)$. In general $P^i \neq P^{-1}$ (=inverse of P in $\mathrm{Sym}(\Omega)$). For example let $\Omega = \{1, 2, \ldots, 6\}$ and $\Phi = \{\mathrm{Id}_\Omega, (1, 2, 4, 5, 3, 6), (1, 3)(2, 5, 4, 6), (1, 4, 2, 6, 3, 5), (1, 5, 6, 4)(2, 3), (1, 6, 5, 2)(3, 4)\}$. It results $R_2^i = R_6 \neq R_2^{-1}$

THEOREM 4. *Let Φ be a r.p.s. on Ω such that $P^{-1} \in \Phi$ for all $P \in \Phi$. Then*

– a) each P has the inverse in $(\Phi, \otimes)$ and $(\Phi, \perp)$ and $P^{-1} = P^i$ in each of them

– b) $(\Phi, \otimes)$ is anti-isomorphic to $(\Phi, \perp)$.

PROOF. a) Trivial. b) It is easy to show (see [4]) that $(\Phi, \otimes)$ satisfies property (6) if and only if $P^{-1} \in \Phi$ for all $P \in \Phi$. So the thesis is true by theorem 3. ∎

THEOREM 5. *Let Φ be a r.p.s. on Ω such that $P^2 \in \Phi$ for all $P \in \Phi$; then $P \otimes P = P \perp P = P^2$.*

PROOF. Let $P = S_a$ in the loop $(\Phi, \perp)$; then $S_a \perp S_a = S_c$ with $c = aS_a^{-1}$. On the other hand, there exists d such that $S_a^2 = S_d$ where, by definition, $dS_a^2 = 1$; but $cS_a^2 = aS_a = 1$. Hence $c = d$, since Φ is a r.p.s. on Ω. In a similar way, we can prove $P \otimes P = P^2$. ■

THEOREM 6. *Let Φ and Ψ be two r.p.s on the same set Ω. Then*

– a) *the capsized permutation loops $(\Phi, \perp)$ and $(\Psi, \perp)$, where $\Phi = \{S_a\}_{a \in \Omega}$ and $\Psi = \{K_a\}_{a \in \Omega}$, are isomorphic if and only if there exists a permutation $\gamma : \Omega \to \Omega$ such that $K_{a\gamma} = \gamma^{-1} S_a \gamma$ for all $a \in \Omega$*

– b) *the upright permutation loops $(\Phi, \otimes)$ and $(\Psi, \otimes))$, where $\Phi = \{R_a\}_{a \in \Omega}$ and $\Psi = \{H_a\}_{a \in \Omega}$, are isomorphic if and only if there exists a permutation $\gamma : \Omega \to \Omega$ such $H_{a\gamma} = \gamma^{-1} R_a \gamma$ for all $a \in \Omega$*

– c) *the capsized permutation loops $(\Phi, \perp)$ and the upright permutation loop $(\Psi, \otimes)$, where $\Phi = \{S_a\}_{a \in \Omega}$ and $\Psi = \{H_a\}_{a \in \Omega}$, are anti-isomorphic if and only if there exists a permutation $\gamma : \Omega \to \Omega$ such that $H_{a\gamma} = \gamma^{-1} S_a^{-1} \gamma$ for all $a \in \Omega$.*

PROOF. Let $\xi : \Phi \mapsto \Psi$ be a isomorphism between the capsized permutation loops $(\Phi, \perp)$ and $(\Psi, \perp)$. We define $\gamma : \Omega \to \Omega$ by $a\gamma = b \Leftrightarrow K_b = S_a \xi$ and we have

$$(S_a \perp S_b)\, \xi = K_{a\gamma} \perp K_{b\gamma} \Leftrightarrow bS_a^{-1} \gamma = b\gamma K_{a\gamma}^{-1}$$

i.e. proposition a); b) and c) can be proved in a similar way. ■

COROLLARY 1 (already known, see [3]). If the r.p.s. Φ is closed under conjugation by its own elements, then the loops derived by distinct choices of the special element are isomorphic.

REMARK. Theorem 6 is analogous to theorem [6, 6] and to theorem [15, 3.1].

3. – Bol loops.

It is known that a loop $(\Gamma, \cdot)$ is called a *Bol loop* if one of the following identities is satisfied

(7) $$[(a \cdot b) \cdot c] \cdot b = a \cdot [(b \cdot c) \cdot b] \quad \text{for all } a, b, c \in \Gamma$$

(8) $$b \cdot [c \cdot (b \cdot a)] = [b \cdot (c \cdot b)] \cdot a \quad \text{for all } a, b, c \in \Gamma$$

REMARK. Identities (8) is the dual of identity (7). In defining Bol loops, some authors (e.g. Robinson or Burn) prefer (7), others (e.g. Karzel or Wefelscheid) prefer (8). In some paper (e.g. see [17, p. 112]), a loop satisfying identity (7) (respectively identity (8)) is called a *right Bol loop* (respectively a *left Bol loop*).

It is known that in a Bol loop $(\Gamma, \cdot)$ each element x has the inverse x^i (see [18, 2.1]).

Let Φ be the set of all right translations of a loop, then it is known (see [3]) that

– property (7) is equivalent to

$$\text{if} \quad P, Q \in \Phi \quad \text{then} \quad PQP \in \Phi \tag{9}$$

– if property (7) is satisfied and $P \in \Phi$, then $P^n \in \Phi$ for all n in Z.

Next theorems expand this result and show the connection between properties (7) and (8).

THEOREM 7. *Let Φ be a r.p.s. on Ω. Then the following statements are equivalent*

– a) *property* (9) *is satisfied in Φ*

– b) *property* (7) *is satisfied in the upright permutation loop* $(\Phi, \otimes)$

– c) *property* (8) *is satisfied in the capsized permutation loop* $(\Phi, \perp)$.

PROOF. In $(\Phi, \otimes)$ we have $[(R_a \otimes R_b) \otimes R_c] \otimes R_b = R_{aR_bR_cR_b}$. If property (9) is satisfied in Φ, then $R_b R_c R_b \in \Phi$ and its label is $bR_c R_b$. On the other hand, also the label of $(R_b \otimes R_c) \otimes R_b$ is $bR_c R_b$. This means that $\forall a \in \Omega$ it results $aR_b R_c R_b = a[R_b \otimes R_c) \otimes R_b]$ i.e. property (7). The converse is trivial.

In a similar way, we can prove the equivalence between a) and c). ■

DEFINITION 3. *Let Φ be a r.p.s. on Ω such that $P^{-1} \in \Phi$ for all $P \in \Phi$. For n in Z, we define by induction $P^{(0, \otimes)} = Id_\Omega$ and $P^{(n, \otimes)} = P^{(n-1, \otimes)} \otimes P$ if $n \geqslant 1$; $P^{(n, \otimes)} = (P^{-1})^{(-n, \otimes)}$ if $n \leqslant -1$.*

DEFINITION 4. *Let Φ be a r.p.s. on Ω such that $P^{-1} \in \Phi$ for all $P \in \Phi$. For n in Z, we define by induction $P^{(0, \perp)} = Id_\Omega$ and $P^{(n, \perp)} = P \perp P^{(n-1, \perp)}$* if $n \geqslant 1$; $P^{(n, \perp)} = (P^{-1})^{(-n, \perp)}$ if $n \leqslant -1$.

THEOREM 8. *Let property* (9) *be fulfilled in the r.p.s Φ on the set Ω and $P, Q \in \Phi$. Then*

– a) $(P \otimes Q) \otimes P) = P \perp (Q \perp P) = PQP$

– b) $P \otimes Q = QMQ$ *where* $M \in \Phi$ *is determined by* $1P = 1QM$
– c) $P \perp Q = PNP$ *where* $N \in \Phi$ *is determined by* $1P^{-1} = 1Q^{-1}N$
– d) $P^{(n, \otimes)} = P^{(n, \perp)} = P^n$ *for* $n \in Z$
– e) $(\Phi, \otimes)$ *and* $(\Phi, \perp)$ *are anti-isomorphic.*

PROOF. a) From the proof of theorem 7 we have $(R_b \otimes R_c) \otimes R_b = R_b R_c R_b$ i.e. $(P \otimes Q) \otimes P) = PQP$. Since property (8) holds in $(\Phi, \perp)$, $\forall a, b, c \in \Omega$ we get $aS_b^{-1} S_c^{-1} S_b^{-1} = a[S_b \perp (S_c \perp S_b)]^{-1}$ i.e. $P \perp (Q \perp P) = PQP$.

b) In $(\Phi, \otimes)$ let M be such that $(Q \otimes M) = P$, i.e. $1QM = 1P$; therefore $P \otimes Q = (Q \otimes M) \otimes Q = QMQ$. Property c) can be proved in a similar way; d) and e) are trivial. ■

REMARKS. 1) By theorem 7 each Bol loop can arise from a r.p.s. which satisfies property (9).

2) Theorem 8 determines the composition in a Bol loop.

3) Obviously $(\Phi, \otimes)$ (respectively $(\Phi, \perp)$ is a group if and only if $\forall P, Q \in \Phi$ the permutation M (resp. N) considered in theorem 8 is $Q^{-1}P$ (resp. QP^{-1}).

As usually, a loop $(\Gamma, \cdot)$ is called a *Moufang loop* if property

$$(a \cdot b) \cdot (c \cdot a) = [a \cdot (b \cdot c)] \cdot a \quad \text{for all } a, b, c \in \Gamma \tag{10}$$

is fulfilled. As it is known (see [18]), if $(\Gamma, \cdot)$ is a loop, then the following statements are equivalent

– $(\Gamma, \cdot)$ is Moufang
– $(\Gamma, \cdot)$ satisfies (7) and (8).
– $(\Gamma, \cdot)$ is Bol and $(x \cdot y)^i = y^i \cdot x^i$ for all $x, y \in \Gamma$.

THEOREM 9. *Let property* (9) *hold in the r.p.s.* Φ *on* Ω. *Then the following statements are equivalent*

– a) Id_Φ *is an isomorphism between* $(\Phi, \otimes)$ *and* $(\Phi, \perp)$
– b) $(\Phi, \perp)$ *is Moufang*
– c) $(\Phi, \otimes)$ *is Moufang.*

PROOF. Since the mapping $P \mapsto P^{-1}$ is an anti isomorphism between $(\Phi, \otimes)$ and $(\Phi, \perp)$, we have $(P \perp Q)^{-1} = Q^{-1} \perp P^{-1} \Leftrightarrow P \perp Q = P \otimes Q$. ■

THEOREM 10. *Let property* (9) *be fulfilled in the r.p.s* Φ *on the set* Ω *and let there exists* $1 \in \Omega$ *such that* $1PQ = 1QP$, *for all* $P, Q \in \Phi$. *If we choose* 1 *as special element, then both* $(\Phi, \otimes)$ *and* $(\Phi, \perp)$ *are commutative Moufang loops.*

PROOF. The considered property is fulfilled if and only if $(\Phi, \otimes)$ is commutative. So the thesis is trivial by theorem 8. ■

By definition, the *automorphic inverse property* holds in the loop $(\Gamma, \cdot)$ if

$$(x \cdot y)^i = x^i \cdot y^i \quad \text{for all } x, y \in \Gamma. \tag{11}$$

Of course property (11) is consistent only if each element of Γ as the inverse.

THEOREM 11. *Let Φ be a r.p.s. on Ω such that $P^{-1} \in \Phi$ for all $P \in \Phi$. Then the following statements are equivalent*

– a) *Id_Φ is an anti-isomorphism between $(\Phi, \otimes)$ and $(\Phi, \perp)$*

– b) *property* (11) *holds in* $(\Phi, \perp)$

– c) *property* (11) *holds in* $(\Phi, \otimes)$

– d) *$P = \psi\chi^{-1} P^{-1} \chi\psi^{-1}$ where $P \in \Phi$ and ψ, χ are defined by* (3) *in section* 2.

PROOF. Let a) hold; then $\forall P; Q \in \Phi$ it follows $T = P \otimes Q = Q \perp P$ where, by definition, $1T = 1PQ$ and $1T^{-1} = 1(QP)^{-1}$. In $(\Phi, \perp)$, the label of T^{-1} is $1T$, while $1PQ$ is the label of $Q^{-1} \perp P^{-1}$: this means $(Q \perp P)^{-1} = Q^{-1} \perp P^{-1}$ and b) holds; c) holds by theorem 4, d) holds by theorem 6. Conversely, let b) hold. Therefore if $T = P \perp Q$ the label of T^{-1} is $1T = 1QP$. On the other hand, in $(\Phi, \otimes)$, the label of $(Q \otimes P)$ is $1QP$: this means $T = Q \otimes P = P \perp Q$ i.e. the identity map is an anti-isomorphism. ■

EXAMPLE 2. Let $\Omega = \{1, 2, \dots, 6\}$ and $\Phi = \{\mathrm{Id}_\Omega, (1, 2, 3, 4, 5, 6), (1, 3)(2, 5)(4, 6), (1, 4, 2, 6, 3, 5), (1, 5, 3, 6, 2, 4), (1, 6, 5, 4, 3, 2)\}$. By theorem 4, $(\Phi, \otimes)$ is anti-isomorphic to $(\Phi, \perp)$; but the identity map is not an anti-isomorphism between them.

As it is well known, a Bol loop $(\Gamma, \cdot)$ is called a *Bruck loop* if (11) is valid.

THEOREM 12. *Let property* (9) *hold in the r.p.s Φ on the set Ω. Then the following statements are equivalent*

– a) *$(\Phi, \otimes)$ is a Bruck loop*

– b) *$(\Phi, \perp)$ is a Bruck loop*

– c) *there exists $1 \in \Omega$ satisfying the following property*
if $P, Q \in \Phi$ then $\exists M \in \Omega$ such that $1P = 1QM$ and $1Q^{-1} = 1P^{-1}M$.

PROOF. By theorems 7 and 8, both $(\Phi, \otimes)$ and $(\Phi, \perp)$ are Bol loops and $\forall P, Q \in \Phi$ we have $P \otimes Q = QMQ$ with $1P = 1QM$ and $Q \perp P = QNQ$ with $1Q^{-1} = 1P^{-1}N$. By theorem 11, a) and b) are equivalent, and from each of them it follows that the identity map is an anti-isomorphism between $(\Phi, \otimes)$ and $(\Phi, \perp)$. If a) or b) holds, we have $QMQ = P \otimes Q = Q \perp P = QNQ$. This means N=M i.e. condition c). The converse is trivial. ■

4. – A-loops.

Let $(\Gamma, \cdot)$ be a loop; $\forall a, b \in \Gamma$ the condition $a \cdot (b \cdot x) = (a \cdot b) \cdot x\delta_{a,b}$ clearly defines a bijective map $\delta_{a,b}\colon \Gamma \mapsto \Gamma$; following Ungar (see [19], [20]), in many papers these maps are called *precession maps*.

Analogously we give the

DEFINITION 5. *Let $(\Gamma, \cdot)$ be a loop; $\forall a, b \in \Gamma$ the condition $(x \cdot a) \cdot b = x\beta_{a,b} \cdot (a \cdot b)$ defines a bijective map $\beta_{a,b}\colon \Gamma \mapsto \Gamma$ called an anti-precession map.*

It is trivial to prove that if in the loop $(\Gamma, \cdot)$ property

$$a \cdot b = 1 \Rightarrow \delta_{a,b} = \mathrm{Id}_\Gamma \tag{12}$$

or property

$$a \cdot b = 1 \Rightarrow \beta_{a,b} = \mathrm{Id}_\Gamma \tag{13}$$

is fulfilled, then there exists the inverse a^i of each element a.

THEOREM 13. *Let Φ be a r.p.s. Then the following statements are equivalent:*

- a) *$P^{-1} \in \Phi$ for all $P \in \Phi$.*
- b) *in the loop $(\Phi, \perp)$ property* (12) *holds*
- c) *in the loop $(\Phi, \otimes)$ property* (13) *holds*

PROOF. a) is equivalent to b): if a) holds, in $(\Phi, \perp)$ we have $P^i = P^{-1}$. Let $X = S_x$: the label of $P \perp (P^{-1} \perp X)$ is $xPP^{-1} = x$, so b) holds. Conversely if b) holds, let $P = S_a$ and $S_b = S_a^i$, then $\forall x \in \Omega$ it follows $xS_b^{-1}S_a^{-1} = x$, i.e. $S_b = S_a^{-1} \in \Phi$. In a similar way we can prove that a) and c) are equivalent. ■

LEMMA 14. *Let Φ be a r.p.s. on the set Ω such that $P^{-1} \in \Phi$ for all $P \in \Phi$. If $\delta_{P,Q}$ (respectively $\beta_{P,Q}$) is a precession map in $(\Phi, \perp)$ (respectively an anti-precession map in $(\Phi, \otimes)$) and $X, P, Q \in \Phi$, then $(X\delta_{P,Q})^{-1} = X^{-1}\beta_{Q^{-1},P^{-1}}$ and $(X\beta_{P,Q})^{-1} = X^{-1}\delta_{Q^{-1},P^{-1}}$.*

PROOF. Recall that the map $\Phi \mapsto \Phi$ defined by $P \mapsto P^{-1}$ is an anti-isomorphism between $(\Phi, \perp)$ and $(\Phi, \otimes)$. ■

THEOREM 15. *Let Φ be a r.p.s. on the set Ω such that $P^{-1} \in \Phi$ for all $P \in \Phi$. If property* (11) *holds in* $(\Phi, \otimes)$ *or in* $(\Phi, \perp)$ *and* $P, Q, X \in \Phi$, *then*

- $(X\delta_{P,Q})^{-1} = X^{-1}\delta_{P^{-1},Q^{-1}}$
- $(X\beta_{P,Q})^{-1} = X^{-1}\beta_{P^{-1},Q^{-1}}$.

PROOF. By theorem 11 property (11) holds in $(\Phi, \otimes)$ if and only if it holds in $(\Phi, \perp)$ and the identity map is an anti-isomorphism between them. From $P \perp (Q \perp X) = (P \perp Q) \perp X\delta_{P,Q}$ we get $(X \otimes Q) \otimes P = X\delta_{P,Q} \otimes (Q \otimes P)$; therefore the thesis follows from lemma 14. ■

As usually a loop $(\Gamma, \cdot)$ is called an *A-loop* if each precession map is an automorphism.

THEOREM 16. *If property* (12) *holds in the A-loop* $(\Gamma, \cdot)$, *then there exists a capsized permutation loop which is isomorphic to* $(\Gamma, \cdot)$.

PROOF. $\forall a, b \in \Gamma$ let us define $a + b = b \cdot a$. Then in the loop $(\Gamma, +)$ each element has the inverse and property (6) holds, so, by theorem 3, the capsized permutation loop $(\Phi, \perp)$ associated to $(\Gamma, +)$ is isomorphic to $(\Gamma, \cdot)$. ■

Since we are interested only in A-loops in which (12) holds, by theorem 16, we can suppose that each A-loop is a capsized permutation loop $(\Phi, \perp)$.

DEFINITION 6. *A loop* $(\Gamma, \cdot)$ *is called a dual A-loop if each anti-precession map is an automorphism.*

THEOREM 17. *Let Φ be a r.p.s. on the set Ω such that $P^{-1} \in \Phi$ for all $P \in \Phi$. Then the following statements are equivalent*

- $(\Phi, \perp)$ *is an A-loop*
- $(\Phi, \otimes)$ *is a dual A-loop.*

PROOF. By theorems 4, the map $\Phi \to \Phi$ defined by $P \to P^{-1}$ is an anti-isomorphism between $(\Phi, \perp)$ and $(\Phi, \otimes)$, therefore each map $\delta : \Phi \to \Phi$ is an automorphism of $(\Phi, \perp)$ if and only if it is an automorphism of $(\Phi, \otimes)$, so our thesis follows from lemma 14. ■

By theorem 6, a simple computation proves the following

COROLLARY 2. The capsized permutation loop $(\Phi, \perp)$ is an A-loop if and only if $\Phi = \gamma_{P,Q}^{-1} \Phi \gamma_{P,Q}$ where $P, Q \in \Phi$ and $\gamma_{P,Q} = Q^{-1} P^{-1} (P \perp Q)$.

COROLLARY 3. The upright permutation loop $(\Phi, \otimes)$ is a dual A-loop if and only if $\Phi = \gamma_{P,Q}^{-1} \Phi \gamma_{P,Q}$ where $P, Q \in \Phi$ and $\gamma_{P,Q} = PQ(P \otimes Q)^{-1}$.

Moreover by theorem 15 we obtain the following

COROLLARY 4. (already known (see [15, 2.9b]): If properties (11) and (12) hold in the A-loop $(\Phi, \perp)$ and $P, Q \in \Phi$, then $\delta_{P,Q} = \delta_{P^{-1}, Q^{-1}}$

COROLLARY 5. If properties (11) and (13) hold in the dual A-loop $(\Phi, \otimes)$ and $P, Q \in \Phi$, then $\beta_{P,Q} = \beta_{P^{-1}, Q^{-1}}$

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