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Normal Generation of Line Bundles on a General k-gonal Algebraic Curve.

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Sunto. – *Sia X una curva k-gonale generale di genere g ed $L \in \mathrm{Pic}^d(X)$ con $d := \deg L > (3g-1)/2$, $h^1(X, L) \neq 0$ ed L molto ampio. In questo lavoro dimostriamo che L è normalmente generato se il luogo base di KL^{-1} ha grado al massimo $c(k-2)/2$ con $c := d - (3g-1)/2$.*

Summary. – *We prove that a very ample special line bundle L of degree $d > (3g-1)/2$ on a general k-gonal curve is normally generated if the degree of the base locus of its dual bundle KL^{-1} does not exceed $c(k-2)/2$, where $c := d - (3g-1)/2$.*

0. – Introduction and notation.

Let C be a smooth irreducible projective algebraic curve of genus $g \geqslant 2$ over an algebraically closed field of characteristic 0. Following Mumford [Mu], a line bundle L on C is called normally generated if L is very ample and C is projectively normal under the associated projective embedding.

Let's briefly recall several known results on normal generation of line bundles on smooth algebraic curves. A classical theorem of M. Noether states that the canonical line bundle on a non-hyperelliptic curve is normally generated, and a well known theorem of Mumford [Mu] asserts that every line bundle of degree more than $2g$ is normally generated. In the past, there have been several achievements in finding criteria on the normal generation of a line bundle in terms of the degree of the line bundle together with other invariants of the given curve C. Most notably, a remarkable result of Green and Lazarsfeld [GL] provides a good criteria for a line bundle being normally generated in terms of the Clifford index of the curve C.

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On the other hand, Lange and Martens [LM] has shown that a hyperelliptic curve has no normally generated line bundle of degree less than or equal to $2g$. Hence Mumford's result is the best possible statement when C is hyperelliptic. Accordingly one may want to ask when a line bundle of degree d with $d < 2g$ on a special curve (in the sense of moduli) is normally generated. In this short note, we are interested in normal generation of special line bundles on a general k-gonal curve. We prove that a very ample line bundle L of degree $d > \frac{3g-1}{2}$ on a general k-gonal curve is normally generated if the degree of the base locus of its dual bundle KL^{-1} does not exceed $\frac{c(k-2)}{2}$, where $c := d - \frac{3g-1}{2}$.

We use standard notation for divisors, linear series and line bundles on algebraic curves following [ACGH]. As usual, g^r_d is an r-dimensional linear series of degree d on C, which may be possibly incomplete. If D is a divisor on C, we write $|D|$ for the associated complete linear series on C. For a line bundle L on C, we also write $|L|$ for the corresponding complete linear series. For two complete linear series $|A|$ and $|B|$, we sometimes write $|A| \subset |B|$ if $B - A$ is linearly equivalent to an effective divisor; in other words, for every divisor $A' \in |A|$, there exists a divisor $B' \in |B|$ such that $B' - A'$ is effective. For two effective divisors A and B, $\gcd(A, B)$ denotes the maximal effective divisor contained in both A and B. By K we denote a canonical divisor on C, and $|K|$ is the canonical linear series on C. For a line bundle L, the Clifford index of L is defined by $\mathrm{Cliff}(L) := \deg L - 2h^0(C, L) + 2$. A base-point-free g^r_d on C defines a morphism $f : C \to \mathbb{P}^r$ onto a non-degenerate irreducible (possibly singular) curve in $\mathbb{P}^r$; in particular the minimal degree d for $r = 1$ is called the gonality of C.

1. – The result.

THEOREM 1. – *Let C be a general k-gonal curve of genus $g \geqslant 4$, $k \geqslant 4$ with $2k - 2 - g < 0$. Let L be a very ample special line bundle of degree $d > \frac{3g-1}{2}$ and set $c := d - \frac{3g-1}{2}$. Let B be the base locus of KL^{-1}. If*

$$\deg B \leqslant \frac{c(k-2)}{2},$$

then L is normally generated.

We first recall the following result which gives a criteria for a line bundle on a general k-gonal curve being compounded of the k-gonal pencil.

LEMMA 2 ([K, Theorem 2.1]). – *Let C be a general k-gonal curve of genus $g \geqslant 4$, $k \geqslant 4$ with the unique pencil $|F|$ of degree k on C. Let L be a line bundle on C with* $\mathrm{Cliff}(L) \leqslant \left[\frac{g-4}{2}\right]$ *and* $\deg L \leqslant g-1$. *Then $|L|$ is compounded of F, i.e. $L \cong \mathcal{O}_C(lF+B)$ where $l = h^0(C, L) - 1$.*

We also need the following lemma which is due to [GL]. It provides a numerical criteria for a line bundle L failing to be normally generated. We omit a proof of the lemma which is explicitly written inside the proof of [GL, Theorem 2.1].

LEMMA 3. – *Let C be a smooth algebraic curve of genus g and L a very ample line bundle on C with* $\deg L > \frac{3g-3}{2} + \varepsilon$; *$\varepsilon = 0$ if L is special, $\varepsilon = 2$ if L is non-special. Suppose that L is not normally generated. Then there exists a non-trivial effective divisor R and a line bundle A such that* (i) $A \cong L(-R)$, (ii) $\deg A \geqslant \frac{g-1}{2}$, (iii) $\mathrm{Cliff}(A) \leqslant \mathrm{Cliff}(L)$, (iv) $h^0(C, A) \geqslant 2$ *and* $h^1(C, A) \geqslant h^1(C, L) + 2$.

PROOF OF THEOREM 1. – Let $|F| = g^1_k$ be the unique pencil of degree k on C. We first note that $KL^{-1} \cong \mathcal{O}_C(lF+B)$ holds, where $l+1 = h^0(C, KL^{-1})$; since $d = \deg L > \frac{3g-1}{2}$, $\deg KL^{-1} < 2g-2-\frac{3g-1}{2} = \frac{g-3}{2} < g-1$, hence $\mathrm{Cliff}(L) = \mathrm{Cliff}(KL^{-1}) < \frac{g-3}{2} - 2l \leqslant \left[\frac{g-4}{2}\right]$ and therefore it follows from Lemma 2 that $KL^{-1} \cong \mathcal{O}_C(lF+B)$. We now suppose that L is not normally generated. By Lemma 3, there exists a line bundle A and a non-trivial effective divisor R satisfying the following conditions;

(i) $A \cong L(-R)$

(ii) $\deg A \geqslant \frac{g-1}{2}$

(iii) $\mathrm{Cliff}(A) \leqslant \mathrm{Cliff}(L)$

(iv) $h^0(C, A) \geqslant 2$ and $h^1(C, A) \geqslant h^1(C, L) + 2$.

Note that $\deg KA^{-1} \leqslant 2g-2-\frac{g-1}{2} = \frac{3g-3}{2}$ and we first consider the case when the degree of KA^{-1} is relatively low.

I. *Assume* $\deg KA^{-1} \leqslant g-1$.

By $\mathrm{Cliff}(KA^{-1}) = \mathrm{Cliff}(A) \leqslant \mathrm{Cliff}(L) \leqslant \left[\frac{g-4}{2}\right]$ and Lemma 2, we also have $KA^{-1} \cong \mathcal{O}_C(mF+E)$ where E is the base locus of $|KA^{-1}|$. By the second inequality in (iv), we have $h^0(C, KA^{-1}) = m+1 \geqslant h^0(C, KL^{-1}) + 2$, hence $\alpha := m - l \geqslant 2$. We also have $\deg B - \deg E \geqslant (k-2)(m-l)$ by the condition (iii). Note that

$$|KL^{-1}| = |lF + B| \subsetneq |KA^{-1}| = |mF + E|, \tag{1}$$

since $A = L(-R)$ for some $R \gneq 0$.

Claim: If B contains a divisor S such that $S \leqslant G$ for some $G \in g_k^1$, then $\deg S \leqslant k - 3$.

It is clear that for such a divisor S, we must have $\deg S \leqslant k - 1$. Assume $\deg S = k - 2$ and choose $P, Q \in C$ such that $S + P + Q \in g_k^1$. Then

$$|lF + B| \subset |lF + S + P + Q + B - S| = |lF + F + B - S| = |(l+1)F + B - S|.$$

Note that $|(l+1)F + B - S|$ is also compounded of g_k^1. Therefore

$$\dim |KL^{-1}(P+Q)| = \dim |(l+1)F + B - S| = l + 1,$$

contradicting L being very ample. If $\deg S = k - 1$, a similar argument leads to a contradiction to the condition L being base-point-free, and this finishes the proof of our claim.

By (1), we have

$$B \leqslant G_1 + \ldots + G_\alpha + E \tag{2}$$

for some $G_1, \ldots, G_\alpha \in g_k^1$. Let $D := \gcd(B, E)$ and set $\widetilde{B} := B - D$, $\widetilde{E} = E - D$. By (2), we have

$$\widetilde{B} \leqslant G_1 + \ldots + G_\alpha + \widetilde{E}$$

and hence

$$\widetilde{B} \leqslant G_1 + \ldots + G_\alpha$$

since $\gcd(\widetilde{B}, \widetilde{E}) = 0$. We take

$$\widetilde{B}_1 = \gcd(\widetilde{B}, G_1), \ldots, \widetilde{B}_\alpha = \gcd(\widetilde{B}, G_\alpha)$$

$$G_1 - \widetilde{B}_1 = T_1, \ldots, G_\alpha - \widetilde{B}_\alpha = T_\alpha,$$

where some of the $\widetilde{B}_i$'s may possibly be empty (zero) divisors. We note that

$\widetilde{B}_1 + \ldots + \widetilde{B}_\alpha = \widetilde{B}$. Furthermore, we have

$$\begin{aligned} lF + D + \widetilde{B} &= lF + B \\ &\lneqq lF + B + T_1 = lF + G_1 + B - \widetilde{B}_1 \\ &\lneqq lF + G_1 + B - \widetilde{B}_1 + T_2 = lF + G_1 + G_2 + B - (\widetilde{B}_1 + \widetilde{B}_2) \\ &\lneqq \ldots \\ &\lneqq lF + G_1 + \ldots + G_{\alpha-1} + B - (\widetilde{B}_1 + \ldots + \widetilde{B}_{\alpha-1}) \\ &\lneqq lF + G_1 + \ldots + G_{\alpha-1} + B - (\widetilde{B}_1 + \ldots + \widetilde{B}_{\alpha-1}) + T_\alpha \\ &= lF + G_1 + \ldots + G_\alpha + D \\ &\leqslant lF + G_1 + \ldots + G_\alpha + E \in |mF + E| . \end{aligned}$$

We note that in each step with strict inequality, the degree of the divisor is increased at least three by the claim, whereas the dimension of the corresponding complete linear system is increased by exactly one, therefore

$$\begin{aligned} \operatorname{Cliff}(L) = \operatorname{Cliff}(KL^{-1}) &= \operatorname{Cliff}(lF + B) \\ &\lneqq \operatorname{Cliff}(mF + E) = \operatorname{Cliff}(KA^{-1}) = \operatorname{Cliff}(A), \end{aligned}$$

which is contradictory to the condition (iii).

II. *Assume* $\deg A \leqslant g - 1$.

Also in this case, A is compounded of g^1_k by Lemma 2 and hence $|A| = |nF + H|$, where $n = h^0(C, A) - 1$ and H is the base locus of $|A|$. By the condition (iii) we have

$$\deg B - \deg H \geqslant (k-2)(n-l) . \tag{3}$$

Set $c := \deg L - \frac{3g-1}{2}$. Note that $\deg A \geqslant \frac{g-1}{2}$ and $\deg KL^{-1} = \frac{g-3}{2} - c$. Hence

$$\deg A - \deg KL^{-1} = (n-l)\,k + \deg H - \deg B \geqslant \frac{g-2}{2} - \frac{g-3}{2} + c > c,$$

and it follows from (3) that

$$(k-2)(n-l) \leqslant \deg B - \deg H < k(n-l) - c,$$

whence $c \lneqq 2(n-l)$. Again by (3),

$$\deg B \geqslant \deg B - \deg H \geqslant (k-2)(n-l) \gneqq \frac{c(k-2)}{2}$$

contradictory to our numerical assumption. ∎

COROLLARY 4. – *Let C be a general k-gonal curve of genus $g \geqslant 4$, $k \geqslant 4$ with the unique pencil $|F|$ of degree k.*

(i) *Let B be an effective divisor which does not contain any degree two divisor in a fiber of g^1_k. Then $|K - tF - B|$ is normally generated if $c := \frac{g-3}{2} - tk - \deg B > 0$ and $\deg B \leqslant \frac{c(k-2)}{2}$.*

(ii) *For $t < \frac{g-3}{2k}$, $|K - tF|$ is normally generated.*

PROOF. – (i) By using [CKM, Proposition 1.1], it is easy to check that $|K-tF-B|$ is very ample, hence is normally generated by Theorem 1.

(ii) This is a special case of (i) for $B = 0$. It should be remarked that this has been observed already in [B, Theorem 0.1] with a slightly different bound for t. ∎

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