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Some Remarks on a Class of Elliptic Equations with Degenerate Coercivity.

LUCIO BOCCARDO - HAÏM BREZIS

Sunto. – *Si studiano problemi ellittici degeneri del tipo*

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(1+|u|)^\theta}\right)=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega\,. \end{cases}$$

Summary. – *We study degenerate elliptic problems of the type*

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(1+|u|)^\theta}\right)=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega\,. \end{cases}$$

1. – Introduction.

In the paper [5], existence and regularity results for the following elliptic problem (with degenerate coercivity) are studied:

$$\begin{cases} -\operatorname{div}(a(x,u)\,\nabla u)=f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, with $N>2$, and $a(x,s):\Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory function (that is, measurable with respect to x for every $s\in\mathbb{R}$, and continuous with respect to s for almost every $x\in\Omega$) satisfying the following conditions:

$$\frac{\alpha}{(1+|s|)^\theta}\leqslant a(x,s)\leqslant\beta\,, \tag{2}$$

for some real number θ such that

$$0\leqslant\theta\leqslant 1, \tag{3}$$

for almost every $x\in\Omega$, for every $s\in\mathbb{R}$, where α and β are positive constants. The datum f belongs to $L^m(\Omega)$, for some $m\geqslant 1$.

The main difficulty in dealing with problem (1) is the fact that, because of assumption (2), the differential operator $A(v) = -\operatorname{div}(a(x, v)\nabla v)$, even if it is well defined between $W_0^{1,2}(\Omega)$ and its dual $W^{-1,2}(\Omega)$, is not coercive on $W_0^{1,2}(\Omega)$ (when v is large, $\frac{1}{(1+|v|)^\theta}$ goes to zero: for an explicit example of the fact that A is not coercive, see [10]).

This implies that the classical methods used in order to prove the existence of a solution for problem (1) cannot be applied, even if the datum f is regular.

In this note, two new and short proofs of the existence and summability theorems, proved in [5], will be presented. The shortness relies on the use of a technique of Guido Stampacchia ([11]); by contrast the original ones of [5] are self-contained.

We will recall here the existence and regularity results proved in [5], then we will give new proofs of the first two theorems.

Moreover we will study the impact on the existence assertion of the presence of a lower order term. Such a term allows to establish existence even if $\theta > 1$. We recall that in [2] a nonexistence result (also for bounded data f) is proved if $\theta > 1$.

The first result concerns the existence of bounded solutions, and coincides with the classical boundedness results for uniformly elliptic operators (that is the case $\theta = 0$, see [11]). The main tool of the proof will be an $L^\infty(\Omega)$ *a priori* estimate, which then implies the $W_0^{1,2}(\Omega)$ estimate, since if u is bounded then the operator A is uniformly elliptic.

THEOREM 1.1. – *Let f be a function in $L^m(\Omega)$, with $m > \frac{N}{2}$. Assume* (2) *and* (3). *Then there exists a weak solution of* (1) *u in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.*

The next result deals with a given f which yields unbounded solutions in $W_0^{1,2}(\Omega)$.

THEOREM 1.2. – *Assume* (2) *and* $0 < \theta < 1$. *Let f be a function in $L^m(\Omega)$, with m such that*

$$\frac{2N}{N+2-\theta(N-2)} \leqslant m < \frac{N}{2}. \tag{4}$$

Then there exists a function u in $W_0^{1,2}(\Omega) \cap L(\Omega) m^{**}(1-\theta)$, which is a weak solution of (1), *where $m^{**} = (m^*)^* = \frac{mN}{N-2m}$.*

Notice that, since $\frac{2N}{N+2-\theta(N-2)} \leqslant m$ and $0 < \theta < 1$, then $m \geqslant \frac{2N}{N+2}$ so that f belongs to the dual of $W_0^{1,2}(\Omega)$.

We recall that Example 1.5 of [5] shows that the result of Theorem 1.2 is sharp.

Observe that if $\theta = 0$, the result of the preceding theorems coincides with the classical regularity results for uniformly elliptic equations (see [11] and [7]).

We refer to [10] for a uniqueness result for (1).

If we decrease the summability of f ($f \in L^m(\Omega)$, $m \leq \dfrac{N}{N+1-\theta(N-1)}$), in [5] it has been proved that there exist solutions which do not in general belong anymore to $W_0^{1,2}(\Omega)$, even if the assumptions on f ($f \in L^m(\Omega)$, $m \geq \dfrac{2N}{N+2}$) imply that $f \in W^{-1,2}(\Omega)$. This is stated in the following theorem proved in [5].

THEOREM 1.3. – *Let* $0 < \theta < 1$ *and* f *be a function in* $L^m(\Omega)$, *with* m *such that*

$$\frac{N}{N+1-\theta(N-1)} < m < \frac{2N}{N+2-\theta(N-2)}. \tag{5}$$

Then there exists a function u *in* $W_0^{1,q}(\Omega)$, *with*

$$q = \frac{Nm(1-\theta)}{N-m(1+\theta)} < 2, \tag{6}$$

which solves (1) *in the sense of distributions, that is,*

$$\int_\Omega a(x, u)\, \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi, \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{7}$$

Moreover, the truncation $T_k(u)$ *belongs to* $W_0^{1,2}(\Omega)$ *for every* $k > 0$, *where*

$$T_k(s) = \max\{-k, \min\{k, s\}\}. \tag{8}$$

Up to now, we have obtained solutions belonging to some Sobolev space. If we weaken the summability hypotheses on f, then the gradient of u (and even u itself) may no longer be in $L^1(\Omega)$. However, it is possible to give a meaning to solution for problem (1) (using the concept of entropy solutions which has been introduced in [3]). The existence result can be found in [5].

2. – Existence results.

In this section, we will prove Theorems 1.1, 1.2. The proofs of the existence results will be obtained by approximation. Let f be a function in $L^m(\Omega)$, with m as in the statements of Theorems 1.1, 1.2. Let $\{f_n\}$ be a sequence of functions such that

$$f_n \in L^{\frac{2n}{N+2}}(\Omega), \qquad f_n \to f \text{ strongly in } L^m(\Omega), \tag{9}$$

and such that

$$\|f_n\|_{L^m(\Omega)} \leqslant \|f\|_{L^m(\Omega)}, \qquad \forall n \in \mathbb{N}. \tag{10}$$

Take, for instance, $f_n = T_n(f)$.

Let us define the following sequence of problems:

$$\begin{cases} -\operatorname{div}(a(x, T_n(u_n))\nabla u_n) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

The existence of weak solutions u_n in $W_0^{1,2}(\Omega)$ of the Dirichlet problem (11) is classical, since the differential operator in (11) is uniformly elliptic.

2.1. *Bounded solutions.*

LEMMA 2.1. – *Assume the same hypotheses of Theorem* 1.1. *Let f be in $L^m(\Omega)$ and let u_n be a solution of* (11) *with $f_n = f$ for every $n \in \mathbb{N}$. Then the norms of u_n in $L^\infty(\Omega)$ and in $W_0^{1,2}(\Omega)$ are bounded by a constant which depends on θ, m, N, α,* meas Ω *and on the norm of f in $L^m(\Omega)$.*

PROOF. – Let us start with the estimate in $L^\infty(\Omega)$. Define, for s in $\mathbb{R}$ and for $k > 0$,

$$G_k(s) = (|s| - k)^+ \operatorname{sgn}(s) = s - T_k(s),$$

and

$$H(s) = \int_0^s \frac{1}{(1+|t|)^\theta}\, dt.$$

For $k > 0$, if we take $G_k(H(u_n))$ as test function in (11) and use assumption (2), we obtain

$$\alpha \int_{\{|H(u_n(x))| > k\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^{2\theta}} \leqslant \int_{\{|H(u_n(x))| > k\}} f G_k(H(u_n));$$

that is

$$\alpha \int_{A_k} |\nabla(H(u_n))|^2 \leqslant \int_{A_k} f G_k(H(u_n)), \tag{12}$$

where we have set

$$A_k = \{x \in \Omega : |H(u_n(x))| > k\}.$$

Inequality (12) is exactly the starting point of Stampacchia's L^∞-regularity proof (see [11], [9]), so that there exists a constant c_1 such that

$$\|H(u_n)\|_{L^\infty(\Omega)} \leq c_1. \tag{13}$$

The properties of the function H (in particular the fact that $\lim_{s \to +\infty} H(s) = +\infty$, $\lim_{s \to -\infty} H(s) = -\infty$) yield a bound for u_n in $L^\infty(\Omega)$ from (13):

$$\|u_n\|_{L^\infty(\Omega)} \leq c_2.$$

The estimate in $W_0^{1,2}(\Omega)$ is now very easy. Taking u_n as test function in (11), one obtains

$$\frac{\alpha}{(1+c_2)^\theta} \int_\Omega |\nabla u_n|^2 \leq \int_\Omega f u_n,$$

and the right hand side is bounded since f belongs, at least, to $L^1(\Omega)$.

REMARK 2.2. – We point out that Lemma 2.1 can be proved under the slightly more general assumption

$$h(s) \leq a(x, s) \leq \beta, \tag{14}$$

where the real function $h(s)$ is continuous, positive and such that its primitive

$$H(s) = \int_0^s h(t)\, dt \tag{15}$$

satisfies $\lim_{t \to +\infty} H(t) = +\infty$, $\lim_{t \to -\infty} H(t) = -\infty$.

Thus it is possible, for instance, to study also problems where $h(s) = \dfrac{1}{(e + |s|) \ln(e + |s|)}$.

PROOF OF THEOREM 1.1. – Since $\|u_n\|_{L^\infty(\Omega)} \leq c_2$, take $\nu > c_2$, so that $T_\nu(u_\nu) = u_\nu$. Then u_ν is a weak solution of (1) in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. ■

2.2. *Unbounded solutions.*

The next result will be used in the proof of Theorem 1.2.

LEMMA 2.3. – *Assume the same hypotheses as in Theorem* 1.2. *Let f belong to $L^\infty(\Omega)$, and let $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ be a solution of* (1) (*which exists by Theorem* 1.1). *Then the norms of u in $L^{m^{**}(1-\theta)}(\Omega)$ and in $W_0^{1,2}(\Omega)$ are* *bounded by constants depending only on* θ, m, N, α, meas Ω *and the norm of* f *in* $L^m(\Omega)$.

PROOF. – Multiplying (11) by

$$((1+|u_n|)^p - 1)\,\mathrm{sign}\,(u_n), \qquad p = \frac{(1-\theta)\,N(m-1)}{N-2m},$$

integrating on Ω and using the standard Sobolev imbedding $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$, yields

$$\|u_n\|_{L^{(p+1-\theta)2^*/2}(\Omega)} \leqslant c_3, \tag{16}$$

and

$$\int_\Omega |\nabla u_n|^2 (1+|u_n|)^{p-1-\theta} \leqslant c_4. \tag{17}$$

It is convenient to observe that $pm' = (p+1-\theta)2^*/2$. So far, we have not used (4). Note that

$$(p+1-\theta)\,2^*/2 = m^{**}(1-\theta),$$

and that (4) is equivalent to $p-1-\theta \geqslant 0$.

Thus, if (4) holds, then (16) implies a bound for ∇u_n in $L^2(\Omega)$. Here the constants c_3 and c_4 depend only on θ, m, N, α, meas Ω and the norm of f in $L^m(\Omega)$. ■

PROOF OF THEOREM 1.2. – The estimates for u_n in $W_0^{1,2}(\Omega)$ imply that there exist a subsequence $\{u_{n_j}\}$ and a function $u \in W_0^{1,2}(\Omega)$, such that u_{n_j} converges weakly in $W_0^{1,2}(\Omega)$ to u. The coefficient $a(x, u_{n_j})$ converges to $a(x, u)$ in any $L^q(\Omega)$. Thus it is possible to pass to the limit in (11) in order to obtain the existence of a weak solution u of (1). ■

3. – Lower order terms.

In this section we will study the Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(a(x, u)\,\nabla u) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{18}$$

where $a(x, s)$ still satisfies the following inequality

$$\frac{\alpha}{(1+|s|)^\gamma} \leqslant a(x, s) \leqslant \beta, \tag{19}$$

for some real number $\gamma > 0$ (for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, α, $\beta > 0$) and f belongs to $L^m(\Omega)$, for some $m \geqslant 1$.

Even in semilinear problems the impact of a lower order term can be important (see [4], [8]). For us, the presence of lower order terms in the boundary value problem (1) may change the nature of the existence results. Let $\{f_n\}$ be the sequence of functions

$$f_n = T_n(f). \tag{20}$$

Define the following sequence of problems:

$$\begin{cases} -\operatorname{div}(a(x, T_n(u_n))\nabla u_n) + u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

The classical result about semilinear equations saying that the lower order term has, at least, the same summability as the right hand side can be repeated here. We recall the following lemma (see [4]).

LEMMA 3.1. – *If* u_n *is a solution of* (21), *then we have*

$$\|u_n\|_{L^m(\Omega)} \leqslant \|f_n\|_{L^m(\Omega)}. \qquad \blacksquare$$

3.1. *Solutions with finite energy.*

In this section we will assume that $f \in L^m(\Omega)$, $m \geqslant \gamma + 2$.

The following lemma gives the a priori estimate in $W_0^{1,2}(\Omega)$.

LEMMA 3.2. – *Assume*

$$m \geqslant \gamma + 2. \tag{22}$$

and (19). *Then the sequence* $\{u_n\}$, *defined by* (21), *is bounded in* $W_0^{1,2}(\Omega)$.

PROOF. – The use of

$$[(1 + |u_n|)^{1+\gamma} - 1] \operatorname{sign}(u_n)$$

as test function in (21) implies that

$$\int_\Omega |\nabla u_n|^2 \leqslant c_5 \{1 + \|f_n\|_{L^m(\Omega)}\} \| |u_n|^{(1+\gamma)} \|_{L^{m'}(\Omega)}.$$

Since $(1+\gamma) m' \leqslant m$ if and only if $m \geqslant 2 + \gamma$, the a priori estimate follows from the result of Lemma 3.1. $\blacksquare$

Then, if f belongs to $L^m(\Omega)$, $m \geqslant \gamma + 2$, the existence of solutions follows as in the proof of Theorem 1.2.

So we can state the following theorem.

THEOREM 3.3. – *If f belongs to $L^m(\Omega)$ and if* (19), (22) *hold, then there exists a solution $u \in W_0^{1,2}(\Omega) \cap L^m(\Omega)$ of the boundary value problem* (18).

3.2. *Bounded solutions.*

Now we assume that (19) holds with $\gamma > 1$. Let $f \in L^m(\Omega)$, with $m > \gamma \frac{N}{2}$; we will prove the existence of bounded solutions.

THEOREM 3.4. – *If f belongs to $L^m(\Omega)$, with $m > \gamma \frac{N}{2}$ and $\gamma > 1$, then there exists a solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of the boundary value problem* (18).

PROOF. – The use of

$$\left[\frac{(1+|u_n|)^{\gamma-1}-(1+k)^{\gamma-1}}{\gamma-1}\right]^+ \operatorname{sgn}(u_n), \quad k > 0,$$

as test function in (21) and Young inequality imply that

$$\int_{A_k} a(x, T_n(u_n))\,|\nabla u_n|^2(1+|u_n|)^{\gamma-2} + \int_{A_k} |u_n|\,\frac{(1+|u_n|)^{\gamma-1}-(1+k)^{\gamma-1}}{\gamma-1} \leqslant$$

$$\frac{1}{\gamma-1}\int_{A_k} |f_n|\,[(1+|u_n|)^{\gamma-1}-(1+k)^{\gamma-1}] \leqslant$$

$$\frac{C_\varepsilon}{\gamma-1}\int_{A_k} |f_n|^\gamma + \frac{\varepsilon}{\gamma-1}\int_{A_k} [(1+|u_n|)^{\gamma-1}-(1+k)^{\gamma-1}]^{\frac{\gamma}{\gamma-1}},$$

where

$$A_k = \{x \in \Omega : |u_n(x)| > k\}.$$

We shall use the inequality

$$[(1+t)^{\gamma-1}-(1+k)^{\gamma-1}] \leqslant \begin{cases} c_\gamma t^{\gamma-1}, & \forall t > k \geqslant 2^{\frac{\gamma-2}{\gamma-1}}-1,\ c_\gamma = 2^{\gamma-2}, \quad \text{if } \gamma \geqslant 2; \\ c_\gamma t^{\gamma-1}, & \forall t > k, \quad c_\gamma = 1, \quad \text{if } 1 < \gamma < 2 \end{cases}$$

and we choose ε such that

$$\varepsilon (c_\gamma)^{\frac{1}{\gamma-1}} = \frac{1}{2}.$$

Then one obtains

$$\int_{A_k} \frac{|\nabla u_n|^2}{(1+|u_n|)^2} \leq c_1(\gamma) \int_{A_k} |f_n|^\gamma,$$

which implies, by Hölder's inequality,

$$\int_{A_k} \left| \nabla \log\left(\frac{1+|u_n|}{1+k}\right)\right|^2 \leq c_1(\gamma) \int_{A_k} |f_n|^\gamma \leq c_2(\gamma, \varrho \|f\|_{L^m(\Omega)}) \ (\text{meas}\, A_k)^{1-\frac{\gamma}{m}}.$$

Then, using Sobolev's inequality, we find

$$\left(\int_{A_k} [\log(1+|u_n|) - \log(1+k)]^{2^*}\right)^{\frac{2}{2^*}} \leq c_2(\gamma, \|f\|_{L^m(\Omega)}) \ (\text{meas}\ A_k)^{1-\frac{\gamma}{m}}.$$

Now we set

$$\log(1+|u_n|) = v_n$$

and

$$\log(1+k) = h.$$

Remark that $A_k = \{x \in \Omega : v_n(x) > h\}$. So the last inequality gives

$$\left(\int_{\{x\in\Omega\,:\, v_n(x)>h\}} (v_n - h)^{2^*}\right)^{\frac{2}{2^*}} \leq c_2(\gamma, \|f\|_{L^m(\Omega)}) (\text{meas}\ \{x \in \Omega : v_n(x) > h\})^{1-\frac{\gamma}{m}}.$$

Note that $\left[1 - \dfrac{\gamma}{m}\right]\dfrac{2^*}{2} > 1$, since $m > \gamma \dfrac{N}{2}$. Then Stampacchia's technique (see [11], [9]) implies that, for some positive constant c_4,

$$\|v_n\|_{L^\infty(\Omega)} = \|\log(1+|u_n|)\|_{L^\infty(\Omega)} \leq c_4,$$

that is $\|u_n\|_{L^\infty(\Omega)}$ is bounded.

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