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Quasimonotone Systems of Higher Order.

MANFRED KRONZ

Sunto. – *Consideriamo sistemi nonlineari quasimonotoni di tipo di divergenza di ordine alto con crescenza di ordine p, $p \geq 2$ e coefficienti di Dini continui. Usando la tecnica dell'approssimazione armonica, diamo una dimostrazione diretta per la regolarità parziale di soluzioni deboli.*

Summary. – *We consider higher order quasimonotone nonlinear systems of divergence type with growth of order p, $p \geq 2$, and Dini continuous coefficients. Using the technique of harmonic approximation we give a direct partial regularity proof for weak solutions.*

1. – Introduction.

In this paper we are concerned with the regularity of weak solutions to quasimonotone nonlinear systems of higher order in divergence form of the type

$$\int_\Omega \mathcal{A}(x, D^m u)\, D^m \varphi \, d\mathcal{L}^n = 0 \,, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, u is a function in the Sobolev space $W^{m,p}(\Omega, \mathbb{R}^N)$, $m \geq 1$, $p \geq 2$, $\mathcal{L}^n$ is the Lebesgue measure and $\mathcal{A}: \Omega \times \odot^m(\mathbb{R}^n, \mathbb{R}^N) \to \mathrm{Hom}(\odot^m(\mathbb{R}^n, \mathbb{R}^N), \mathbb{R})$, $(x, Q) \mapsto \mathcal{A}(x, Q)$, is Dini continuous with respect to x, strictly quasimonotone and C^1 with respect to Q. Here $\odot^m(\mathbb{R}^n, \mathbb{R}^N) \cong \mathbb{R}^{N\binom{n+m-1}{m}}$ denotes the vectorspace of symmetric m-linear $\mathbb{R}^N$-valued functions on $\mathbb{R}^n$.

In the case of weak solutions $u \in W^{m,2}(\Omega, \mathbb{R}^N)$ to inhomogeneous nonlinear elliptic systems of the type

$$\int_\Omega \mathcal{A}(x, \delta u, D^m u)\, D^m \varphi \, d\mathcal{L}^n = \int_\Omega a(x, \delta u)\, D^m \varphi \, d\mathcal{L}^n + \sum_{k=0}^{m-1} b_k(x, \delta u, D^m u)\, D^k \varphi \, d\mathcal{L}^n \tag{2}$$

of higher order in divergence form (here $\delta u = (u, Du, \ldots, D^{m-1}u)$), it has been proved by Giaquinta and Modica [GM1], assuming natural hypotheses on the regularity and the growth of $\mathcal{A} = \mathcal{A}(x, q, Q)$ with respect to Q, Hölder continuity with respect to x and q and suitable growth and continuity assumptions on the inhomogenities and regularity assumptions on u (i.e. $u \in W^{m-1,\infty}(\Omega, \mathbb{R}^N)$), that u has Hölder continuous m^{th}-order derivatives outside a singular set of Lebesgue measure 0 for some positive Hölder exponent. Recently, Duzaar, Gastel and Grotowski [DGG] gave a partial regularity proof for the homogeneous elliptic system (2), which yields, using the *technique of harmonic approximation*, the optimal partial regularity result (with respect to the Hölder exponent). Partial regularity results for minimizers of the functional

$$F(u) = \int_\Omega f(Du)\, d\mathcal{L}^n \tag{3}$$

of second order, whose integrand satisfies a quasiconvexity condition in the sense of Morrey were first proved by Evans [E], using an indirect blow-up argument. A direct proof, using higher integrability arguments, for minimizers of quasiconvex variational integrals $\int_\Omega f(x, u, Du)\, d\mathcal{L}^n$ was supplied by Giaquinta and Modica [GM2]. In [K] the current author obtained a partial regularity result for minimizers of quasiconvex functionals $\int_\Omega f(D^m u)\, d\mathcal{L}^n$ of higher order with the technique of harmonic approximation. Guidorzi [Gu] proves the same result with an indirect blow-up argument. In attempting to extend Evans' partial regularity result for minimizers of the variational integral (3) to weak solutions of second order nonlinear systems of type (1), Hamburger [H] replaced the quasiconvexity condition on the integrand in (3) by a quasimonotonicity condition on $\mathcal{A}$. This corresponds to the fact that quasiconvexity of a function is implied by quasimonotonicity of its gradient. Moreover he assumes Hölder continuity of $\mathcal{A}$ with respect to x with Hölder exponent α and proves partial $C^{m,\alpha}$-regularity of u. (For the concept of quasimonotonicity and relations to monotonicity, convexity and quasiconvexity see Hamburger [H] and also Fuchs [F] and Zhang [Z].)

In this paper we extend Hamburger's result on weak solutions of quasimonotone systems of second order to those of quasimonotone systems of arbitrary order. To prove partial regularity results for weak solutions of system (1) we assume that $\mathcal{A}$ is uniformly strictly quasimonotone, which means that for all balls $B_\varrho(x_0) \subset\subset \Omega$, any $Q \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$ and all test functions $\varphi \in C_0^\infty(B_\varrho(x_0), \mathbb{R}^N)$ there holds

$$\int_{B_\varrho(x_0)} \mathcal{A}(x_0, Q + D^m\varphi)\, D^m\varphi\, d\mathcal{L}^n \geq \lambda \int_{B_\varrho(x_0)} \left(|D^m\varphi|^2 + |D^m\varphi|^p\right) d\mathcal{L}^n.$$

Moreover we assume, that $\mathcal{A}$ is Dini continuous with respect to x. This means that there exists a modulus of continuity $\omega : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\mathcal{W}(\varrho) := \int_0^\varrho \frac{\omega(s)}{s} ds < \infty$ for some $\varrho > 0$ such that for all $x_1, x_2 \in \Omega$ and $Q \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$ there holds

$$|\mathcal{A}(x_1, Q) - \mathcal{A}(x_2, Q)| \leq \omega(|x_1 - x_2|)(1 + |Q|^{p-1}).$$

Our main result can be stated as follows: Assuming the additional standard hypothesis $|D_Q \mathcal{A}(x, Q)| \leq L(1 + |Q|^{p-2})$, a weak solution $u \in W^{m,p}(\Omega, \mathbb{R}^N)$, $p \geq 2$, is C^m outside a (relatively) closed singular set Σ of Lebesgue measure 0 and $D^m u$ has the modulus of continuity $\varrho \mapsto \mathcal{W}(\varrho)$ in a neighborhood of every point x_0 in the regular set $\Omega \backslash \Sigma$.

As in the case of second order systems, the uniformly quasimonotonicity condition for $\mathcal{A}$ implies that $D_Q \mathcal{A}$ is elliptic in the sense of Legendre-Hadamard. With regard to regularity theory, the uniform strict quasimonotonicity permits the proof of a *Caccioppoli inequality* for weak solutions of (1). Both allow us to apply the technique of harmonic approximations and to give a direct proof of our partial regularity result.

The point of this technique is the fact, that for a bilinear form $\mathcal{B}$ on $\odot^m(\mathbb{R}^n, \mathbb{R}^N)$ which is elliptic in the sense of Legendre-Hadamard, an «approximately $(\mathcal{B}, m)$-harmonic» function v — that means $\int_\Omega \mathcal{B}(D^m v, D^m \varphi) \, d\mathcal{L}^n$ is sufficiently small for all test function φ — lies L^2-close to some function h with $\int_\Omega \mathcal{B}(D^m h, D^m \varphi) \, d\mathcal{L}^n = 0$. Then, standard a priori sestimates for h lead to estimates for excess terms.

In this paper we deal with the bilinear form $\mathcal{B} = D_Q \mathcal{A}(x_0, D^m P_{x_0, \varrho})$, where $x_0 \in \Omega$ and $P_{x_0, \varrho}$ is a special polynomial of degree at most m which is associated to a weak solution of system (1) on a ball $B_\varrho(x_0)$. Applying the technique of harmonic approximation we derive decay estimates for our excess term

$$\Psi(x_0, \varrho, P_{x_0, \varrho}) =$$

$$\left(\varrho^{-2} \fint_{B_\varrho(x_0)} |D^{m-1}(u - P_{x_0, \varrho})|^2 \, d\mathcal{L}^n + \varrho^{-p} \fint_{B_\varrho(x_0)} |D^{m-1}(u - P_{x_0, \varrho})|^p \, d\mathcal{L}^n \right)^{1/2}$$

in points $x_0 \in \Omega$ where $\Psi(x_0, \varrho, P_{x_0, \varrho})$ is sufficiently small. Iterating this excess term, we derive an estimate for $\int_{B_r(x)} |D^{m-1}(u - P_{x,r})|^2 \, d\mathcal{L}^n$ which is valid for all $0 < r \leq \varrho$ and all x in a sufficiently small neighborhood of x_0 and which implies our partial regularity result. Here, even in the case $p \neq 2$ of non-linear growth of $\mathcal{A}$, we only use L^2-estimates, i.e. standard a priori estimates for solutions of constant coefficient elliptic systems.

2. – Hypotheses and statements of results.

Assume that $\mathcal{A}$ in (1) fulfills the following conditions:

(H1) The function $\mathcal{A}$ is C^1 with respect to Q and *strictly quasimonotone*, i.e. there exist an exponent $p \geqslant 2$ and a constant $\lambda > 0$ such that for all balls $B_\varrho(x_0) \subset\subset \Omega$, $Q \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$ and $\varphi \in C_0^\infty(B_\varrho(x_0), \mathbb{R}^N)$ there holds

$$\int_{B_\varrho(x_0)} \mathcal{A}(x_0, Q + D^m \varphi) D^m \varphi \, d\mathcal{L}^n \geqslant \lambda \int_{B_\varrho(x_0)} (|D^m \varphi|^2 + |D^m \varphi|^p) \, d\mathcal{L}^n;$$

(H2) there exists a nondecreasing concave function $\omega : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ with $\omega(0) = 0$ such that for all $x_1, x_2 \in \Omega$ and $Q \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$ there holds

$$|\mathcal{A}(x_1, Q) - \mathcal{A}(x_2, Q)| \leqslant \omega(|x_1 - x_2|)\,(1 + |Q|^{p-1});$$

moreover we assume that $s \mapsto s^{-\beta}\omega(s)$ is nonincreasing for some exponent $\beta \in]0, 1[$ and that ω fulfills *Dini's condition*

$$\mathcal{W}(\varrho) := \int_0^\varrho \frac{\omega(s)}{s} ds < \infty \quad \text{for some} \quad \varrho > 0;$$

(H3) there exists $L > 0$ such that for all $x \in \Omega$, and $Q \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$ there holds

$$|D_Q \mathcal{A}(x, Q)| \leqslant L(1 + |Q|^{p-2}).$$

REMARK 1. – **(1)** Notice that the conditions **(H2)** and **(H3)** imply that there exists $\kappa > 0$ such that

$$(4) \qquad |\mathcal{A}(x, Q)| \leqslant |\mathcal{A}(x, Q) - \mathcal{A}(x_0, Q)| + |\mathcal{A}(x_0, 0)| + \left| \int_0^1 D_Q \mathcal{A}(x_0, tQ)\, dt \right| |Q| \leqslant \kappa\,(1 + |Q|^{p-1})$$

for all $x \in \Omega$ and $Q \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$. From this we infer that hypothesis **(H1)** is valid for all $\varphi \in W_0^{m,p}(B_\varrho(x), \mathbb{R}^N)$.

(2) From **(H3)** we infer the existence of a continuous, non-negative function $\nu(t, s)$, for fixed s monotone increasing in t and vice versa, concave in s with $\nu(t, 0) = 0$ and

$$|D_Q \mathcal{A}(x_1, Q_1) - D_Q \mathcal{A}(x_2, Q_2)| \leqslant (1 + |Q_1|^{p-2} + |Q_2|^{p-2})\, \nu(|Q_1|, |x_1 - x_2| + |Q_1 - Q_2|)$$

for all $x_1, x_2 \in \Omega$ and $Q_1, Q_2 \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$. This implies that there exists a monotone increasing and concave function $\nu_M : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\nu_M(0) = 0$ and

$$(5) \qquad |D_Q \mathcal{A}(x_1, Q_1) - D_Q \mathcal{A}(x_2, Q_2)| \leq (1 + M^{p-2} + |Q_2|^{p-2})\, \nu_M(|x_1 - x_2| + |Q_1 - Q_2|)$$

for all $x_1, x_2 \in \Omega$ and $Q_1, Q_2 \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$ with $|Q_1| \leq M$.

(3) In view of hypotheses **(H1)** and **(H3)** we see that for given $\varphi \in W_0^{m,p}(B_\varrho(x_0), \mathbb{R}^N)$

$$\int_{B_\varrho(x_0)} \mathcal{A}(x_0, Q + D^m\varphi)\, D^m\varphi\, d\mathcal{L}^n = \int_{B_\varrho(x_0)} (\mathcal{A}(x_0, Q + D^m\varphi) - \mathcal{A}(x_0, Q))\, D^m\varphi\, d\mathcal{L}^n =$$

$$\int_{B_\varrho(x_0)} \int_0^1 D_Q \mathcal{A}(x_0, Q + t D^m\varphi)(D^m\varphi, D^m\varphi)\, dt\, d\mathcal{L}^n \geq \lambda \int_{B_\varrho(x_0)} (|D^m\varphi|^2 + |D^m\varphi|^p)\, d\mathcal{L}^n.$$

Rescaling φ to $\varepsilon\varphi$ and letting $\varepsilon \searrow 0$, we obtain that **(H1)-(H3)** imply

$$(6) \qquad \int_{B_\varrho(x_0)} D_Q \mathcal{A}(x_0, Q)(D^m\varphi, D^m\varphi)\, d\mathcal{L}^n \geq \lambda \int_{B_\varrho(x_0)} |D^m\varphi|^2 d\mathcal{L}^n$$

for all balls $B_\varrho(x_0) \subset\subset \Omega$, $Q \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$ and $\varphi \in W_0^{m,p}(B_\varrho(x_0), \mathbb{R}^N)$. This integral condition is — as in the case $m = 1$ — equivalent to the Legendre-Hadamard condition

$$D_Q \mathcal{A}(x_0, Q)(\zeta^m \otimes \eta, \zeta^m \otimes \eta) \geq \frac{\lambda}{m!} |\zeta|^{2m} |\eta|^2$$

for all $x_0 \in \Omega$, $Q \in \odot^m(\mathbb{R}^n, \mathbb{R}^N)$, $\zeta \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^N$, see [M], Theorem 7. Here $\zeta^m = \zeta \otimes \ldots \otimes \zeta$.

(4) From hypotheses **(H2)** we infer $\int_0^\infty \omega(\vartheta^t \varrho)\, dt \leq \sum_{j=0}^\infty \omega(\vartheta^j \varrho) \leq \int_0^\infty \omega(\vartheta^{t-1} \varrho)\, dt$ for $\vartheta \in]0, 1[$ and $\varrho > 0$. This implies (using $1/\vartheta > 1$ and the concavity of ω)

$$(7) \qquad -\frac{\mathcal{W}(\varrho)}{\log(\vartheta)} \leq \sum_{j=0}^\infty \omega(\vartheta^j \varrho) \leq -\frac{\mathcal{W}(\varrho)}{\vartheta \log(\vartheta)}.$$

THEOREM 1. – *Suppose that* $\mathcal{A}$ *in* (1) *satisfies* **(H1)**, **(H2)** *and* **(H3)** *and* $u \in W^{m,p}(\Omega, \mathbb{R}^N)$ *is a weak solution of* (1). *Then there exists an open subset* $U \subset \Omega$ with

$$\mathcal{L}^n(\Omega \backslash U) = 0$$

and $u \in C^m_{\mathrm{loc}}(U, \mathbb{R}^N)$. *In addition, for* $1 > \alpha > \beta$, $D^m u$ *has the modulus of continuity* $s \mapsto s^\alpha + \mathcal{W}(s)$ *in a neighborhood of* $x_0 \in U$.

Furthermore $\Omega \backslash U \subset \Sigma_1 \cup \Sigma_2$ *where*

$$\Sigma_1 = \left\{ x_0 \in \Omega : \liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} |D^m u - (D^m u)_{x_0, \varrho}|^p \, d\mathcal{L}^n > 0 \right\},$$

$$\Sigma_2 = \left\{ x_0 \in \Omega : \limsup_{\varrho \searrow 0} |(D^m u)_{x_0, \varrho}| = \infty \right\}.$$

3. – Polynomials with good properties.

Let P be a $\mathbb{R}^N$-valued polynomial of degree at most m with

$$P(x) = \sum_{l=0}^m \frac{1}{l!} D^l P(x_0)(\underbrace{x - x_0, \ldots, x - x_0}_{l \text{ times}}) =: \sum_{l=0}^m \frac{1}{l!} D^l P(x_0)(x - x_0)^l,$$

where $D^l P(x_0) \in \odot^l(\mathbb{R}^n, \mathbb{R}^N)$. For $u \in W^{m,p}(B_\varrho(x), \mathbb{R}^N)$ let P be a polynomial satisfying

$$\int_{B_\varrho(x_0)} D^k(u - P) \, d\mathcal{L}^n = 0 \quad \text{for all} \quad 0 \leq k \leq l - 1 \ (l \leq m). \tag{8}$$

Then a repeated application of Poincaré's inequality shows the existence of a constant $C_{\mathrm{poin}} = C_{\mathrm{poin}}(n, N, m)$ such that

$$\left(\int_{B_\varrho(x_0)} |D^k(u-P)|^p \, d\mathcal{L}^n(x) \right)^{1/p} \leq C_{\mathrm{poin}} \, \varrho^{l-k} \left(\int_{B_\varrho(x_0)} |D^l(u-P)|^p \, d\mathcal{L}^n(x) \right)^{1/p} \tag{9}$$

holds for every $0 \leq k < l \leq m$. Analogously we derive Sobolev inequalities for $u \in W^{m,p}(B_\varrho(x_0), \mathbb{R}^N)$ and polynomials P satisfying (8). If

$$p^* = \begin{cases} \dfrac{np}{n-p} & \text{if } 1 \leq p < n \\ p^* \in [1, \infty[\text{ fixed} & \text{if } p \geq n, \end{cases} \tag{10}$$

then there exist a constant $C_{\mathrm{sob}} = C_{\mathrm{sob}}(n, N, m, p^*)$ such that

$$\left(\fint_{B_\varrho(x_0)} |D^k(u-P)|^{p^*} d\mathcal{L}^n(x) \right)^{1/p^*} \leq C_{\mathrm{sob}} \, \varrho^{l-k} \left(\fint_{B_\varrho(x_0)} |D^l(u-P)|^p \, d\mathcal{L}^n(x) \right)^{1/p} \tag{11}$$

holds every $0 \leq k < l \leq m$.

In this paper we will work with special polynomials $P_{x_0, \varrho}$ of degree $\leq m$, associated with $u \in W^{m,p}(B_\varrho(x_0), \mathbb{R}^N)$, which fulfill condition (8). Hence

Poincaré's and Sobolev's inequality (9), (11) are valid for these polynomials. They are defined in the following way:

Let $p_{x_0,\varrho}$ be the unique polynomial of degree $\leqslant 1$ minimizing $p \mapsto \int_{B_\varrho(x_0)} |v-p|^2 \, d\mathcal{L}^n$, $v \in L^2(B_\varrho(x_0), \mathbb{R}^M)$ given, among all polynomials $p : B_\varrho(x_0) \to \mathbb{R}^M$ of degree $\leqslant 1$. An explicit formula for $p_{x_0,\varrho}$ (see [K], § 2) is given by $p_{x_0,\varrho}(x) = q_{x_0,\varrho} + Q_{x_0,\varrho}(x-x_0)$, where

$$(12) \qquad q_{x_0,\varrho} = \fint_{B_\varrho(x_0)} v \, d\mathcal{L}^n \quad \text{and} \quad Q_{x_0,\varrho} = \frac{n+2}{\varrho^2} \fint_{B_\varrho(x_0)} v(x) \otimes (x-x_0) \, d\mathcal{L}^n(x).$$

From [K], Lemma 2, we have

LEMMA 1. – *Let* $v \in L^2(B_\varrho(x_0))$, $0 < \vartheta \leqslant 1$, $q_{x_0,\varrho}$, $Q_{x_0,\varrho}$ *and* $Q_{x_0,\vartheta\varrho}$ *be as in* (12). *Moreover let* $p_{x_0,\varrho}$ *be the polynomial of degree* 1 *with* $p_{x_0,\varrho}(x) = q_{x_0,\varrho} + Q_{x_0,\varrho}(x-x_0)$. *Then the following estimates hold:*

(i) $|Q_{x_0,\vartheta\varrho} - Q_{x_0,\varrho}|^2 \leqslant n(n+2)(\vartheta\varrho)^{-2} \fint_{B_{\vartheta\varrho}(x_0)} |v - p_{x_0,\varrho}|^2 \, d\mathcal{L}^n$

(ii) $|Q_{x_0,\varrho} - (Dv)_{x_0,\varrho}|^2 \leqslant C_{\text{poin}}^2 n(n+2) \fint_{B_\varrho(x_0)} |Dv - (Dv)_{x_0,\varrho}|^2 \, d\mathcal{L}^n$

Here we have used the abbreviation $(Dv)_{x_0,\varrho} = \fint_{B_\varrho(x_0)} Dv \, d\mathcal{L}^n$. ■

A fruitful way, showing that a given L^2-function has a Hölder continuous representative is Campanato's integral characterization of Hölder continuous functions (see [G], Theorem 3.1). Inspecting the proof it is obvious (using Remark 4) that an L^2-function w has the modulus of continuity $s \mapsto \mathcal{W}(s)$ in a neighborhood of $x_0 \in \Omega$ if

$$(13) \qquad \left(\fint_{B_\varrho(x)} |w - w_{x,\varrho}|^2 \, d\mathcal{L}^n \right)^{1/2} \leqslant C\omega(\varrho)$$

holds for all balls $B_\varrho(x)$, $\varrho \leqslant \varrho_0$, with center x in a neighborhood of x_0, where ω and $\mathcal{W}$ satisfy hypothesis **(H2)**. It is then straightforward to show (using Lemma 1) that this integral characterization (13) with $w = Dv$ is equivalent to

$$(14) \qquad \left(\varrho^{-2} \fint_{B_\varrho(x)} |v(y) - q_{x,\varrho} - Q_{x,\varrho}(y-x)|^2 \, d\mathcal{L}^n(y) \right)^{1/2} \leqslant C\omega(\varrho).$$

Hence, if (14) holds for all balls $B_\varrho(x)$, $\varrho \leqslant \varrho_0$, with center x in a neighborhood of x_0, then Dv is continuous in a neighborhood of x_0 with the modulus of continuity $s \mapsto \mathcal{W}(s)$ (see also Campanato's integral characterization for $C^{m,\alpha}$-functions [C1], [C2]).

For $u \in W^{m,2}(\Omega, \mathbb{R}^N)$ and a ball $B_\varrho(x_0) \subset\subset \Omega$ let $P_{x_0,\varrho}$ be the unique polynomial of degree $\leqslant m$ such that $D^{m-1} P_{x_0,\varrho}$ minimizes

$$(15) \quad P \mapsto \int_{B_\varrho(x_0)} |D^{m-1}(u-P)|^2 \, d\mathcal{L}^n$$

$$\text{and} \quad \int_{B_\varrho(x_0)} D^k(u - P_{x_0,\varrho}) \, d\mathcal{L}^n = 0 \quad \text{for } 0 \leqslant k \leqslant m-2.$$

From (12) we see $D^{m-1} P_{x_0,\varrho}(x) = q_{x_0,\varrho} + Q_{x_0,\varrho}(x - x_0)$, where $q_{x_0,\varrho} = (D^{m-1}u)_{x_0,\varrho}$ and

$$(16) \qquad D^m P_{x_0,\varrho} = Q_{x_0,\varrho} = \frac{n+2}{\varrho^2} \fint_{B_\varrho(x_0)} D^{m-1} u(x) \otimes (x - x_0) \, d\mathcal{L}^n(x).$$

This special structure of $P_{x_0,\varrho}$ implies $\int_{B_\varrho(x_0)} D^{m-1}(u - P_{x_0,\varrho}) \, d\mathcal{L}^n = 0$ and $P_{x_0,\varrho}$ fulfills condition (8) with $l = m-1$. Moreover we can apply Lemma 1 with $D^m P_{x_0,\varrho}$ instead of $Q_{x_0,\varrho}$, $(D^m u)_{x_0,\varrho}$ instead of $(Dv)_{x_0,\varrho}$ and $D^m u$ instead of Dv. This leads to

$$|D^m P_{x_0,\vartheta\varrho} - D^m P_{x_0,\varrho}|^2 \leqslant n(n+2)(\vartheta\varrho)^{-2} \fint_{B_{\vartheta\varrho}(x_0)} |D^{m-1}(u - P_{x_0,\varrho})|^2 \, d\mathcal{L}^n,$$

$$|D^m P_{x_0,\varrho} - (Du)_{x_0,\varrho}|^2 \leqslant C^2_{\text{poin}} n(n+2) \fint_{B_\varrho(x_0)} |D^m u - (D^m u)_{x_0,\varrho}|^2 \, d\mathcal{L}^n.$$

Further we can apply integral characterization (14) with $D^{m-1}(u - P_{x,\varrho})$ instead of $v - q_{x,\varrho} - Q_{x,\varrho}(y - x)$.

4. – Caccioppoli inequality.

In this section we prove a Caccioppoli inequality for solutions $u \in W^{m,p}(\Omega, \mathbb{R}^N)$ of (1).

First we demonstrate the following technical lemma which is an extension of Lemma 5.1 in [Gi]:

LEMMA 2. – *Let $0 < \vartheta < 1$, $A_k \geqslant 0$, $\alpha_k > 0$ for $k = 0, 1, \ldots, l$, $B \geqslant 0$ and $f \geqslant 0$ a bounded function satisfying*

$$f(t) \leqslant \vartheta f(s) + \sum_{k=0}^{l} A_k (s-t)^{-\alpha_k} + B$$

for all $0 < r \leqslant t < s \leqslant \varrho$. *Then there exists a constant* $C_{\mathrm{tech}} = C_{\mathrm{tech}}(\alpha_0, \dots, \alpha_l, \vartheta)$ *such that*

$$f(r) \leqslant C_{\mathrm{tech}} \left(\sum_{k=0}^{l} A_k (\varrho - r)^{-\alpha_k} + B \right).$$

PROOF OF LEMMA 2. – For $0 < \tau < 1$ satisfying $\tau^{-\alpha_k} \vartheta < 1$ for $k = 0, \dots, l$ (i.e. $1 > \tau^{\max(\alpha_k)} > \vartheta$) define the recursive sequence $t_0 = r$ and $t_{n+1} = t_n + (1-\tau)\, \tau^n (\varrho - r) \leqslant \varrho$. We get

$$f(r) = f(t_0) \leqslant \vartheta^n f(t_n) + \sum_{j=0}^{n-1} \vartheta^j \left(\sum_{k=0}^{l} A_k [(1-\tau)\, \tau^j (\varrho - r)]^{-\alpha_k} + B \right)$$

$$= \vartheta^n f(t_n) + \sum_{k=0}^{l} A_k [(1-\tau)(\varrho - r)]^{-\alpha_k} \sum_{j=0}^{n-1} (\vartheta \tau^{-\alpha_k})^j + B \sum_{j=0}^{n-1} \vartheta^j .$$

Define $C_{\mathrm{tech}} = \max\left(\frac{(1-\tau)^{-\alpha_0}}{1 - \vartheta \tau^{-\alpha_0}}, \dots, \frac{(1-\tau)^{-\alpha_l}}{1 - \vartheta \tau^{-\alpha_l}}, \frac{1}{1-\vartheta} \right)$ and get the desired estimate letting $n \to \infty$. ■

For a ball $B_\varrho(x_0) \subset \Omega$, $u \in W^{m,p}(B_\varrho(x_0), \mathbb{R}^N)$ and a polynomial $P : B_\varrho(x_0) \to \mathbb{R}^N$ of degree $\leqslant m$ we define the quantities

$$\Phi(x_0, \varrho, P) := \left(\fint_{B_\varrho(x_0)} |D^m(u-P)|^2 \, d\mathcal{L}^n + \fint_{B_\varrho(x_0)} |D^m(u-P)|^p \, d\mathcal{L}^n \right)^{1/2}, \tag{17}$$

$$\Psi(x_0, \varrho, P) = \tag{18}$$

$$\left(\varrho^{-2} \fint_{B_\varrho(x_0)} |D^{m-1}(u-P)|^2 \, d\mathcal{L}^n + \varrho^{-p} \fint_{B_\varrho(x_0)} |D^{m-1}(u-P)|^p \, d\mathcal{L}^n \right)^{1/2}.$$

Now we state a Caccioppoli inequality.

LEMMA 3. – *Let* $u \in W^{m,p}(\Omega, \mathbb{R}^N)$ *be a weak solution of equation* (1), *where* $\mathcal{A}$ *satisfies* **(H1)**, **(H2)**, **(H3)** *and let* $M > 0$. *Then there exist constants* $C_{\mathrm{cac}} = C_{\mathrm{cac}}(L, \lambda, p, m, n, N, M) \geqslant 1$ *and* $\varrho_0 = \varrho_0(\omega, \lambda, p, M) \leqslant 1$ *such that*

$$\Phi^2(x_0, \varrho/2, P) \leqslant C_{\mathrm{cac}}^2 \big(\Psi^2(x_0, \varrho, P) + \omega^2(\varrho) \big) \tag{19}$$

holds for all balls $B_\varrho(x_0) \subset\subset \Omega$ *with* $\varrho \leqslant \varrho_0$ *and all polynomials* $P : B_\varrho(x_0) \to \mathbb{R}^N$ *of degree* $\leqslant m$ *satisfying* $|D^m P| \leqslant M$ *and* $\int_{B_\varrho(x_0)} D^k(u-P)\, d\mathcal{L}^n = 0$ *for* $0 \leqslant k \leqslant m-2$. *(Here* $\Phi(x_0, \varrho/2, P)$ *and* $\Psi(x_0, \varrho, P)$ *are as in* (17) *respectively* (18).)

PROOF. – Take a polynomial P of degree at most m fulfilling the assumptions of the lemma and choose the functions

$$\varphi = \eta(u-P) \quad \text{and} \quad \psi = (1-\eta)(u-P),$$

where $\eta \in C_0^\infty(B_\varrho(x_0))$ satisfies $(0 < t < s \leqslant \varrho)$ $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B_t(x_0)$, $\eta \equiv 0$ on $\Omega \backslash B_s(x_0)$ and $|D^k \eta| \leqslant (C_\eta(s-t))^{-k}$ for all $0 \leqslant k \leqslant m$. From the product formula $D(f \otimes g) = Df \otimes g + f \otimes Dg$ for tensor products and our choice of φ and ψ we get for both $|D^m \varphi|$ and $|D^m \psi|$ the estimate

$$|D^m \varphi| \leqslant \eta |D^m(u-P)| + \mathrm{LOT}, \qquad |D^m \psi| \leqslant (1-\eta)|D^m(u-P)| + \mathrm{LOT} \tag{20}$$

where LOT are lower order terms with

$$\mathrm{LOT} = \sum_{k=0}^{m-1} \binom{m}{k} (C_\eta(s-t))^{k-m} |D^k(u-P)|. \tag{21}$$

Later we will use the estimates:

$$\mathrm{LOT}^2 \leqslant m \sum_{k=0}^{m-1} \binom{m}{k}^2 (C_\eta(s-t))^{2(k-m)} |D^k(u-P)|^2, \tag{22}$$

$$\mathrm{LOT}^p \leqslant m^{p-1} \sum_{k=0}^{m-1} \binom{m}{k}^p (C_\eta(s-t))^{p(k-m)} |D^k(u-P)|^p. \tag{23}$$

Using the quasimonotonicity condition (**H1**), $D^m P + D^m \varphi = D^m u - D^m \psi$ and (1) we obtain

$$\begin{aligned} \lambda \int_{B_s(x_0)} [|D^m\varphi|^2 + |D^m\varphi|^p] \, d\mathcal{L}^n & \leqslant \int_{B_s(x_0)} \mathcal{A}(x_0, D^m P + D^m \varphi) \, D^m \varphi \, d\mathcal{L}^n \\ &= \int_{B_s(x_0)} (\mathcal{A}(x_0, D^m u - D^m \psi) - \mathcal{A}(x_0, D^m u)) \, D^m \varphi \, d\mathcal{L}^n \\ &\quad + \int_{B_s(x_0)} (\mathcal{A}(x_0, D^m u) - \mathcal{A}(x, D^m u)) \, D^m \varphi \, d\mathcal{L}^n \\ &= I + II. \end{aligned} \tag{24}$$

From the estimate

$$(a+b)^{p-2} \leqslant 2^{p-2}(a^{p-2} + b^{p-2}) \tag{25}$$

for $a, b \geqslant 0$ and $p \geqslant 2$, hypothesis (**H3**), $|\psi| \equiv 0$ on $B_t(x_0)$ and Young's inequality we now get

$$(26)\quad I \leqslant \int\limits_{B_s(x_0)} \int\limits_0^1 |D_Q \mathcal{A}(x_0, D^m u - \tau D^m \psi))|d\tau |D^m \psi||D^m \varphi| d\mathcal{L}^n$$

$$\leqslant 4^{p-2} L \int\limits_{B_s(x_0)\setminus B_t(x_0)} (1 + M^{p-2} + |D^m(u-P)|^{p-2} + |D^m \psi|^{p-2})|D^m \psi||D^m \varphi| d\mathcal{L}^n$$

$$\leqslant 4^{p-2} L (1 + M^{p-2}) \Bigg[\int\limits_{B_s(x_0)\setminus B_t(x_0)} |D^m \psi||D^m \varphi| d\mathcal{L}^n + \int\limits_{B_s(x_0)\setminus B_t(x_0)} |D^m(u-P)|^p d\mathcal{L}^n$$

$$+ \int\limits_{B_s(x_0)\setminus B_t(x_0)} (|D^m \psi||D^m \varphi|)^{p/2} d\mathcal{L}^n + \int\limits_{B_s(x_0)\setminus B_t(x_0)} |D^m \psi|^{p-1} |D^m \varphi| d\mathcal{L}^n \Bigg].$$

Using (20) and Hölder's inequality we see

$$\int\limits_{B_s(x_0)\setminus B_t(x_0)} |D^m \psi||D^m \varphi| d\mathcal{L}^n \leqslant 2 \Bigg(\int\limits_{B_s(x_0)\setminus B_t(x_0)} |D^m(u-P)|^2 d\mathcal{L}^n + \int\limits_{B_s(x_0)\setminus B_t(x_0)} \mathrm{LOT}^2 d\mathcal{L}^n \Bigg).$$

Similiary we obtain

$$\int\limits_{B_s(x_0)\setminus B_t(x_0)} (|D^m \psi||D^m \varphi|)^{p/2} d\mathcal{L}^n + \int\limits_{B_s(x_0)\setminus B_t(x_0)} |D^m \psi|^{p-1} |D^m \varphi| d\mathcal{L}^n$$

$$\leqslant 2^p \Bigg(\int\limits_{B_s(x_0)\setminus B_t(x_0)} |D^p(u-P)|^p d\mathcal{L}^n + \int\limits_{B_s(x_0)\setminus B_t(x_0)} \mathrm{LOT}^p d\mathcal{L}^n \Bigg).$$

Inserted in (26) this yields the estimate

$$I \leqslant C_I \Bigg(\int\limits_{B_s(x_0)\setminus B_t(x_0)} [|D^m(u-P)|^2 + |D^m(u-P)|^p] d\mathcal{L}^n + \int\limits_{B_s(x_0)} [\mathrm{LOT}^2 + \mathrm{LOT}^p] d\mathcal{L}^n \Bigg)$$

where $C_I = C_I(p, L, M) = 4^{p-2} L \quad (1 + M^{p-2})(2^p + 1)$.

Next we estimate II, using **(H2)**, (21), Hölder's and Young's inequality for $\varepsilon > 0$ arbitary:

$$II \leqslant \int\limits_{B_s(x_0)} \omega(|x - x_0|)(1 + |D^m u|^{p-1})|D^m \varphi| d\mathcal{L}^n$$

$$\leqslant 2^{p-2}(1 + M^{p-1})\, \omega(s) \int\limits_{B_s(x_0)} (|D^m(u-P)|^{p-1} |D^m \varphi| + |D^m \varphi|)\, d\mathcal{L}^n$$

$$\leqslant 2^{p-2}(1 + M^{p-1})\, \omega(s) \Bigg[\int\limits_{B_s(x_0)} |D^m(u-P)|^p d\mathcal{L}^n$$

$$+ \int\limits_{B_s(x_0)} |D^m(u-P)|^{p-1} \cdot \mathrm{LOT}\, d\mathcal{L}^n + \int\limits_{B_s(x_0)} \left[|D^m(u-P)| + \mathrm{LOT}\right] d\mathcal{L}^n \Bigg]$$

$$\leqslant C_{II}(\omega(s)+\varepsilon)\Bigg[\int\limits_{B_s(x_0)} \left[|D^m(u-P)|^2 + |D^m(u-P)|^p\right] d\mathcal{L}^n + \int\limits_{B_s(x_0)} \left[\mathrm{LOT}^2 + \mathrm{LOT}^p\right] d\mathcal{L}^n\Bigg]$$

$$+ \frac{C_{II}}{\varepsilon}\omega^2(s)\, \alpha_n s^n,$$

where $C_{II} = C_{II}(p, M) = 2^{p-1}(1 + M^{p-1})$ and α_n is the volume of the unit ball in $\mathbb{R}^n$. Since $D^m\varphi = D^m(u-P)$ on $B_t(x_0)$ we find by (24), $s \leqslant \varrho$ and the estimates on I and II:

$$\lambda \int\limits_{B_t(x_0)} \left[|D^m(u-P)|^2 + |D^m(u-P)|^p\right] d\mathcal{L}^n \leqslant$$

$$C_I \int\limits_{B_s(x_0)\setminus B_t(x_0)} \left[|D^m(u-P)|^2 + |D^m(u-P)|^p\right] d\mathcal{L}^n +$$

$$C_{II}(\omega(s)+\varepsilon)\Bigg[\int\limits_{B_s(x_0)} \left[|D^m(u-P)|^2 + |D^m(u-P)|^p\right] d\mathcal{L}^n\Bigg] +$$

$$(C_I + C_{II}(\omega(s)+\varepsilon)) \int\limits_{B_s(x_0)} \left[\mathrm{LOT}^2 + \mathrm{LOT}^p\right] d\mathcal{L}^n + \frac{C_{II}}{\varepsilon}\alpha_n \varrho^n \omega^2(\varrho).$$

«Filling the hole» on the right-hand side and choosing ϱ and ε sufficiently small with $C_{II}\,\omega(\varrho) \leqslant \lambda/4$ and $C_{II}\,\varepsilon = \lambda/4$ — this fixes $\varrho_0 = \varrho_0(\omega, \lambda, p, M)$ — we derive

$$\int\limits_{B_t(x_0)} \left[|D^m(u-P)|^2 + |D^m(u-P)|^p\right] d\mathcal{L}^n \leqslant$$

$$\frac{C_I + \lambda/2}{C_I + \lambda} \int\limits_{B_s(x_0)} \left(|D^m(u-P)|^2 + |D^m(u-P)|^p\right) d\mathcal{L}^n +$$

$$\frac{C_I + \lambda/2}{C_I + \lambda} \int\limits_{B_s(x_0)} \left[\mathrm{LOT}^2 + \mathrm{LOT}^p\right] d\mathcal{L}^n + \frac{C_{II}\,\alpha_n}{\varepsilon(C_I + \lambda)} \varrho^n \omega^2(\varrho).$$

Using (22) and (23), Lemma 2 and Poincaré's inequality (9) we obtain the desired result after taking integral mean values. The dependence of C_{cac} on L, λ, p, m, n, N and M follows from (22), (23), the dependences of C_I, C_{II}, C_{poin} in (9), and C_{tech} in Lemma 2 on $\vartheta = \frac{C_I + \lambda/2}{C_I + \lambda}$. Note, that $M \to \infty$ implies $\vartheta \to 1$ and $C_{\mathrm{tech}} \to \infty$. ■

5. – The harmonic approximation lemma.

The result of this section, the $(\mathcal{B}, m)$-harmonic approximation lemma, is central to our technique. In the case $m = 1$ the result was given in [DS] Lemma 3.3 (cf. [S], Section 1.6 for the case of Laplace's equation and harmonic approximation). The point of this technique is to show that for a bilinear form $\mathcal{B}$, which is elliptic in the sense of Legendre-Hadamard (compare Remark 1 (3)), a function v which is «approximately $(\mathcal{B}, m)$-harmonic» — that means $\int_\Omega \mathcal{B}(D^m v, D^m \varphi)\, d\mathcal{L}^n$ is sufficiently small for all test function φ — lies $W^{m-1,2}$-close to some $(\mathcal{B}, m)$-harmonic function h — that means a function h with $\int_\Omega \mathcal{B}(D^m h, D^m \varphi)\, d\mathcal{L}^n = 0$ for all test functions.

LEMMA 4 ($(\mathcal{B}, m)$-harmonic approximation lemma). – *For any given $\varepsilon > 0$ there exists $\delta = \delta(n, N, \lambda, L, m, \varepsilon) \in]0, 1]$ with the following property: for any given $\mathcal{B} \in \odot^2(\odot^m(\mathbb{R}^n, \mathbb{R}^N), \mathbb{R})$ satisfying*

$$\int_\Omega \mathcal{B}(D^m w, D^m w)\, d\mathcal{L}^n \geq \lambda \int_\Omega |D^m w|^2\, d\mathcal{L}^n \ \textit{for all}\ w \in W_0^{m,2}(\Omega, \mathbb{R}^N) \tag{27}$$

and

$$\mathcal{B}(A, B) \leq L|A|\,|B| \ \textit{for all}\ A, B \in \odot^m(\mathbb{R}^n, \mathbb{R}^N), \tag{28}$$

and for any $v \in W^{m,2}(B_\varrho(x_0), \mathbb{R}^N)$ satisfying $⨍_{B_\varrho(x_0)} |D^m v|^2 \leq 1$ and

$$\left| ⨍_{B_\varrho(x_0)} \mathcal{B}(D^m v, D^m \varphi)\, d\mathcal{L}^n \right| \leq \delta \sup_{B_\varrho(x_0)} |D^m \varphi| \ \textit{for all}\ \varphi \in C_0^\infty(B_\varrho(x_0), \mathbb{R}^N),$$

there exists a function $h \in W^{m,2}(B_\varrho(x_0), \mathbb{R}^N)$ such that $⨍_{B_\varrho(x_0)} |D^m h|^2\, d\mathcal{L}^n \leq 1$, $⨍_{B_\varrho(x_0)} \mathcal{B}(D^m h, D^m \varphi)\, d\mathcal{L}^n = 0$ for all $\varphi \in C_0^m(B_\varrho(x_0), \mathbb{R}^N)$ and

$$\varrho^{-2} ⨍_{B_\varrho(x_0)} \sum_{\gamma=0}^{m-1} |D^\gamma (h - v)|^2\, d\mathcal{L}^n \leq \varepsilon\,. \qquad \blacksquare$$

For the proof of this Lemma 4 we refer to [K], Lemma 6. The next result is a standard estimate for solutions of systems with constant coefficients (see [C3], [C4]).

LEMMA 5. – *Consider $\mathcal{B} \in \odot^2(\odot^m(\mathbb{R}^n, \mathbb{R}^N), \mathbb{R})$ satisfying* (27) *and* (28) *and $h \in W^{m,2}(B_\varrho(x_0), \mathbb{R}^N)$ with $\int_{B_\varrho(x_0)} \mathcal{B}(D^m h, D^m \varphi)\, d\mathcal{L}^n = 0$ for all $\varphi \in C_0^\infty(B_\varrho(x_0), \mathbb{R}^N)$. Then there exists a constant $C_{\mathrm{harm}} = C_{\mathrm{harm}}(n, N, m, \lambda, \Lambda)$ such that the following estimate holds:*

$$\varrho^{-2} \sup_{B_{\varrho/2}(x_0)} |D^m h|^2 + \sup_{B_{\varrho/2}(x_0)} |D^{m+1} h|^2 \leq C_{\mathrm{harm}}^2\, \varrho^{-2} ⨍_{B_\varrho(x_0)} |D^m h|^2\, d\mathcal{L}^n. \qquad \blacksquare$$

The last lemma in this section is required in order to be able to apply the $(\mathcal{B}, m)$-harmonic approximation technique. In the proof of Theorem 1, an application of this lemma will show that a in terms of Caccioppoli's inequality rescaled solution of (1) is approximately harmonic in the sense of Lemma 4. Then, the standard a priori estimates Lemma 5 will be used to derive estimates for an excess-decay term.

LEMMA 6. – *Let $M > 0$ be a constant and let $\mathcal{A}$ satisfy hypotheses* **(H2)** *and* **(H3)**. *Then there exists a constant $C_{\mathrm{eu}} = C_{\mathrm{eu}}(p, L, M) \geq 1$ such that the following holds: If $u \in W^{m,p}(\Omega, \mathbb{R}^N)$ is a weak solution of equation* (1) *and if P is a polynomial of degree at most m fullfilling $\fint_{B_\varrho(x_0)} D^l(u-P)\, d\mathcal{L}^n = 0$ for $l = 0, \ldots, m-1$, $\Phi(x_0, \varrho, P) \leq 1$ and $|D^m P| \leq M$, then* [1]

$$\left| \fint_{B_\varrho(x_0)} D_Q \mathcal{A}(x_0, D^m P)(D^m(u-P), D^m \varphi)\, d\mathcal{L}^n \right| \leq$$

$$C_{\mathrm{eu}} \big(\nu_M(\Phi(x_0, \varrho, P))^{1/p} \Phi(x_0, \varrho, P) + \omega(\varrho) \big) \sup_{B_\varrho(x_0)} |D^m \varphi|$$

for all $B_\varrho(x_0) \subset\subset \Omega$ and $\varphi \in C_0^\infty(B_\varrho(x_0), \mathbb{R}^N)$. (The constant L is from hypothesis **(H3)** *and ν_M is from Remark* 1 (2). *$\Phi(x_0, \varrho, P)$ is defined in* (17).)

PROOF. – We may assume $\sup_{B_\varrho(x_0)} |D^m \varphi| \leq 1$ and abbreviate $\Phi = \Phi(x_0, \varrho, P)$. Since u is a weak solution of equation (1) and given the fact that $\int_{B_\varrho(x_0)} \mathcal{A}(x_0, D^m P)\, D^m \varphi\, d\mathcal{L}^n = 0$ we infer

$$\fint_{B_\varrho(x_0)} D_Q \mathcal{A}(x_0, D^m P)(D^m(u-P), D^m \varphi)\, d\mathcal{L}^n$$

$$= \fint_{B_\varrho(x_0)} \int_0^1 [D_Q \mathcal{A}(x_0, D^m P) - D_Q \mathcal{A}(x_0, D^m P + t D^m(u-P))]\, dt (D^m(u-P), D^m \varphi)\, d\mathcal{L}^n$$

$$+ \fint_{B_\varrho(x_0)} [\mathcal{A}(x_0, D^m u) - \mathcal{A}(x, D^m u)]\, D^m \varphi\, d\mathcal{L}^n$$

$$= I + II\,.$$

[1] Here we use the notation $D_Q \mathcal{A}(x_0, D^m P)(A, B) := D_Q \mathcal{A}(x_0, D^m P)\, A \cdot B$, where «$\cdot$» is the dot product in $\odot^m(\mathbb{R}^n, \mathbb{R}^N) \cong \mathbb{R}^{N\binom{n+m-1}{m}}$, which regards $D_Q \mathcal{A}(x_0, D^m P)$ as a symmetric bilinar form on $\odot^m(\mathbb{R}^n, \mathbb{R}^N)$.

For $0 \leqslant t \leqslant 1$, using in turn **(H3)**, (25) and $|D^m P| \leqslant M$, we see

$$|D_Q \mathcal{A}(x_0, D^m P) - D_Q \mathcal{A}(x_0, D^m P + t D^m(u - P))| \leqslant$$
$$2^{p-1} L \big(1 + M^{p-2} + |D^m(u - P)|^{p-2}\big)$$

as well as, via (5), (25) and $|D^m P| \leqslant M$,

$$|D_Q \mathcal{A}(x_0, D^m P) - D_Q \mathcal{A}(x_0, D^m P + t D^m(u - P))| \leqslant$$
$$2^{p-1}\big(1 + M^{p-2} + |D^m(u-P)|^{p-2}\big)\, \nu_M(|D^m(u-P)|).$$

This gives the estimate

$$I \leqslant 2^{p-1} L^{\frac{p-1}{p}} (1 + M^{p-2}) \fint_{B_\varrho(x_0)} \nu_M(|D^m(u-P)|)^{1/p} \cdot$$
$$\big(|D^m(u-P)| + |D^m(u-P)|^{p-1}\big)\, d\mathcal{L}^n.$$

Therefore Hölder's, Jensen's and Minkowski's inequalities together with $\dfrac{p}{p-1} \leqslant 2$ and $\Phi \leqslant 1$ imply

$$I \leqslant 2^p L^{\frac{p-1}{p}} (1 + M^{p-2})\, \nu_M(\Phi)^{1/p}\, \Phi\,. \tag{29}$$

Next we estimate II using **(H2)**, $|D^m P| \leqslant M$, Hölder's inequality and $\Phi \leqslant 1$:

$$\begin{aligned} II &\leqslant \fint_{B_\varrho(x_0)} |\omega(|x - x_0|)(1 + |D^m u|^{p-1})|\, d\mathcal{L}^n \\ &\leqslant 2^{p-2}(1 + M^{p-1})\, \omega(\varrho)(1 + \Phi) \\ &\leqslant 2^{p-1}(1 + M^{p-1})\, \omega(\varrho). \end{aligned} \tag{30}$$

Combining (29), (30) we obtain the desired estimate with $C_{\mathrm{eu}} = \max\big(2^p L^{\frac{p-1}{p}}(1 + M^{p-2}),\ 2^{p-1}(1 + M^{p-1})\big)$. ■

6. – Proof of the theorem.

Let $u \in W^{m,p}(\Omega, \mathbb{R}^N)$ be a weak solution of (1), where $\mathcal{A}$ satisfies hypotheses **(H1)**, **(H2)** and **(H3)**. For a given Ball $B_\varrho(x_0)$ choose the unique polynomial $P_{x_0, \varrho}$ of degree at most m specified in (15). Let $\Phi(\varrho/2) := \Phi(x_0, \varrho/2, P_{x_0, \varrho})$ and $\Psi(\varrho) := \Psi(x_0, \varrho, P_{x_0, \varrho})$ where Φ and Ψ are defined in (17) respectively (18). First we prove the following decay estimate:

LEMMA 7. – *Let $M > 0$ fixed. For any $\vartheta \in]0, 1/4]$ there exist constants $\delta = \delta(n, N, \lambda, L, m, \vartheta)$, $q = q(n, p)$ and $C_{\text{dec}}(M) = C_{\text{dec}}(n, N, \lambda, L, m, M, p)$ such that if $\nu_M(\Phi(\varrho/2))^{1/p} \leq \delta/2$, $\Psi^2(\varrho) + \dfrac{4\omega^2(\varrho)}{\delta^2} \leq \vartheta^q$, $|D^m P_{x_0, \varrho}| \leq M$ and $\varrho \leq \varrho_0(M)$ then*

$$\Psi^2(\vartheta\varrho) \leq C_{\text{dec}}^2(M)\, \vartheta^2 \left(\Psi^2(\varrho) + \frac{4\omega^2(\varrho)}{\delta^2} \right).$$

Here n, N, λ, L, m are as in hypotheses **(H1)** *and* **(H3)**, *ω is as in* **(H2)**, *ν_M is the modulus of continuity for $D_Q \mathcal{a}$ from Remark* 1 (2). *The constant $\varrho_0(M) = \varrho_0(\omega, \lambda, p, M) \leq 1$ is the constant from Lemma* 3.

PROOF. – For $\vartheta \in]0, 1/4]$ choose $\varepsilon = \vartheta^{n+4}$ and $\delta = \delta(n, N, \lambda, L, m, \varepsilon) = \delta(n, N, \lambda, L, m, \vartheta) \leq 1$ from Lemma 4. We define

$$\Gamma(\varrho) := \sqrt{\Psi^2(\varrho) + \frac{4\omega^2(\varrho)}{\delta^2}} \quad \text{and set} \quad v = \frac{u - P_{x_0, \varrho}}{C_{\text{cac}} C_{\text{eu}} \Gamma(\varrho)}.$$

Here the constants $C_{\text{cac}} = C_{\text{cac}}(M)$ and $C_{\text{eu}} = C_{\text{eu}}(M)$ are from Lemma 3 and Lemma 6. The assumption $\varrho \leq \varrho_0(M)$ and the choice of $P = P_{x_0, \varrho}$ allows us to apply Lemma 3, i.e. (19), and Lemma 6. In view of $\delta \leq 1$ we get from (19):

$$\Phi^2(\varrho/2) \leq C_{\text{cac}}^2(\Psi^2(\varrho) + \omega^2(\varrho)) \leq C_{\text{cac}}^2 \Gamma^2(\varrho). \tag{31}$$

This leads to $\displaystyle\fint_{B_{\varrho/2}(x_0)} |D^m v|^2 \, d\mathcal{L}^n \leq C_{\text{eu}}^{-2} \leq 1$ and

$$\left| \fint_{B_{\varrho/2}(x_0)} D_Q \mathcal{a}(x_0, D^m P_{x_0, \varrho})(D^m v, D^m \varphi)\, d\mathcal{L}^n \right| \leq \left(\nu_M(\Phi(\varrho/2))^{1/p} + \delta/2\right) \sup_{B_{\varrho/2}(x_0)} |D^m \varphi|.$$

By our smallness condition $(\nu_M(\Phi(\varrho/2))^{1/p} \leq \delta/2$ the assumptions of Lemma 4 are fulfilled. This implies that there exists a $(D_Q \mathcal{a}(x_0, D^m P_{x_0, \varrho}|, m)$-harmonic function h with

$$\fint_{B_{\varrho/2}(x_0)} D_Q \mathcal{a}(x_0, D^m P_{x_0, \varrho})(D^m h, D^m \varphi)\, d\mathcal{L}^n = 0, \quad \fint_{B_{\varrho/2}(x_0)} |D^m h|^2 \, d\mathcal{L}^n \leq 1 \tag{32}$$

and

$$\left(\frac{\varrho}{2}\right)^{-2} \fint_{B_{\varrho/2}(x_0)} |D^{m-1} v - D^{m-1} h|^2 \, d\mathcal{L}^n \leq \varepsilon = \vartheta^{n+4}. \tag{33}$$

The standard a priori estimate (cf.Lemma 5) for solutions of the linear system (32) applied to h on $B_{\varrho/2}(x_0)$ yields

$$\left(\frac{\varrho}{2}\right)^{-2} \sup_{B_{\varrho/4}(x_0)} |D^m h|^2 + \sup_{B_{\varrho/4}(x_0)} |D^{m+1} h|^2 \leq$$

$$C_{\text{harm}}^2 \left(\frac{\varrho}{2}\right)^{-2} \fint_{B_{\varrho/2}(x_0)} |D^m h|^2 \, d\mathcal{L}^n \leq C_{\text{harm}}^2 (\varrho/2)^{-2}$$

where the last inequality follows from (32). Applying Taylor's theorem to h on $B_{\vartheta\varrho}(x_0)$ we deduce

$$\sup_{B_{\vartheta\varrho}(x_0)} |D^{m-1}h(x) - D^{m-1}h(x_0) - D^m h(x_0)(x-x_0)|^2 \leqslant$$

$$\left(\frac{(\vartheta\varrho)^2}{2!}\sup_{B_{\varrho/4}(x_0)}|D^{m+1}h|\right)^2 \leqslant C_{\mathrm{harm}}^2\vartheta^4\varrho^2.$$

Denoting by $P_{x_0,\,\vartheta\varrho}$ the unique polynomial associated to u on $B_{\vartheta\varrho}(x_0)$, we infer using the minimal property (15) of $P_{x_0,\,\vartheta\varrho}$, (33) and the above mentioned a priori estimate for h:

$$\left(\fint_{B_{\vartheta\varrho}(x_0)} |D^{m-1}(u-P_{x_0,\,\vartheta\varrho})|^2\,d\mathcal{L}^n\right)^{1/2} \leqslant$$

$$\left(\fint_{B_{\vartheta\varrho}(x_0)} |D^{m-1}u - D^{m-1}P_{x_0,\varrho} - C_{\mathrm{cac}}C_{\mathrm{eu}}\Gamma(\varrho)(D^{m-1}h(x_0) + D^m h(x_0)(x-x_0))|^2\,d\mathcal{L}^n\right)^{1/2} \leqslant$$

$$\left(2^{-\frac{n+2}{2}} + C_{\mathrm{harm}}\right)\vartheta^2\varrho C_{\mathrm{cac}}C_{\mathrm{eu}}\Gamma(\varrho)$$

respectively with $C_1 = \left(2^{-\frac{n+2}{2}} + C_{\mathrm{harm}}\right)C_{\mathrm{cac}}C_{\mathrm{eu}}$

$$\fint_{B_{\vartheta\varrho}(x_0)} |D^{m-1}(u-P_{\vartheta x_0,\,\varrho})|^2\,d\mathcal{L}^n \leqslant C_1^2(\vartheta^2\varrho)^2\Gamma^2(\varrho). \tag{34}$$

Next we derive an estimate for the second term

$$\fint_{B_{\vartheta\varrho}(x_0)} |D^{m-1}(u-P_{x_0,\,\vartheta\varrho})|^p\,d\mathcal{L}^n$$

in the definition of $\Psi(\vartheta\varrho)$. For this we let (in the case $p>2$)

$$p^* = \begin{cases} \dfrac{np}{n-p} > p & \text{the Sobolev conjugateto } p \text{ in the case } 2<p<n \\ \quad p^*>p & \text{fixed in the case } p \geqslant n \end{cases}$$

with $\frac{1}{2} > \frac{1}{p} > \frac{1}{p^*}$. Therefore we can find $t \in]0,1[$ such that

$$\frac{1}{p} = (1-t)\frac{1}{2} + t\frac{1}{p^*}. \tag{35}$$

Using Hölder's inequality, Sobolev's inequality (11) and (34) we compute:

$$(36)\qquad \fint_{B_{\vartheta\varrho}(x_0)} |D^{m-1}(u-P_{x_0,\vartheta\varrho})|^p \, d\mathcal{L}^n \leqslant$$

$$\left(\fint_{B_{\vartheta\varrho}(x_0)} |D^{m-1}(u-P_{x_0,\vartheta\varrho})|^2 \, d\mathcal{L}^n\right)^{(1-t)\frac{p}{2}} \left(\fint_{B_{\vartheta\varrho}(x_0)} |D^{m-1}(u-P_{x_0,\vartheta\varrho})|^{p^*} \, d\mathcal{L}^n\right)^{t\frac{p}{p^*}} \leqslant$$

$$\left(C_1^2(\vartheta^2\varrho)^2\Gamma^2(\varrho)\right)^{(1-t)p/2} C_{\mathrm{sob}}^{tp}(\vartheta\varrho)^{tp}\left(\int_{B_{\vartheta\varrho}(x_0)} |D^m(u-P_{x_0,\vartheta\varrho})|^p \, d\mathcal{L}^n\right)^t =$$

$$C_2^p \mathcal{L}^n(B_{\vartheta\varrho}(x_0))^{-t}(\vartheta\varrho)^p \vartheta^{(1-t)p}\Gamma(\varrho)^{(1-t)p}\left(\int_{B_{\vartheta\varrho}(x_0)} |D^m(u-P_{x_0,\vartheta\varrho})|^p \, d\mathcal{L}^n\right)^t$$

where $C_2 = C_1^{(1-t)} C_{\mathrm{sob}}^t = C_2(n, N, \lambda, L, m, M, p, p^*)$. From Lemma 1 (i), Caccioppoli's inequality (19) and (31) we deduce

$$\left(\int_{B_{\vartheta\varrho}(x_0)} |D^m(u-P_{x_0,\vartheta\varrho})|^p \, d\mathcal{L}^n\right)^{1/p} \leqslant$$

$$\left(\int_{B_{\vartheta\varrho}(x_0)} |D^m(u-P_{x_0,\varrho})|^p \, d\mathcal{L}^n\right)^{1/p} + \mathcal{L}^n(B_{\vartheta\varrho}(x_0))^{1/p}\,|D^m P_{x_0,\vartheta\varrho} - D^m P_{x_0,\varrho}| \leqslant$$

$$\left(\mathcal{L}^n(B_{\varrho/2}(x_0))\,\Phi^2(\varrho/2)\right)^{1/p} + \sqrt{n(n+2)}\,(\vartheta\varrho)^{-1}\left(\int_{B_\varrho(x_0)} |D^{m-1}(u-P_{x_0,\varrho})|^p d\mathcal{L}^n\right)^{1/p} \leqslant$$

$$\mathcal{L}^n(B_\varrho(x_0))^{1/p}\vartheta^{-1}\left((2^{-n}C_{\mathrm{cac}}^2)^{1/p} + \sqrt{n(n+2)}\right)\Gamma(\varrho)^{2/p}.$$

From this estimate, inserted in inequality (36) and Young's inequality we obtain, letting $C_3^2 = C_2^p\,((2^{-n}C_{\mathrm{cac}}^2)^{1/p} + \sqrt{n(n+2)})^{tp} = C_3^2(n, N, \lambda, L, m, M, p, p^*)$

$$(37)\qquad (\vartheta\varrho)^{-p}\fint_{B_{\vartheta\varrho}(x_0)} |D^{m-1}(u-P_{x_0,\varrho})|^p d\mathcal{L}^n \leqslant C_3^2\vartheta^{(1-t)p}\Gamma(\varrho)^{(1-t)p}\vartheta^{-(n+p)t}\Gamma(\varrho)^{2t} \leqslant$$

$$C_3^2\left(\frac{(1-t)p}{2}\vartheta^2\Gamma^2(\varrho) + \frac{tp}{p^*}\vartheta^{-\frac{(n+p)p^*}{p}}\Gamma(\varrho)^{2p^*/p}\right) \leqslant C_3^2\vartheta^2\Gamma^2(\varrho),$$

provided $\Gamma(\varrho)^{2(p^*-p)/p} \leqslant \vartheta^{(n+p)p^*/p+2}$. If we choose $q = \dfrac{(n+p)\,p^* + 2p}{p^* - p}$ the desired estimate follows from (34) and (37) with $C_{\mathrm{dec}}^2 = (C_1^2 + C_3^2)$. Note that $C_{\mathrm{dec}} = C_{\mathrm{dec}}(M)$. ∎

Given $M > 0$ and $0 < \beta < 1$ from hypotheses **(H2)** we choose $1 > \alpha > \beta$ and fix $\vartheta \leqslant 1/4$ sufficiently small such that

$$C_{\mathrm{dec}}(2M)\vartheta \leqslant \vartheta^{\alpha}.$$

This also fixes $\varepsilon = \vartheta^{n+4}$ and $\delta = \delta(n, N, \lambda, L, m, \vartheta)$. Then we choose $S > 0$ and $R > 0$ (depending on $2M$ also) such that $\nu_{2M}(C_{\mathrm{cac}}(2S + R))^{1/p} \leqslant \delta/2$ and $4S^2 + \dfrac{4R^2}{\delta^2} \leqslant \vartheta^q$. Then, by Caccioppoli's inequality (19), if for some ball $B_\varrho(x_0) \subset\subset \Omega$ the conditions

$$\Psi(\varrho) \leqslant 2S, \quad \omega(\varrho) \leqslant R, \quad |D^m P_{x_0, \varrho}| \leqslant 2M \quad \text{and} \quad \varrho \leqslant \varrho_0(2M) \tag{38}$$

are satisfied, we have the decay estimate

$$\Psi^2(\vartheta\varrho) \leqslant \vartheta^{2\alpha}\Psi^2(\varrho) + C_\omega^2(2M)\,\omega^2(\varrho), \tag{39}$$

where $C_\omega = C_\omega(n, N, \lambda, L, m, 2M, p, \alpha) = 2\vartheta^\alpha/\delta$ depends on $C_{\mathrm{dec}}(2M)$ and δ.

LEMMA 8. – *Let $M > 0$ fixed. Suppose $\Psi(\varrho) \leqslant S$, $\omega(\varrho) \leqslant R$, $\varrho \leqslant \varrho_0(2M)$ and $|D^m P_{x_0, \varrho}| \leqslant M$ for some ball $B_\varrho(x_0) \subset\subset \Omega$. Then there exist constants $C_{it}(2M) = C_{it}(n, N, \lambda, L, m, 2M, p, \alpha, \beta)$ and $C_{\wp}(2M) = C_{\wp}(n, N, \lambda, L, m, 2M, p, \alpha, \beta)$ such that $C_{it}(2M)\omega(\varrho) \leqslant S$ and $C_{\wp}(2M)(\Psi(\varrho) + \wp(\varrho)) \leqslant M$ imply*

$$\Psi^2(\vartheta^k\varrho) \leqslant \vartheta^{2\alpha k}\Psi^2(\varrho) + C_{it}^2(2M)\,\omega^2(\vartheta^k\varrho)$$

for all $k \in \mathbb{N}$. Here $\wp$ is from hypothesis **(H2)**.

PROOF. – For $k = 1$ this follows from the decay estimate (39). Suppose that the conditions (38) of the decay estimate (39) are fulfilled on the balls $B_{\vartheta^j\varrho}(x_0)$ for $j \leqslant k - 1 \in \mathbb{N}_0$. Firstly, we will show that this implies that these conditions are also fulfilled on the ball $B_{\vartheta^k\varrho}(x_0)$. Therefore it remains to show that $\Psi(\vartheta^k\varrho) \leqslant 2S$ and $|D^m P_{x_0, \vartheta^k\varrho}| \leqslant 2M$. In view of hypotheses **(H2)** we know that $s \mapsto s^{-\beta}\omega(s)$ is nonincreasing for some $\beta < 1$ (note also $\beta < \alpha < 1$).

$$\begin{aligned}
\Psi^2(\vartheta^k\varrho) &\leqslant \vartheta^{2\alpha k}\Psi^2(\varrho) + C_\omega^2(2M)\sum_{j=0}^{k-1}\vartheta^{2\alpha j}\omega^2(\vartheta^{(k-j-1)}\varrho) \\
&\leqslant \vartheta^{2\alpha k}\Psi^2(\varrho) + C_\omega^2(2M)\,\omega^2(\vartheta^k\varrho)\,\vartheta^{-2\beta}\sum_{j=0}^{k-1}\vartheta^{2(\alpha-\beta)j} \\
&\leqslant \vartheta^{2\alpha k}\Psi^2(\varrho) + C_{it}^2(2M)\,\omega^2(\vartheta^k\varrho)
\end{aligned} \tag{40}$$

where $C_{it}(2M) = \dfrac{C_\omega(2M)}{\sqrt{\vartheta^{2\alpha} - \vartheta^{2\beta}}}$. By our assumptions, this implies $\Psi(\vartheta^k\varrho) \leqslant \vartheta^{\alpha k}\Psi(\varrho) + C_{it}(2M)\,\omega(\vartheta^k\varrho) \leqslant 2S$.

From (16), Lemma 1 (i) and (7) we infer

$$\begin{aligned} |D^m P_{x_0,\,\vartheta^k\varrho}| &\leq M + \sum_{j=0}^{k-1} |D^m P_{x_0,\,\vartheta^{j+1}\varrho} - D^m P_{x_0,\,\vartheta^j\varrho}| \\ &\leq M + \sqrt{n(n+2)}\,\vartheta^{-(n+2)/2} \sum_{j=0}^{k-1} \Psi(\vartheta^j\varrho) \\ &\leq M + \sqrt{n(n+2)}\,\vartheta^{-(n+2)/2} \left(\Psi(\varrho) \sum_{j=0}^{k-1} \vartheta^{\alpha j} + C_{it}(2M) \sum_{j=0}^{k-1} \omega(\vartheta^j\varrho) \right) \\ &\leq M + C_{\mathfrak{W}}(2M)(\Psi(\varrho) + \mathfrak{W}(\varrho)) \leq 2M\,, \end{aligned}$$

provided $C_{\mathfrak{W}}(2M)(\Psi(\varrho) + \mathfrak{W}(\varrho)) \leq M$. Here

$$C_{\mathfrak{W}}(2M) = \sqrt{n(n+2)}\,\vartheta^{-(n+2)/2} \left(\frac{1}{1-\vartheta^\alpha} - \frac{C_{it}(2M)}{\vartheta \log(\vartheta)} \right).$$

We have demonstrated that our smallness conditions (38) are fulfilled on the ball $B_{\theta^k\varrho}(x_0)$. Hence we conclude that (40) holds with $k+1$ instead of k. ■

For $0 < r < \varrho$ fix $k \in \mathbb{N}_0$ with $\vartheta^{k+1}\varrho < r \leq \vartheta^k\varrho$. Then, if the assumptions of Lemma 8 are fulfilled, we deduce, using (15) and $\omega(s\varrho) \leq s\omega(\varrho)$ for $s \geq 1$

$$\begin{aligned} \fint_{B_r(x_0)} |D^{m-1}(u - P_{x_0,r})|^2 d\mathcal{L}^n &\leq \frac{(\vartheta^k\varrho)^n}{r^n} (\vartheta^k\varrho)^2 \Psi^2(\vartheta^k\varrho) \\ &\leq \vartheta^{-(n+2+2\alpha)} \left(\frac{r}{\varrho}\right)^{2\alpha} r^2 \Psi^2(\varrho) + \vartheta^{-(n+4)} C_{it}^2(2M)\, r^2 \omega(r)\,. \end{aligned}$$

respectively

$$r^{-2} \fint_{B_r(x_0)} |D^{m-1}(u - P_{x_0,\,r})|^2 d\mathcal{L}^n \leq \vartheta^{-(n+4)} \left(\left(\frac{r}{\varrho}\right)^{2\alpha} \Psi^2(\varrho) + C_{it}^2(2M)\, \omega^2(r) \right). \tag{41}$$

To show that the assumptions of Lemma 8 are satisfied locally for all x_0 in the regular set $\Omega \backslash (\Sigma_1 \cup \Sigma_2)$ we made the following observations: From Lemma 1 (ii), Hölder's inequality, (15) and (9) we infer

$$|D^m P_{x_0,\varrho}| \leq |(D^m u)_{x_0,\varrho}| + C_{\text{poin}} \sqrt{n(n+2)} \left(\fint_{B_\varrho(x_0)} |D^m u - (D^m u)_{x_0,\varrho}|^p d\mathcal{L}^n \right)^{1/p},$$

$$\varrho^{-2} \fint_{B_\varrho(x_0)} |D^{m-1}(u - P_{x_0,\,\varrho})|^2 d\mathcal{L}^n \leq C_{\text{poin}}^2 \left(\fint_{B_\varrho(x_0)} |D^m u - (D^m u)_{x_0,\,\varrho}|^p d\mathcal{L}^n \right)^{2/p},$$

$$\varrho^{-p} \fint_{B_\varrho(x_0)} |D^{m-1}(u - P_{x_0,\varrho})|^p \, d\mathcal{L}^n \leqslant$$

$$C_{\text{poin}}^p (1 + C_{\text{poin}} \sqrt{n(n+2)})^p \fint_{B_\varrho(x_0)} |D^m u - (D^m u)_{x_0,\varrho}|^p \, d\mathcal{L}^n.$$

Using these estimates it is standard to show that (41) holds locally on the regular set. This implies the partial regularity statement of Theorem 1 by (14).

REFERENCES

[C1] S. Campanato, *Proprietà di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa **18** (1964), 137-160.

[C2] S. Campanato, *Teoremi di interpolazione per tranformazioni che applicano L^p in $C^{h,\alpha}$*, Ann. Scuola Norm. Sup. Pisa, **18** (1964), 345-360.

[C3] S. Campanato, *Equazioni ellitichi del II^e ordine e spazi $\mathcal{L}^{2,\lambda}$*, Ann. Mat. Pura Appl., **69** (1965), 321-381.

[C4] S. Campanato, *Alcune osservazioni relative alle soluzioni di equazioni ellittiche di ordine 2m*, Atti Convegno Equaz. Der. Parz., Bologna, 1967, 17-25.

[DS] F. Duzaar - K. Steffen, *Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals*, Preprint.

[DGG] F. Duzaar - A. Gastel - J. Grotowski, *Optimal partial regularity for nonlinear elliptic systems of higher order*, Preprint.

[E] L. C. Evans, *Quasiconvexitity and Partial Regularity in the Calculus of Variations*, Arch. Rat. Mech. Anal., **95** (1986), 227-252.

[F] M. Fuchs, *Regularity theorems for nonlinear systems of partial differential equations under natural ellipticity conditions*, Analysis **7** (1987), 83-93.

[Gi] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Basel-Boston-Berlin, Birkhäuser, 1993.

[Gu] M. Guidorzi, *A Remark on Partial Regularity of Minimizers of Quasiconvex Integrals of Higher Order*, Rend. Istit. Mat. Univ. Trieste, **32** (2000), 1-24.

[GM1] M. Giaquinta - G. Modica, *Almost-everywhere regularity results for solutions of nonlinear elliptic systems*, Manuscripta Math., **28** (1979), 109-158.

[GM2] M. Giaquinta - G. Modica, *Partial regularity of minimizers of quasiconvex integrals*, Ann. Inst. Henri Poincaré, Analyse non linéaire, **3** (1986), 185-208.

[H] C. Hamburger, *Quasimonotonicity, regularity and duality for nonlinear systems of partial differential equations*, Ann. Mat. Pura Appl., IV. Ser., **169** (1995), 321-354.

[K] M. Kronz, *Partial Regularity Results for Quasiconvex Functionals of Higher Order*, Ann. Inst. Henri Poincaré, Analyse non linéaire, **19** (2002), 81-112.

[M] N. G. Meyers, *Quasi-Convexity and Lower Semi-Continuity of Multiple Variational Integrals of Any Order*, Trans. Am. Math. Soc., **119** (1965), 125-149.

[S] L. SIMON, *Theorems on Regularity and Singularity of Energy Minimizing Maps*, Basel-Boston-Berlin, Birkhäuser, 1996.

[Z] ZHANG KE-WEI, *On the Dirichlet problem for a class of quasilinear elliptic systems of partial differential equations in divergence form*, Partial differential equations, Proc. Symp., Tianjin/China, 1986, Lect. Notes Math., **1306** (1988), 262-277.

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