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Composition operators on Banach spaces of formal power series


Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2003_8_6B_2_481_0>
Composition Operators on Banach Spaces of Formal Power Series.

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dedicated to the memory of Karim Seddighi

Sunto. – Supponiamo che \{\beta(n)\}_{n=0}^{\infty} sia una successione di numeri positivi e 1 \leq p < \infty. Consideriamo lo spazio \(H^p(\beta)\) di tutte le serie di potenze \(f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n\), tali che \(\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty\). Supponiamo che \(\frac{1}{p} + \frac{1}{q} = 1\) e \(\frac{\sum_{n=1}^{\infty} \frac{n \cdot n^q}{n^p} \beta(n)^p}{\sum_{n=1}^{\infty} \frac{n \cdot n^q}{n^p} \beta(n)^p} = \infty\) per un intero non-negativo \(j\). Dimostriamo che se \(C_\varphi\) è compatto su \(H^p(\beta)\), allora il limite non-tangenziale di \(\phi^{(j+1)}\) ha modulo maggiore di uno, in ogni punto della frontiera del disco unitario aperto. Dimostriamo anche che se \(C_\varphi\) è di Fredholm su \(H^p(\beta)\), allora \(\varphi\) deve essere un automorfismo del disco unitario aperto.

Summary. – Let \{\beta(n)\}_{n=0}^{\infty} be a sequence of positive numbers and 1 \leq p < \infty. We consider the space \(H^p(\beta)\) of all power series \(f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n\) such that \(\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty\). Suppose that \(\frac{1}{p} + \frac{1}{q} = 1\) and \(\sum_{n=1}^{\infty} \frac{n \cdot n^q}{n^p} \beta(n)^p = \infty\) for some non-negative integer \(j\). We show that if \(C_\varphi\) is compact on \(H^p(\beta)\), then the non-tangential limit of \(\varphi^{(j+1)}\) has modulus greater than one at each boundary point of the open unit disc. Also we show that if \(C_\varphi\) is Fredholm on \(H^p(\beta)\), then \(\varphi\) must be an automorphism of the open unit disc.

Introduction.

First in the following, we generalize the defintions coming in [5]. Let \{\beta(n)\} be a sequence of positive numbers with \(\beta(0) = 1\) and 1 \leq p < \infty. We consider the space of sequences \(f = \{\hat{f}(n)\}_{n=0}^{\infty}\) such that

\[
\|f\|^p = \|f\|^p_\beta = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.
\]

The notation \(f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n\) shall be used whether or not the series converges for any value of \(z\). These are called formal power series. Let \(H^p(\beta)\) denotes the space of such formal power series. These are reflexive Banach spaces with the norm \(\|\cdot\|_\beta ([4])\) and the dual of \(H^p(\beta)\) is \(H^q(\beta^{p/q})\) where \(\frac{1}{p} +\)
$\frac{1}{q} = 1$ and $\beta^{p/q} = \{\beta(n)^{p/q}\}_n$ ([6]). Also if $g(z) = \sum_{n=0}^{\infty} \tilde{g}(n) z^n \in H^q(\beta^{p/q})$, then $\|g\|^q = \sum_{n=0}^{\infty} |\tilde{g}(n)|^q \beta(n)^p$. The Hardy, Bergman and Dirichlet spaces can be viewed in this way when $p = 2$ and respectively $\beta(n) = 1$, $\beta(n) = (n + 1)^{-1/2}$ and $\beta(n) = (n + 1)^{1/2}$. If $\lim \frac{\beta(n+1)}{\beta(n)} = 1$ or $\lim \inf \beta(n)^{1/n} = 1$, then $H^p(\beta)$ consists of functions analytic on the open unit disc $U$. It is convenient and helpful to introduce the notation $\langle f, g \rangle$ to stand for $g(f)$ where $f \in H^p(\beta)$ and $g \in H^p(\beta)^*$. Note that $\langle f, g \rangle = \sum_{n=0}^{\infty} \tilde{f}(n) \tilde{g}(n) \beta(n)^p$. Let $\tilde{f}_k(n) = \delta_k(n)$. So $\tilde{f}_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = \beta(k)$. Clearly $M_z$, the operator of multiplication by $z$ on $H^p(\beta)$ shifts the basis $\{f_k\}_k$.

Remember that a complex number $\lambda$ is said to be a bounded point evaluation on $H^p(\beta)$ if the functional of point evaluation at $\lambda$, $e_\lambda$, is bounded. The functional of evaluation of the $j$-th derivative at $\lambda$ is denoted by $e^{(j)}_\lambda$.

The function $\varphi$ in $H^p(\beta)$ that maps the unit disc $U$ into itself induces a composition operator $C_\varphi$ on $H^p(\beta)$ defined by $C_\varphi f = f \circ \varphi$. The operator $C_\varphi$ is Fredholm, if it is invertible modulo the compact operators. If $C_\varphi$ is a bounded invertible operator, then $\varphi$ must be an automorphism of $U$, that is a one to one map of $U$ onto $U$.

We say an analytic self-map $\varphi$ of $U$ has an angular derivative at $w \in \partial U$, if for some $\eta \in \partial U$ the non-tangential limit of $\frac{\varphi(z) - \eta}{z - w}$ when $z \to w$, exists and is finite. We call this limit the angular derivative of $\varphi$ at $w$ and denoted it by $\varphi'(w)$.

**Main results.**

We suppose that $H^p(\beta)$ consists of functions analytic on the open unit disc $U$. We study the Fredholm composition operator $C^\beta$ and investigate the compactness and essential norm of $C_\varphi$ acting on the Banach space $H^p(\beta)$.

**Lemma 1.** Let $X$ be a Banach space of analytic functions on a domain $\Omega$ in $\mathbb{C}$. If there exists a sequence of functions $g_k$ in the dual space $X^*$ such that $\|g_k\| = 1$ and $g_k \to 0$ weakly with $\|C^\beta (g_k)\| \to 0$, then $C_\varphi$ is not Fredholm on $X$.

**Proof.** Suppose $S$ is any bounded operator on $X^*$. Then by the hypothesis $\|SC^\beta (g_k)\| \leq \|S\| \|C^\beta (g_k)\| \to 0$ as $k \to \infty$. Now let $Q$ be an arbitrary compact operator on $X^*$. Since $Q$ is necessarily completely continuous, then we have $\|Q(g_k)\| \to 0$ ([2, p. 177, Proposition 3.3]). Thus $\|(I + Q)g_k\| \to 1$ for every compact operator $Q$ on $X^*$. This implies that $SC^\beta - I$ cannot be compact, since else it should be $\|(I + (SC^\beta - I)g_k)\| \to 1$ that is a contradiction. Thus $C^\beta$, and hence $C_\varphi$, is not Fredholm. \[ \square \]
In the following we use the fact that \( e_w \in H^q(\beta^{1/q}) \) and \( \|e_w\|^q = \sum_{n=0}^{\infty} \frac{|w|^n}{\beta(n)^q} < \infty \) for all \( w \) in \( U \) ([6]).

**Theorem 2.** – Let \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \sum_{n=0}^{\infty} \frac{n^q}{\beta(n)^q} = \infty \) for some non-negative integer \( j \). If \( C_\varphi \) is Fredholm on \( H^p(\beta) \), then \( \varphi \) is an automorphism of the disc.

**Proof.** – It is well known that if \( C_\varphi \) is Fredholm, then \( \varphi \) is univalent since else the kernel of \( C_\varphi^* \) will contain an infinite linearly independent set whose elements are differences of evaluation functionals. This is a contradiction, since \( \dim \ker C_\varphi^* < \infty \). So we need only show that \( \varphi \) maps \( U \) onto \( U \). If not, there exists \( v \in \partial \varphi(U) \cap U \) and \( z_k \in U \) such that \( \varphi(z_k) \to v \). By the Open Mapping Theorem it should be \( |z_k| \to 1 \).

Let \( j \) be the least non-negative integer such that the sum \( \sum_{n=0}^{\infty} \frac{n^q}{\beta(n)^q} = \infty \). If \( j = 0 \), set \( e_k = \frac{e_{zk}}{\|e_{zk}\|} \). Then \( \|e_k\| = 1 \). But

\[
\lim_k \|e_{zk}\|^q = \lim_k \sum_{n=0}^{\infty} \frac{|z_k|^n \beta(n)^q}{\beta(n)^q} = \sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} = \infty
\]

and so if \( p \) is a polynomial in \( H^p(\beta) \), then \( \lim_k \langle p, e_k \rangle = \lim_k \frac{p(z_k)}{\|e_{zk}\|} = 0 \). But polynomials are dense in \( H^p(\beta) \), thus \( e_k \to 0 \) weakly as \( k \to \infty \). Since \( v \) is in \( U \) and \( \varphi(z_k) \to v \), we have \( e_{\varphi(z_k)} \to e_v \). Since we also have \( \|e_{zk}\| \to \infty \), we conclude that \( \|C_\varphi e_{zk}\| = \|e_{\varphi(z_k)}\|/\|e_{zk}\| \) tends to zero. So by Lemma 1, \( C_\varphi \) is not Fredholm that is a contradiction.

If \( j > 0 \), let \( e_k = \frac{e_{zk}^{(j)}}{\|e_{zk}^{(j)}\|} \) where \( e_{zk}^{(j)} \) is the functional of evaluation of the \( j \)-th derivative at \( z_k \). Note that \( e_w(z) = \sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} \overline{w}^n z^n \) and \( e_{w}^{(j)} = \frac{d^j}{dw^j} e_w \). Thus

\[
e_{zk}^{(j)} = \sum_{n=0}^{\infty} n(n-1)(n-2)\ldots(n-j+1) \frac{(\overline{z}_k)^{n-j}}{\beta(n)^q} z^n.
\]

Since \( |z_k| \to 1 \) and \( \sum_{n=0}^{\infty} \frac{n^q}{\beta(n)^q} = \infty \), we have

\[
\lim_k \|e_{zk}^{(j)}\|^q = \lim_k \sum_{n=0}^{\infty} (n(n-1)\ldots(n-j+1)) \frac{|z_k|^q \beta(n)^q}{\beta(n)^q} = \infty.
\]

Since polynomials are dense in \( H^p(\beta) \), by the same manner as in the previous case, we can see that \( e_k \to 0 \) weakly as \( k \to \infty \). Now we show that \( \|C_\varphi^* e_k\| \to 0 \) as
A straightforward computation gives the following equalities:

\[ C_q^* e_k^{(1)} = \varphi'(z_k) e_{q(z_k)}^{(1)} \]
\[ C_q^* e_k^{(2)} = \varphi'(z_k)^2 + e_{q(z_k)}^{(2)}(z_k) + \varphi''(z_k) e_{q(z_k)}^{(1)} \]
\[ C_q^* e_k^{(3)} = \varphi'(z_k)^3 e_{q(z_k)}^{(3)}(z_k) + \varphi''(z_k) e_{q(z_k)}^{(1)}(z_k) + 2 \varphi''(z_k) e_{q(z_k)}^{(2)}(z_k) + \varphi'(z_k) \varphi''(z_k) e_{q(z_k)}^{(2)}(z_k) \]

\[ \vdots \]
\[ C_q^* e_k^{(j)} = \varphi'(z_k)^j e_{q(z_k)}^{(j)}(z_k) + \varphi^{(j)}(z_k) e_{q(z_k)}^{(1)}(z_k) + \text{lower order terms} \]

where the lower order terms involves functionals of evaluation of derivatives of order less than \( j \) at \( q(z_k) \) with coefficients involving products of derivatives of \( q \) at \( z_k \) of order less than \( j \). From this it follows that \( \|C_q^* e_k\| \to 0 \) as \( k \to 0 \).

To see this first suppose that \( j = 1 \). Thus we have

\[ C_q^* e_k = \frac{\varphi'(z_k) e_{q(z_k)}^{(1)}}{\|e_k^{(1)}\|} = \langle \varphi, e_k \rangle e_{q(z_k)}^{(1)}. \]

But \( \varphi(z_k) \to \nu \), where \( \nu \in U \). So \( \|e_{q(z_k)}^{(1)}\| \to \|e_k^{(1)}\| < \infty \). Also since \( e_k \to 0 \) weakly, \( \langle \varphi, e_k \rangle \to 0 \) as \( k \to \infty \). Thus indeed \( \|C_q^* e_k\| \to 0 \) as \( k \to \infty \).

If \( j > 1 \), remark that for all \( i < j \) we have

\[ e_{q(z_k)}^{(j)} = \sum_{n=1}^{\infty} \frac{n!}{(n-i)!} \frac{\varphi(z_k)^n}{\beta(n)^q} \]

and so

\[ \|e_{q(z_k)}^{(i)}\| = \sum_{n=i}^{\infty} (n(n-1)\ldots(n-i+1))^{\frac{1}{q}} \frac{|v(z_k)|^{n-i}}{\beta(n)^q} \]

\[ \leq \sum_{n=i}^{\infty} \frac{n^{i-1}q}{\beta(n)^q} \leq \sum_{n=i}^{\infty} \frac{n^{j-1}q}{\beta(n)^q} < \infty, \]

since \( j \) is the least non-negative integer such that \( \sum_{n=0}^{\infty} \frac{n^{j-1}q}{\beta(n)^q} = \infty \). Thus the limit of the norms of the functionals of evaluation of derivatives at \( q(z_k) \) of order less than \( j \) remain bounded in \( U \). Also, by the Principle of Uniform Boundedness Theorem \( \sup_k \|e_{z_k}^{(j)}\| < \infty \) for \( i < j \) and all derivatives of \( q \) at \( z_k \) of order less than \( j \) are bounded. Note that \( \|e_{z_k}^{(j)}\| \to \infty \) and \( \varphi(z_k) \to \nu \in U \). Thus we have \( \lim_{k \to \infty} \|C_q^* e_k\| = 0 \) provided that

\[ \lim_{k \to \infty} \frac{1}{\|e_{z_k}^{(j)}\|} \|(\varphi'(z_k))^{j} e_{q(z_k)}^{(j)}(z_k) + \varphi^{(j)}(z_k) e_{q(z_k)}^{(1)}(z_k)\| = 0. \]
Clearly
\[
\left( * \right) \quad \frac{1}{\|e_{z_k}^{(j)}\|} \left\| (q' (z_k)) e_{q(z_k)}^{(j)} + q^{(j)} (z_k) e_{q(z_k)}^{(1)} \right\| \leq \frac{|q' (z_k)|^j}{\|e_{z_k}^{(j)}\|} \left\| e_{q(z_k)}^{(j)} \right\| + \frac{|q^{(j)} (z_k)|}{\|e_{z_k}^{(j)}\|} \left\| e_{q(z_k)}^{(1)} \right\| \leq \frac{\|q\|_{H^p (\beta)}}{\|\|e_{z_k}^{(j)}\|\|} \left\| e_{q(z_k)}^{(j)} \right\| + \left| \langle q', e_k \rangle \right| \left\| e_{q(z_k)}^{(1)} \right\|.
\]

Note that \( \|e_{z_k}^{(j)}\| \to \infty \) and \( \lim_{k \to \infty} \|e_{q(z_k)}^{(1)}\| < \infty \), since \( 1 < j \). Also \( \|e_{q(z_k)}^{(j)}\| \to \|e_{q(z_k)}^{(1)}\| < \infty \) for \( i = 1, j \) and \( \langle q', e_k \rangle \to 0 \), since \( e_k \to 0 \) weakly. Thus indeed the term in \( * \) tends to zero as \( k \to \infty \) and so \( \|C_q^* e_k\| \to 0 \) which by the lemma implies that \( C_q \) is not Fredholm that is a contradiction.

Note that by the Julia Carathéodory Theorem ([3]), \( q \) has an angular derivative at \( w \in \partial U \) if and only if \( q' \) has non-tangential limit at \( w \), and \( q \) has non-tangential limit of modulus one at \( w \). Consider the open Euclidean disc, Julia disc, \( J(\xi, a) = \{ z \in U; \ |\xi - z|^2 < a (1 - |z|^2) \} \) of radius \( \frac{a}{1 + a} \) and center at \( \frac{\xi}{1 + a} \), whose boundary is tangent to \( \partial U \) at \( \xi \). By the Julia's Lemma ([1]), if \( \xi \in \partial U \) and \( q \) is an analytic function such that \( B_q = \inf_{\xi \in \partial U} |q' (\xi)| < \infty \), then \( q(J(\xi, a)) \subset J(q(\xi), aB_q) \).

Recall that the essential norm of \( C_q \) is denoted by \( \|C_q\|_e \) and is the distance in the operator norm from \( C_q \) to the compact operators.

**Theorem 3.** – Let \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \sum_{n=1}^{\infty} \frac{n^q}{\beta(n)^p} = + \infty \) for some non-negative integer \( j \). Also for \( 0 < i < j \) let \( q^{(i)} \) be an analytic self map of the unit disc \( U \). If \( C_q \) is a bounded operator on \( H^p (\beta) \) and \( |q^{(j+1)}(\xi)| \leq 1 \) for some \( \xi \in \partial U \), then \( \|C_q\|_e \geq 1 \) and \( C_q \) is not compact.

**Proof.** – Let \( \{ z_k \} \) be any sequence in \( U \) with \( z_k \to \xi \). Also let \( j \) be the least non-negative integer such that the sum \( \sum_{n=1}^{\infty} \frac{n^q}{\beta(n)^p} = + \infty \). Set \( e_k = \frac{e_{z_k}^{(j)}}{\|e_{z_k}^{(j)}\|}. \) Then \( \|e_k\| = 1 \) and by the same method used in the proof of Theorem 2, \( e_k \to 0 \) weakly as \( k \to \infty \). If \( K \) is any compact operator, then \( K^* \) is completely continuous and since \( e_k \to 0 \) weakly, it should be \( \|K^* e_k\| \to 0 \). By definition \( \|C_q - K\| : K \) is compact and
\[
\|C_q - K\| = \|(C_q - K)^*\| \geq \|(C_q - K)^* e_k\| \geq \|C_q^* e_k\| - \|K^* e_k\|.
\]
If \( k \to \infty \), then since \( \|K^* e_k\| \to 0 \), we have \( \|C_q\| \geq \lim_k \|C_q^* e_k\| \). Now we show that

\[
\lim_k \|C_q^* e_k\| = \lim_k \frac{\|e_{(j)}^{(j)}(z_k)\|}{\|e_{z_k}^{(j)}\|}.
\]

Note that since \( |\varphi^{(j+1)}(\xi)| \leq 1 \), by the Julia’s Caratheodory theorem the non-tangential limit of \( \varphi^{(i)}(\xi) \) have modulus one for \( i = 0, 1, \ldots, j \).

If \( j = 0 \), then \( e_k = \frac{e_{z_k}}{\|e_{z_k}\|} \) and \( C_q^* e_k = \frac{e_{\varphi(z_k)}}{\|e_{z_k}\|} \). If \( j = 1 \), then \( e_k = \frac{e_{z_k}^{(1)}}{\|e_{z_k}^{(1)}\|} \) and \( C_q^* e_k = \frac{e_{\varphi(z_k)}^{(1)}}{\|e_{z_k}^{(1)}\|} \). But the non-tangential limit of \( \varphi'(\xi) \) has modulus one and so \( \lim_k \|C_q^* e_k\| = \lim_k \|e_{\varphi(z_k)}^{(1)}\|/\|e_{z_k}^{(1)}\| \).

If \( j > 1 \), then \( e_k = e_{z_k}^{(j)}/\|e_{z_k}^{(j)}\| \) and

\[
C_q^* e_k = \frac{1}{\|e_{z_k}^{(j)}\|} (\varphi'(z_k)^{e_{\varphi(z_k)}^{(j)}} + L_{j, k})
\]

where \( L_{j, k} \) is the sum of lower order terms and involves derivatives of order less than \( j \) at \( \varphi(z_k) \), i.e., terms of the type \( e_{\varphi(z_k)}^{(i)} \) \((i < j)\), with coefficients involving product of derivatives of \( \varphi \) at \( z_k \) of order less than or equal to \( j \). Remark that since \( j \) is the least non-negative integer such that \( \sum_{n=0}^{\infty} \frac{\beta(n)^n}{n^\beta} = +\infty \), then we have

\[
\|e_{\varphi(z_k)}^{(i)}\|^q = \sum_{n=1}^{\infty} (n(n-1) \ldots (n-i+1))^{q-1} \frac{\|\varphi^{(i)}(z_k)\|^{n-i}}{\beta(n)^q} \leq \sum_{n=1}^{\infty} \frac{n^{(j-1)}}{\beta(n)^q} < \infty
\]

for \( i < j \). So \( \lim_{k \to \infty} \|e_{\varphi(z_k)}^{(i)}\| \) remains bounded for all \( i \) less than \( j \). Also since \( \|e_{z_k}^{(j)}\| \to \infty \) and \( \varphi^{(i)}(\xi) \) has the non-tangential limit of modulus one for all \( i \leq j \), thus indeed \( \lim_k \|L_{j, k}\|/\|e_{z_k}^{(j)}\| = 0 \).

Therefore

\[
\lim_k \|C_q^* e_k\| = \lim_k \frac{\varphi'(z_k)^{e_{\varphi(z_k)}^{(j)}}}{\|e_{z_k}^{(j)}\|}.
\]

Now to complete the proof it is sufficient to show that \( \lim_k \frac{e_{\varphi(z_k)}^{(j)}}{\|e_{z_k}^{(j)}\|} \geq 1 \). For this set \( z_k = \left(1 - \frac{1}{k}\right) \xi \). Then there exists a sequence \( \{r_k\} \) of non-negative numbers such that \( z_k \) is the point on \( \partial J(\xi, r_k) \) closest to 0. Therefore by the Ju-
lia’s Lemma

\[ \varphi(z_k) \in \varphi(\partial J(\xi, r_k)) \subseteq \partial \varphi J(\xi, r_k) \subseteq \partial J(\varphi(\xi), r_k). \]

It follows that \(|\varphi(z_k)| \geq |z_k|\) for all \(k\). Now since
\[ \|e_z^{(j)}\| = \sum_{n=0}^{\infty} \frac{n!}{(n-j)!} |z|^{n-j} \beta(n, j), \]
the norm \(\|e_z^{(j)}\|\) increases with \(|z|\). Thus for all \(k\), \(\|e_{\varphi(z_k)}^{(j)}\| \geq 1\) and indeed \(\|C_\varphi\| \geq 1\). This implies that \(C_\varphi\) is not compact and so the proof is complete.

**Corollary 4.** – Let \(\frac{1}{p} + \frac{1}{q} = 1\) and \(\sum_{n=1}^{\infty} \frac{n^q}{\beta(n, n)} = +\infty\) for some non-negative integer \(j\). If \(C_\varphi\) is compact on \(H^p(\beta)\), then \(|\varphi^{(j+1)}(\xi)| > 1\) for all \(\xi \in \partial U\) such that \(\varphi^{(j+1)}(\xi)\) exists.

**References**


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*Pervenuta in Redazione
il 5 marzo 2002*