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The p-Laplacian in Domains with Small Random Holes.

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Sunto. – *Attraverso un metodo variazionale, si studia un processo di omogeneizzazione relativo al p-Laplaciano in regioni perforate in maniera stocastica. Per particolari distribuzioni aleatorie dei «buchi» si caratterizza pienamente il problema limite.*

Summary. – *We investigate sequences of nonlinear Dirichlet problems of the form*

$$(P_h) \qquad \begin{cases} -\operatorname{div}(|Du_h|^{p-2}Du_h) = g & \text{in } D\backslash E_h \\ u_h \in H_0^{1,p}(D\backslash E_h). \end{cases}$$

where $2 \leqslant p \leqslant n$ and E_h are random subsets of a bounded open set D of $\boldsymbol{R}^n$ $(n \geqslant 2)$. By means of a variational approach, we study the asymptotic behaviour of solutions of (P_h), characterizing the limit problem for suitable sequences of random sets.

1. – Introduction.

A variational framework has been proposed in [2], for studying the asymptotic behaviour of sequences of nonlinear Dirichlet problems in randomly perforated domains of the form

$$\min_{u \in H_0^{1,p}(D\backslash E_h)} \int_{D\backslash E_h} f(x, Du)\, dx + \int_{D\backslash E_h} gu\, dx\,,$$

where (E_h) is a sequence of closed random subsets of a bounded open set $D \subseteq \boldsymbol{R}^n$, $n \geqslant 2$, $1 < p \leqslant n$ and $g \in L^q(D)$, with $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

In this paper, by using the abstract setting established in [2], we analyze the p-Laplacian operator

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$$

in domains with randomly distributed small holes, when p takes values in the interval [2,n]. More specifically, we deal with sequences of problems of the form

$$(1.1) \qquad \begin{cases} -\Delta_p u_h = g & \text{in } D\backslash E_h \\ u_h \in H_0^{1,p}(D\backslash E_h). \end{cases}$$

The problem (1.1) is the Euler equation of the minimization problem

$$\min_{u \in H_0^{1,p}(D\backslash E_h)} \int_{D\backslash E_h} |Du|^p \, dx - p \int_{D\backslash E_h} gu \, dx .$$

The probabilistic problem that we are going to consider can be rigorously stated as follows. Let β be a nonnegative finite Radon measure on D such that $\beta \in H^{-1,q}(D)$ and define

$$(1.2) \qquad \mathcal{E}_\beta^p(U) = \begin{cases} \displaystyle\int\int_{U \times U} \frac{d\beta(x)\, d\beta(y)}{|x-y|^{n-p}} & \text{if } \quad 2 \leqslant p < n \\ \displaystyle\int\int_{U \times U} \ln \frac{1}{|x-y|} d\beta(x)\, d\beta(y) & \text{if } \quad n = p \end{cases}$$

for every open set U of D.

We assume that there exists a strictly monotone and continuous function $f: R^+ \to R$ with $f(0) = 0$ such that

$$\mathcal{E}_\beta^p(U) \leqslant f(\operatorname{diam} U)\, \beta(U)$$

for every open set U of D. For every $h \in \boldsymbol{N}$, let

$$x_i^h : \Omega \to D, \qquad 1 \leqslant i \leqslant h,$$

be a family of independent, identically distributed random variables defined on a probability space (Ω, Σ, P), whose distributions are given by

$$P\{\omega \in \Omega : x_i^h \in B\} = \beta(B), \qquad 1 \leqslant i \leqslant h$$

for every Borel set $B \subseteq D$. Furthermore, we consider a sequence of positive numbers (ϱ_h) such that

$$(1.3) \qquad l = \begin{cases} \lim\limits_{h \to +\infty} h \varrho_h^{n-p} & \text{if } 2 \leqslant p < n \\ \lim\limits_{h \to +\infty} h(-\ln \varrho_h)^{1-n} & \text{if } p = n \end{cases}$$

is finite and strictly positive. Finally, we define

$$E_h = \bigcup_{i=1}^{h} (x_i^h + \varrho_h F)$$

where F is an arbitrary closed subset contained in the unit ball, such that the interior of F is not empty. We prove that the sequence (u_h) of weak solutions of (1.1) converges (strongly in $L^p(D)$) in probability to the solution of the relaxed Dirichlet problem

$$\begin{cases} -\Delta_p U + cl\beta |U|^{p-2} U = g & \text{in } D \\ U \in H_0^{1,p}(D) \end{cases}$$

where

$$c = \begin{cases} \min\left\{ \displaystyle\int_{\boldsymbol{R}^n} |Du|^p\, dx : u \in H^{1,p}(\boldsymbol{R}^n),\ u \geqslant 1 \ \ p\text{-q.e. on } F \right\} & \text{if } 2 \leqslant p < n \\ \omega_{n-1} & \text{if } p = n, \end{cases}$$

l is given by (1.3) and ω_{n-1} is the area of the unit sphere of $\boldsymbol{R}^n$.

In the linear stochastic case $p = 2$, the result is well-known. It has been investigated in [10], [11], [4] by Brownian motion methods, in [12], [7] by Green function methods, in [1], [3] by a variational method. To the best of our knowledge, any result exists on the p-Laplacian operator in randomly perforated domains with Dirichlet boundary conditions. Also the corresponding deterministic case has been analyzed by many authors; we refer, for a wide bibliografy on the subject, to [5]. Our paper is organized as follows. Section 2 provides the necessary preliminaries. In Section 3 we give the formulation of the problem and state the main result (Th. 3.5) of the paper. Section 4 is completely devoted to the proof of Theorem 3.5; some of the results in this section, in particular Lemma 4.2, may be of independent interest. In that Lemma we construct an explicit supersolution relative to the p-Laplacian in a perforated domain.

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2. – Notation and preliminaries.

Let D be a bounded open subset of $\boldsymbol{R}^n$ with diameter less than or equal to one. In all that follows we shall assume $n \geqslant 2$. We denote the family of all open sets $U \subseteq D$ by $\mathcal{U}$, the family of all compact sets $K \subseteq D$ by $\mathcal{K}$ and the family of all closed sets $F \subseteq D$ by $\mathcal{F}$. Moreover, we indicate the σ-field of all Borel subsets of

D by $\mathcal{B}$. For every $x \in \boldsymbol{R}^n$ and $r > 0$ we set

$$B_r(x) = \{y \in \boldsymbol{R}^n : |x - y| < r\},$$

and for every Borel set $B \subset \boldsymbol{R}^n$ we denote its Lebesgue measure by $|B|$. Moreover, for every set $E \subseteq \boldsymbol{R}^n$ and $x \in \boldsymbol{R}^n$ we set

$$x + E = \{y \in \boldsymbol{R}^n : x - y \in E\}.$$

The symbol $\#(I)$ indicates the number of elements of the set I.

Throughout this paper we shall indicate a real constant such that $2 \leqslant p \leqslant n$ by p. Further, we denote the Sobolev space of all functions in $L^p(D)$ with first order distributional derivatives in $L^p(D)$ by $H^{1,p}(D)$ and the closure of $C_0^\infty(D)$ in $H^{1,p}(D)$ by $H_0^{1,p}(D)$. For all q such that $\frac{1}{q} + \frac{1}{p} = 1$, we denote the dual of $H_0^{1,p}(D)$ by $H^{-1,q}(D)$. For every $K \in \mathcal{K}$, we define the p-capacity of K with respect to D by

$$C_p(K, D) = \inf\left\{ \int_D |D\varphi|^p \, dx : \varphi \in C_0^\infty(D), \ \varphi \geqslant 1 \ \text{ on } \ K\right\}.$$

The definition is extended to the sets $U \in \mathcal{U}$ by

$$C_p(U, D) = \sup\{C_p(K); \ K \subseteq U, \ K \in \mathcal{K}\}$$

and to arbitrary sets $E \subseteq D$ by

$$C_p(E, D) = \inf\{C_p(U); \ U \supseteq E, \ U \in \mathcal{U}\}.$$

The basic properties of the variational capacity so defined can be found, for example, in [9], Th. 2.2. We say that a property $P(x)$ holds for p-quasi every $x \in E$ (or p-quasi-everywhere in E) if

$$C_p(\{x \in E : P(x) \ \text{ is not verified}\}, D) = 0$$

Note that the property of being of p-capacity zero is independent of the open set D. It can be proven that there exists one and only one $u \in H_0^{1,p}(D)$ such that $u \geqslant 1$ p-quasi-everywhere on E such that

$$C_p(E, D) = \int_D |Du|^p \, dx.$$

We shall call such a u the *p-capacitary potential* of E with respect to D. The next Lemma is needed in order to identify a class of random sets. The proof can be obtained adapting to the case of the p-capacity that one of Lemma 4.1 in [1].

LEMMA 2.1. – *Let F be a closed set of $\boldsymbol{R}^n$. For every $K \in \mathcal{K}$ and $h \in \boldsymbol{N}$, the real-valued function*

$$(x_1, x_2, \ldots, x_h) \to C_p\left(\bigcup_{i=1}^{h} (x_i + F) \cap K, D\right)$$

is upper semicontinuous in $(\boldsymbol{R}^n)^h$.

A nonnegative countably additive set function μ defined on $\mathcal{B}$ and with value in $[0, +\infty]$ such that $\mu(\emptyset) = 0$ is called a *Borel measure* on D. A Borel measure which assigns finite value to every compact subset of D is called a *Radon measure.*

DEFINITION 2.2. – Let $\beta \in H^{-1,q}(D)$. In the following, we need the set function so defined

$$\mathcal{E}_\beta^p(A) = \begin{cases} \displaystyle\int\int_{A \times A} \frac{d\beta(x)\, d\beta(y)}{|x-y|^{n-p}} & \text{if } \; 2 \leqslant p < n \\ \displaystyle\int\int_{A \times A} \ln \frac{1}{|x-y|}\, d\beta(x)\, d\beta(y) & \text{if } \; n = p \end{cases}$$

for every $A \in \mathcal{U}$.

REMARK 2.3. – Let $\beta \in H^{-1,q}(D)$. Defining the measure σ on the Borel family of $D \times D$ by

$$\sigma(E) = \begin{cases} \displaystyle\iint_E \frac{d\beta(x)\, d\beta(y)}{|x-y|^{n-p}} & \text{if } 2 \leqslant p < n \\ \displaystyle\iint_E \ln \frac{1}{|x-y|}\, d\beta(x)\, d\beta(y) & \text{if } n = p \end{cases}$$

we can check (e.g. see Remark 5.1 in [3]) that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $E \subseteq D \times D$ with diam $E < \delta$ we have $\sigma(E) < \varepsilon$.

Let (Ω, Σ, P) be a probability space.

DEFINITION 2.4. – A function $F : \Omega \to \mathcal{F}$ is called a p-*random set* if the function

$$\omega \in \Omega \to C_p(F(\omega) \cap K) \in \boldsymbol{R}$$

is Σ-measurable for every $K \in \mathcal{K}$.

EXAMPLE 2.5. – In order to identify a class of random sets according to the previous definition, let us consider a family of vector-random variables, namely a family of Σ-measurable functions $x_i^h \colon \Omega \to D$, $h \in \boldsymbol{N}$, $1 \leq i \leq h$.

Let F be a closed set of $\boldsymbol{R}^n$ such that $F \subseteq B_1(0)$ and the interior of F is not empty; for any $h \in \boldsymbol{N}$, $1 \leq i \leq h$, $\omega \in \Omega$ and $r > 0$, we denote by $F_{i,h}^r(\omega)$ the following set

$$F_{i,h}^r(\omega) = \left\{ x \in D : \frac{1}{r}(x - x_i^h(\omega) \in F) \right\}$$

we note that $F_{i,h}^r(\omega) \subseteq B_r(x_i^h(\omega))$. Finally, we denote by F_h^r the random set

$$F_h^r = \bigcup_{i=1}^{h} F_{i,h}^r .$$

By Lemma 2.1 the sets F_h^r are actually random sets in according to the Definition 2.4.

For every Σ-measurable real-valued function X we define the expectation of X by

$$\boldsymbol{E}[X] = \int_{\Omega} X dP .$$

Let X, Y be two real-valued functions in $L^2(\Omega)$. Then the covariance of X and Y is defined by

$$\mathbf{Cov}[X, Y] = \boldsymbol{E}[XY] - \boldsymbol{E}[X]\,\boldsymbol{E}[Y] .$$

Let (F_h) be a sequence of p-random sets. We shall need the following set functions defined on $\mathcal{U}$

$$\alpha'(U) = \liminf_{h \to \infty} \boldsymbol{E}[C_p(F_h \cap U)] \tag{2.1}$$

$$\alpha''(U) = \limsup_{h \to \infty} \boldsymbol{E}[C_p(F_h \cap U)] \tag{2.2}$$

Next we consider the inner regularizations α'_- and α''_- of the set functions α' and α'', defined for every $U \in \mathcal{U}$ by

$$\begin{cases} \alpha'_-(U) = \sup\{\alpha'(V) \colon V \in \mathcal{U},\ V \subset\subset U\}, \\ \alpha''_-(U) = \sup\{\alpha''(V) \colon V \in \mathcal{U},\ V \subset\subset U\}. \end{cases} \tag{2.3}$$

Then we extend the definitions of α'_- and α''_- to the Borel sets $B \in \mathcal{B}$ by:

$$\begin{cases} \alpha'_-(B) = \inf\{\alpha'_-(U) \colon\ U \in \mathcal{U},\ U \supseteq B\}, \\ \alpha''_-(B) = \inf\{\alpha''_-(U) \colon\ U \in \mathcal{U},\ U \supseteq B\}. \end{cases} \tag{2.4}$$

Finally, we denote by ν' and ν'' the least superadditive set functions on $\mathcal{B}$ greater than or equal to α'_- and α''_-, respectively.

3. – Formulation of the problem and statement of the main result.

We are interested in analyzing the asymptotic behaviour of sequences of quasi-linear problems in randomly perforated domains of the form

$$\begin{cases} -\Delta_p u_h = g \quad \text{in } D\backslash E_h \\ u_h \in H_0^{1,p}(D\backslash E_h). \end{cases} \tag{3.1}$$

where E_h is a sequence of random subsets of D and $g \in L^q(D)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and Δ_p is the *p-Laplacian operator*, that is

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du).$$

Problem (3.1) is the Euler equation of the random minimization problem

$$\min_{u \in H_0^{1,p}(D\backslash E_h)} \int_{D\backslash E_h} |Du|^p\,dx - p \int_{D\backslash E_h} gu\,dx, \tag{3.2}$$

which is equivalent to the following problem

$$\min\left\{ \int_D |Du|^p - p\int_D gu\,dx : u \in H_0^{1,p}(D),\ u = 0 \ \ p\text{-q.e. on } E_h \right\}. \tag{3.3}$$

REMARK 3.1. – For every $\omega \in \Omega$ there exists a unique $u_h(\omega) \in H_0^{1,p}(D)$, $u_h(\omega) = 0$ p-q.e. on E_h solution of problem (3.3).

Let β be a Borel measure on $\mathcal{B}$. For a weak solution of the problem

$$\begin{cases} -\Delta_p U + \beta|U|^{p-2}U = g \quad \text{in } D \\ U \in H_0^{1,p}(D). \end{cases} \tag{3.4}$$

we mean the unique solution of the minimum problem

$$\min_{u \in H_0^{1,p}(D)} \int_D |Du|^p\,dx + \int_D |u|^p\,d\beta(x) - p\int_D gu\,dx\,. \tag{3.5}$$

Problems of this type have been extensively studied in [6].

In what follows, we want to study the behaviour of the sequence $(u_h(\omega))$ of solutions of (3.3) as $h \to +\infty$. In particular we would like to identify the limit problem of the sequence of random minimization problems (3.3).

THEOREM 3.2. – *Let* (E_h) *be a sequence of p-random sets, with* $2 \leqslant p \leqslant n$. *Let* α' *and* α'' *be the set functions defined in* (2.1), (2.2), *and let* ν' *and* ν'' *be* *the least superadditive set functions on $\mathcal{B}$ greater than or equal to α'_- and α''_-, i.e. the set functions defined in* (2.3) *and* (2.4).

Assume that

$$\text{i)} \qquad \nu'(B) = \nu''(B) < \infty \quad \textit{for every} \quad B \in \mathcal{B}$$

and denote by $\nu(B)$ the common value of $\nu'(B)$ and $\nu''(B)$ for every $B \in \mathcal{B}$; further, there exist $\eta > 0$, a continuous function $\xi : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ with $\xi(0, 0) = 0$ and a Radon measure β on $\mathcal{B}$ such that

$$\text{ii)} \quad \limsup_{h \to +\infty} |\mathbf{Cov}[C_p(E_h \cap U_1), C_p(E_h \cap U_2)]| \leq \xi(diam\, U_1, diam\, U_2)\, \beta(U_1)\, \beta(U_2)$$

for every $U_1, U_2 \in \mathcal{U}$ with $\overline{U}_1 \cap \overline{U}_2 = \emptyset$ and $\operatorname{diam}(U_1) < \eta$, $\operatorname{diam}(U_2) < \eta$. *Let*

$$m_h(\omega) = \min_{u \in H_0^{1,p}(D \setminus E_h(\omega))} \int_{D \setminus E_h(\omega)} |Du|^p\, dx - p \int_{D \setminus E_h(\omega)} gu\, dx \tag{3.6}$$

for any $g \in L^q(D)$, with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $\omega \in \Omega$.

Then ν is finite Borel measure on $\mathcal{B}$ and (m_h) converges in probability, as $h \to +\infty$, to

$$m_0 = \min_{u \in H_0^{1,p}(D)} \int_D |Du|^p\, dx + \int_D |u|^p\, d\nu - p \int_D gu\, dx, \tag{3.7}$$

that is, for any $\varepsilon > 0$,

$$\lim_{h \to +\infty} P\{\omega \in \Omega,\ |m_h(\omega) - m_0| > \varepsilon\} = 0\,.$$

Moreover, if $U_h(\omega)$ is the unique minimum point in $H_0^{1,p}(D \setminus E_h(\omega))$ of problem (3.6) *for every $\omega \in \Omega$, and U_0 is the unique minimum point in $H_0^{1,p}(D)$ of problem* (3.5), *we also have, for any $\varepsilon > 0$,*

$$\lim_{h \to \infty} P\{\omega \in \Omega : \|U_h(\omega) - U_0\|_{L^p(D)} > \varepsilon\} = 0\,.$$

PROOF. – The proof can be deduced,by means minor changes, from Proposition 3.3, Theorem 4.10 and Corollary 4.11 in [2]. An inspection of those proofs, in particular, that one of Proposition 3.3, shows that the more general assumption (ii) above is sufficient to get the result.

REMARK. – 3.3. – We could interpret the assumption (*ii*) as a sort of «asymptotic weak correlation» of the random variables $C_p(E_h \cap U_1)$ and $C_p(E_h \cap U_2)$ on disjoint sets U_1, U_2 in $\mathcal{U}$.

Our aim is to characterize, by applying the previous result, a class of problems, concerning the p-Laplacian operator in randomly perforated domains, for which the measure appearing in the limit problem can be explicitely computed.

ASSUMPTIONS 3.4. – Let us assume the following hypotheses:

i_1) let $\beta \in H^{-1,q}(D)$ such that $\beta(D) = 1$. Furthermore, there exists a continuous function $f : R^+ \to R$ with $f(0) = 0$, strictly monotone in a neighborhood $\mathcal{O}$ of t=0, such that

$$\mathcal{E}^p_\beta(U) \leq f(\operatorname{diam} U)\, \beta(U)$$

for every $U \in \mathcal{U}$;

i_2) for every $h \in \boldsymbol{N}$ we set $I_h = 1, \dots, h$ and we consider h measurable functions $x_i^h : \Omega \to D$, $i \in I_h$, such that $(x_i^h)_{i \in I_h}$ is a family of independent, identically distributed random variables with probability distribution β, that is

$$P\{\omega \in \Omega : x_i^h \in B\} = \beta(B), \qquad i \in I_h$$

for every Borel set $B \in \mathcal{B}$;

i_3) let ϱ_h be a sequence of positive numbers such that $0 < \varrho_h < 1$ and the limit

$$l = \begin{cases} \lim\limits_{h \to +\infty} h \varrho_h^{n-p} & \text{if } 2 \leq p < n \\ \lim\limits_{h \to +\infty} h(-\ln \varrho_h)^{1-n} & \text{if } p = n \end{cases}$$

is finite and strictly positive.

REMARK 3.5. – In this remark some significant examples of measures satisfying hypothesis i_1) of Assumptions 3.4 are given.

(**a**) Let M be a smooth, compact manifold in D (with or without boundary), whose dimension is equal to $n-1$. We denote the $(n-1)$-dimensional Hausdorff measure by $\mathcal{H}^{n-1}$. Let us consider a non-negative function $V \in L^r(M, \mathcal{H}^{n-1})$, such that $\int_M V(x)\, d\mathcal{H}^{n-1}(x) = 1$, with $r > \dfrac{n-1}{p-1}$ and $2 \leq p < n$. Let us define the measure on D

$$\beta(B) = \int_{B \cap M} V(x)\, d\mathcal{H}^{n-1}(x),$$

for every $B \in \mathcal{B}$.

If we set $t = \operatorname{diam} U$, with $U \in \mathcal{U}$, we have

$$\mathcal{E}_\beta^p(U) = \int\int_{U \times U} \frac{d\beta(x)\, d\beta(y)}{|x-y|^{n-p}} \leqslant$$

$$\leqslant \int_{U \cap M} \left(\int_{B(x,t) \cap M} \frac{V(y)}{|x-y|^{n-p}}\, d\mathcal{H}^{n-1}(y) \right) V(x)\, d\mathcal{H}^{n-1}(x).$$

Moreover, by Hölder's inequality, we obtain

$$\int_{B(x,t) \cap M} \frac{V(y)}{|x-y|^{n-p}}\, d\mathcal{H}^{n-1}(y) \leqslant$$

$$\leqslant \|V\|_{L^r(M, \mathcal{H}^{n-1})} \left[\int_{B(x,t) \cap M} \frac{1}{|x-y|^{\frac{(n-p)r}{r-1}}}\, d\mathcal{H}^{n-1}(y) \right]^{\frac{r-1}{r}}.$$

By using the elementary formula

$$\int_{B(x,t) \cap M} \frac{1}{|x-y|^{\alpha}}\, d\mathcal{H}^{n-1}(y) =$$

$$= \alpha \int_0^t \frac{\mathcal{H}^{n-1}(B(x,\varrho) \cap M)}{\varrho^{\alpha+1}}\, d\varrho + \frac{\mathcal{H}^{n-1}(B(x,\varrho) \cap M)}{t^{\alpha}}$$

with $\alpha = \dfrac{(n-p)r}{r-1}$, and by noticing that, for any $x \in M$ and $\varrho > 0$,

$$\mathcal{H}^{n-1}(B(x,\varrho) \cap M) \leqslant C \varrho^{n-1}$$

where C is a constant independent of x and ϱ, it is easy to get

$$f(t) = k \left[C \frac{n-1}{n-\alpha-1} \right]^{\frac{r-1}{r}} t^{(n-\alpha-1)\frac{r-1}{r}}$$

where $k = \|V\|_{L^r(M, \mathcal{H}^{n-1})}$.

(**b**) Consider a measure defined, for every $B \in \mathcal{B}$, as

$$\beta(B) = \int_B V(x)\, dx\,,$$

where $V(x)$ is a non-negative function such that $\int_D V(x)\, dx = 1$.

If $V(x)$ is a continuous function of compact support in D, an easy computation gives

$$f(t) = \begin{cases} k\dfrac{\omega_{n-1}}{p} t^p & \text{if } 2 \leqslant p < n \\ k\dfrac{\omega_{n-1}}{n} t^n \left(\dfrac{1}{n} - \ln t\right) & \text{if } p = n. \end{cases}$$

where $k = \max\{V(x) : x \in D\}$.

If $V \in L^r(D)$, with $r > \dfrac{n}{p}$ in the case $2 \leqslant p < n$ or $r > 1$ in the case $p = n$, with a computation similar to that developed in (**a**), we obtain

$$f(t) = \begin{cases} k\left(\omega_{n-1}\dfrac{r-1}{rp-n}\right)^{\frac{r-1}{r}} t^{\frac{rp-n}{r}} & \text{if } 2 \leqslant p < n \\ k\omega_{n-1}^{\frac{r-1}{r}}\left(\displaystyle\int_0^t \left(\ln\frac{1}{\varrho}\right)^{\frac{r}{r-1}} \varrho^{n-1}\,d\varrho\right)^{\frac{r-1}{r}} & \text{if } p = n, \end{cases}$$

where $k = \|V\|_{L^r(D)}$.

From now on we shall consider the sequence of random sets (F_h) defined in Example 2.5, with $r = \varrho_h$, that is, by setting

$$F_i^h(\omega) = \left\{x \in D : \frac{1}{\varrho_h}(x - x_i^h(\omega) \in F)\right\}$$

we define

$$F_h(\omega) = \bigcup_{i \in I_h} F_i^h(\omega). \tag{3.8}$$

Finally, denoting by ω_{n-1} the area of the unit sphere of $\boldsymbol{R}^n$, we set

$$c = \begin{cases} \min\left\{\displaystyle\int_{\boldsymbol{R}^n} |Du|^p\,dx : u \in H^{1,p}(\boldsymbol{R}^n),\ u \geqslant 1 \text{ p-q.e. on } F\right\}, & \text{if } 2 \leqslant p < n \\ \omega_{n-1}, & \text{if } p = n. \end{cases} \tag{3.9}$$

The next theorem is the main result of the paper.

THEOREM 3.6. – *Let* (E_h) *be the sequence of random sets, as defined in* (3.8). *Assume that the hypotheses* $(i_1), (i_2)$ *and* (i_3) *hold. Moreover, suppose that*

$2 \leqslant p \leqslant n$. *For every* $h \in \boldsymbol{N}$ *and* $\omega \in \Omega$, *let* $U_h(\omega)$ *be the weak solution of the problem*

$$\begin{cases} -\Delta_p U_h = g & \text{in } D \backslash E_h(\omega) \\ U_h \in H_0^{1,p}(D \backslash E_h(\omega)), \end{cases}$$

where $g \in L^q(D)$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then, for every* $\varepsilon > 0$,

$$\lim_{h \to \infty} P\{\omega \in \Omega : \|U_h(\omega) - U_0\|_{L^p(D)} > \varepsilon\} = 0$$

where U_0 *is the unique weak solution of the relaxed Dirichlet problem*

$$\begin{cases} -\Delta_p U + cl\beta |U|^{p-2} U = g & \text{in } D \\ U \in H_0^{1,p}(D), \end{cases}$$

where c *is the constant defined in* (3.9).

4. – Proof of the main result.

By Theorem 3.2, Theorem 3.6 is an immediate consequence of the following proposition.

PROPOSITION 4.1. – *Let* (F_h) *be the sequence of random sets, as defined in* (3.8). *Let* α' *and* α'' *be the set functions defined in* (2.1), (2.2), *and let* ν' *and* ν'' *be the least superadditive set functions on* $\mathcal{B}$ *greater than or equal to* α'_- *and* α''_-, *i.e. the set functions defined in* (2.3) *and* (2.4). *If hypotheses* $(i_1), (i_2)$ *and* (i_3) *are satisfied and* $2 \leqslant p \leqslant n$, *we have:*

t_1) $$\nu'(B) = \nu''(B) = \mathrm{cl}\beta(B) \quad \text{for every } B \in \mathcal{B},$$

where c *is defined in* (3.9).

Moreover, there exist $\eta > 0$, *a continuous function* $\xi : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ *with* $\xi(0, 0) = 0$ *and a Radon measure* β_1 *on* $\mathcal{B}$ *such that*

t_2) $$\limsup_{h \to +\infty} |\mathbf{Cov}[C_p(E_h \cap U_1), C_p(E_h \cap U_2)]| \leqslant$$

$$\leqslant \xi(\mathrm{diam}\, U_1, \mathrm{diam}\, U_2)\, \beta_1(U_1)\, \beta_1(U_2)$$

for every $U_1, U_2 \in \mathcal{U}$ *with* $\overline{U}_1 \cap \overline{U}_2 = \emptyset$ *and* $\mathrm{diam}(U_1) < \eta$, $\mathrm{diam}(U_2) < \eta$.

The next two lemmas will be essential in the proof of Proposition 4.1. In the first one, we identify a suitable supersolution of the p-Laplacian (for a definition see, for example, [8]) in perforated domains; in the second one, we give a result which allows us to estimate from below the p-capacity of the union of a family $(E_i)_{i \in I}$ by means of the sum of p-capacities of the sets E_i.

LEMMA 4.2. – *Let $(E_i)_{i \in I}$ be a family of closed subsets of D and let $E = \bigcup_{i \in I} E_i$. Assume that there exist a finite family $(x_i)_{i \in I}$ of points in D and a real number ϱ such that $0 < \varrho < 1$ and*

$$E_i \subseteq B_\varrho(x_i) \subseteq D \quad \textit{for } i \in I\,.$$

Further, for every $x \in \boldsymbol{R}^n$ and $i \in I$, set

$$z_i(x) = \begin{cases} \left(\dfrac{\varrho}{|x - x_i|}\right)^{\frac{n-p}{p-1}} \wedge 1 & \textit{if } 2 \leqslant p < n \\ (-\ln \varrho)^{-1} \ln(|x - x_i|)^{-1} \wedge 1 & \textit{if } p = n. \end{cases}$$

Finally, let

$$z(x) = \sum_{i \in I} z_i(x).$$

Then $z \in H^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^n \setminus E)$, $z \geqslant 0$ on ∂D, $z \geqslant 1$ on E, and it satisfies the following condition

$$\int_{D \setminus E} |Dz|^{p-2} Dz\, D\varphi\, dx \geqslant 0 \tag{4.1}$$

for every non-negative $\varphi \in C_0^\infty(D \setminus E)$.

PROOF. – We consider the case $2 \leqslant p < n$. The case $p = n$ can be proven in the same way. It is easy to see that the hardest part of the proof is to show that the condition (4.1) holds. Let us set

$$\gamma = \frac{n-p}{p-1}\,.$$

First, we establish that, for every $\varepsilon > 0$,

$$\operatorname{div}\left[\left(|Dz(x)|^2 + \varepsilon\right)^{\frac{p-2}{2}} Dz(x)\right] \leqslant 0, \tag{4.2}$$

for all $x \in \boldsymbol{R}^n \setminus E$.

Let us define, for every $x \in \boldsymbol{R}^n \setminus E$ and $\varepsilon > 0$, the function

$$a_\varepsilon(x) = \left(|Dz(x)|^2 + \varepsilon\right)^{\frac{p-2}{2}}.$$

Note that

$$\operatorname{div}(a_\varepsilon Dz) = \langle Da_\varepsilon, Dz\rangle + a_\varepsilon \Delta z\,. \tag{4.3}$$

where $\langle , \rangle$ is the scalar product in $\boldsymbol{R}^n$, and Δ is the Laplace operator in $\boldsymbol{R}^n$.

A simple computation gives

$$\langle Da_\varepsilon, Dz\rangle = (p-2)\,\frac{a_\varepsilon}{|Dz|^2+\varepsilon}\sum_{i,h=1}^{n}\frac{\partial z}{\partial x_i}\,\frac{\partial z}{\partial x_h}\,\frac{\partial^2 z}{\partial x_i\,\partial x_h}\,.$$

Moreover, we have that

$$\begin{aligned}
&\sum_{i,h=1}^{n}\frac{\partial z}{\partial x_i}\,\frac{\partial z}{\partial x_h}\,\frac{\partial^2 z}{\partial x_i\,\partial x_h}\\
&= \varrho^{\gamma}\gamma(\gamma+2)\Big\langle\sum_{j\in I}|x-x_j|^{-(\gamma+4)}(x-x_j)\otimes(x-x_j)\,Dz,\,Dz\Big\rangle\\
&\quad -\gamma r^{n-p}\sum_{j\in I}\frac{|Dz|^2}{|x-x_j|^{\gamma+2}}\\
&\leqslant \varrho^{\gamma}\gamma(\gamma+1)\;|Dz|^2\sum_{j\in I}|x-x_j|^{-(\gamma+2)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle Da_\varepsilon, Dz\rangle &\leqslant (p-2)\,\gamma(\gamma+1)\,a_\varepsilon\,\frac{|Dz|^2}{|Dz|^2+\varepsilon}\sum_{j\in I}|x-x_j|^{-(\gamma+2)}\\
&\leqslant (p-2)\,\gamma(\gamma+1)\,a_\varepsilon\sum_{j\in I}|x-x_j|^{-(\gamma+2)}.
\end{aligned}\tag{4.4}$$

It is also straightforward to show that

$$\Delta z = \varrho^{\gamma}\gamma(\gamma+2-n)\sum_{j\in I}|x-x_j|^{-(\gamma+2)}.\tag{4.5}$$

Thus, by (4.3), (4.4) and (4.5), we get

$$\begin{aligned}
&\operatorname{div}\Big[\big(|Dz(x)|^2+\varepsilon\big)^{\frac{p-2}{2}}\,Dz(x)\Big]\leqslant\\
&\gamma\varrho^{\gamma}\big(|Dz(x)|^2+\varepsilon\big)^{\frac{p-2}{2}}\sum_{j\in I}|x-x_j|^{-(\gamma+2)}[(\gamma+2-n)+(p-2)(\gamma+1)].
\end{aligned}\tag{4.6}$$

By applying the definition of γ, we see that the quantity in bracket on the right-hand side of (4.6) is equal to zero and so (4.2) is proven.

Now we are in a position to prove condition (4.1). Indeed, from (4.2), integrating by part, we obtain

$$\int_{D\setminus E}(|Dz|^2+\varepsilon)^{\frac{p-2}{2}}\,Dz\,D\varphi\,dx\geqslant 0\tag{4.7}$$

for every non-negative $\varphi\in C_0^\infty(D\setminus E)$.

Finally, by taking the limit for $\varepsilon \to 0$ in (4.7) and by applying Lebesgue's dominated convergence theorem, we get (4.1) and the proof is accomplished.

LEMMA 4.3. – *Let $(E_i)_{i \in I}$ be a family of closed subsets of D and let $E = \bigcup_{i \in I} E_i$. Assume that there exist a finite family $(x_i)_{i \in I}$ of points in D and two real positive numbers ϱ and R such that*

(*i*) $$0 < \varrho < R < 1;$$

(*ii*) $$E_i \subseteq B_\varrho(x_i) \subseteq B_R(x_i) \subseteq D \quad \textit{for } i \in I;$$

(*iii*) $$|x_i - x_j| \geq 2R \quad \textit{for } i \neq j.$$

Define

$$r = \begin{cases} \varrho^{\frac{1}{p-1}} & \textit{if } 2 \leq p < n \\ e^{-(\ln \frac{1}{\varrho})^{\frac{1}{n-1}}} & \textit{if } p = n. \end{cases}$$

Let us set

$$\delta = \begin{cases} 2\left(\dfrac{r}{R}\right)^{n-p} & \textit{if } 2 \leq p < n \\ 2(-\ln r)^{1-n} \ln R^{-1} & \textit{if } p = n. \end{cases} \tag{4.8}$$

If, in addition, we suppose

(*iv*) $$\begin{cases} \displaystyle\sum_{i \neq j} \frac{r^{n-p}}{(|x_i - x_j| - R)^{n-p}} < \frac{\delta}{2} & \textit{if } 2 \leq p < n \\ \displaystyle(-\ln r)^{1-n} \sum_{i \neq j} \ln(|x_i - x_j| - R)^{-1} < \frac{\delta}{2} & \textit{if } p = n. \end{cases}$$

then, for $\delta < 1$,

$$C_p(E) \geq (1-\delta)^p \sum_{i \in I} C_p(E_i, B_R(x_i)).$$

PROOF. – Let $u \in H_0^{1,p}(D)$ be the capacitary potential of E with respect to D. We claim that the proof is achieved, whenever $u \leq \delta$ on $\partial B_R(x_i)$ for every $i \in I$. Indeed, if this is the case, let us define the function $v = (1-\delta)^{-1}(u - \delta)^+$. By definition of capacitary potential, it is easy to see that $v \in H_0^{1,p}(D)$, $v \geq 1$ p-q.e. on E and $v = 0$ p-q.e. on $\partial B_R(x_i)$, for every $i \in I$. Since (*ii*) holds, we have

$$C_p(E_i, B_R(x_i)) \leqslant \int_{B_R(x_i)} |Dv|^p \, dx$$

for every $i \in I$. Hence,

$$\int_D |Dv|^p \, dx \geqslant \sum_{i \in I} \int_{B_R(x_i)} |Dv|^p \, dx \geqslant \sum_{i \in I} C_p(E_i, B_R(x_i)) \,. \tag{4.9}$$

By definition of v, we have also

$$\begin{aligned} \int_D |Dv|^p \, dx &= \frac{1}{(1-\delta)^p} \int_D |D(u-\delta)^+|^p \, dx \\ &\leqslant \frac{1}{(1-\delta)^p} \int_D |Dv|^p \, dx = \frac{1}{(1-\delta)^p} C_p(E). \end{aligned} \tag{4.10}$$

We obtain the assertion by (4.9) and (4.10). Now, it remains to prove that $u \leqslant \delta$ on $\partial B_R(x_i)$ for every $i \in I$. We shall give the details only for the case $2 < p < n$. The case $p = n$ is obtained in the same way. Consider the function $z(x)$ defined in Lemma 4.2.

The function z is a supersolution relative to the p-Laplacian operator in $D \backslash E$ (see Remark 4.3), such that $z \geqslant 0$ on ∂D and $z \geqslant 1$ on E. Since the capacitary potential u is a weak solution in $D \backslash E$ relative to the p-Laplacian, that is

$$\int_{D \backslash E} |Du|^{p-2} Du \, D\varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(D \backslash E)$, we can apply the comparison principle for supersolutions relative to the p-Laplacian in $D \backslash E$ (see [9] Lemma 3.18), which gives

$$u \leqslant z \qquad \text{a.e. in } D \backslash E \,. \tag{4.11}$$

Finally, it is easy to see that, for every $i \in I$, $z \leqslant \delta$ on $\partial B_R(x_i)$. For a fixed $i \in I$, let $y \in \partial B_R(x_i)$. By definition of function z and by assumption (iv) we obtain

$$
\begin{aligned}
z(y) &\leqslant \sum_{j \in I} \frac{r^{n-p}}{(|y-x_j|)^{\frac{n-p}{p-1}}} \\
&\leqslant \left(\frac{r}{R}\right)^{n-p} + \sum_{i \neq j} \frac{r^{n-p}}{(|y-x_j|)^{\frac{n-p}{p-1}}} \\
&\leqslant \frac{\delta}{2} + \sum_{i \neq j} \frac{r^{n-p}}{(|y-x_j|)^{n-p}} \\
&\leqslant \frac{\delta}{2} + \sum_{i \neq j} \frac{r^{n-p}}{(|x_i-x_j|-R)^{n-p}} \leqslant \delta .
\end{aligned}
$$

This inequality, together with (4.11), shows that the assumption $u \leqslant \delta$ on $\partial B_R(x_i)$, for every $i \in I$, is always satisfied and so the proof is complete.

For our purposes we also need a suitable probabilistic result. In order to state it, we have to introduce some more notation.

Let $(\xi_i)_{i \in I}$ be a finite family of independent, identically distributed random variables with values in D, and with distribution given by

$$P\{\omega \in \Omega : \xi_i(\omega) \in B\} = \beta(B) \qquad \text{for every } B \in \mathcal{B},$$

where $\beta \in H^{-1,q}(D)$.

For $0 < r < R < 1$ and for any subset $Z \subseteq D$, let us introduce the following random sets of indices

$$
\begin{aligned}
&N(Z) = \{i \in I : \xi_i \in Z\}, \\
&I(Z) = \{i \in I : B_R(\xi_i) \subset Z,\ |\xi_i - \xi_j| \geqslant 2R,\ \forall j \in I, j \neq i\}, \\
&J(Z) = \{i \in I : B_R(\xi_i) \subset Z,\ \exists j \in I, j \neq i,\ |\xi_i - \xi_j| \leqslant 2R\},
\end{aligned}
$$

and for every $\eta > 0$

$$
I_\eta(Z) = \begin{cases} \left\{ i \in I(Z) : \displaystyle\sum_{i \neq j} \frac{r^{n-p}}{(|\xi_i - \xi_j| - R)^{n-p}} < \frac{\eta}{2} \right\} & \text{if } 2 \leqslant p < n \\ \left\{ i \in I(Z) : (-\ln r)^{1-n} \displaystyle\sum_{i \neq j} \ln(|\xi_i - \xi_j| - R)^{-1} < \frac{\eta}{2} \right\}, & \text{if } p = n ; \end{cases}
$$

and finally

$$J_\eta(Z) = I(Z) \setminus I_\eta(Z).$$

We are now in a position to state the Lemma announced above. Its proof can be obtained by adapting to our case the proofs of (i) and (ii) of Lemma 5.1 in [3].

LEMMA 4.4. – *For any* $0 < \varrho < R < 1$, *let* δ *be the positive real number defined in* (4.8) *of Lemma* 4.3. *Then, for every* $A \in \mathcal{U}$, *the expectation of the random variable* $\#(J_\delta(A))$ *satisfies the inequality*

$$(i) \qquad \boldsymbol{E}[\#(J_\delta(A))] \leq \begin{cases} \left(\dfrac{2}{\delta}\right)(2\varrho)^{n-p}(\#(I))^2\,\mathcal{E}^p_\beta(A) & \text{if } 2 \leq p < n \\[2ex] \left(\dfrac{2}{\delta}\right)(-\ln\varrho)^{1-n}(\#(I))^2\,\mathcal{E}^p_\beta(A) & \text{if } p = n, \end{cases}$$

the expectation of the random variable $\#(J(A))$ *satisfies the inequality*

$$(ii) \qquad \boldsymbol{E}[\#(J(A))] \leq \begin{cases} (2R)^{n-p}(\#(I))^2\,\mathcal{E}^p_\beta(\widetilde{A}) & \text{if } 2 \leq p < n \\ (-\ln 2R)^{1-n}(\#(I))^2\,\mathcal{E}^p_\beta(\widetilde{A}) & \text{if } p = n, \end{cases}$$

where $\mathcal{E}^p_\beta$ *is the set function as in Definition* 2.2 *and* $\widetilde{A} = \{y \in D :$ $\operatorname{dist}(y, A) < 2R\}$, *with* $R < \frac{1}{2}$.

PROOF OF PROPOSITION 4.1. – To get the proof, we can apply exactly the same scheme of the proof of Proposition 5.1 in [3]. For the readers convenience, we repeat the basic steps in our case. We shall prove the proposition when $2 \leq p < n$. The case $n = p$, can be adapted in a straightforward way.

For $0 < \delta < 1$ and $h \in \boldsymbol{N}$, we choose $R_h > 0$ such that $\varrho_h < R_h$ and

$$\delta = 2\left(\frac{r_h}{R_h}\right)^{n-p},$$

where r_h is defined as in Lemmma 4.3 (in the definition of the quantity r put $\varrho = \varrho_h$). For every $U \in \mathcal{U}$ and $h \in \boldsymbol{N}$, let us introduce the following families of random indices

$$\begin{aligned} N_h(U) &= \{i \in I : \xi_i \in U\}, \\ I_h(U) &= \{i \in I : B_{R_h}(\xi_i) \subset U,\ |\xi_i - \xi_j| \geq 2R_h,\ \forall j \in I_h,\ j \neq i\}, \\ J_h(U) &= \{i \in I_h : B_{R_h}(\xi_i) \subset U,\ \exists j \in I_h,\ j \neq i,\ |\xi_i - \xi_j| \leq 2R_h\}, \end{aligned}$$

and

$$I^h_\delta(U) = \left\{ i \in I_h(U) : \sum_{i \neq j} \frac{r_h^{n-p}}{(|\xi_i - \xi_j| - R_h)^{n-p}} < \frac{\delta}{2} \right\}.$$

Furthermore, we set

$$J_\delta^h(U) = I_h(U) \backslash I_\delta^h(U).$$

and

$$\widehat{U}_h = \{y \in U : \operatorname{dist}(y, \partial U) > R_h\}.$$

It is not difficult to see that

$$I_h(U) = N_h(\widehat{U}_h) \backslash J_h(U)$$

$$I_\delta^h(U) = (N_h(\widehat{U}_h) \backslash J_h(U)) \backslash J_\delta^h(U).$$

Denote by F_h' the random set

$$F_h'(\omega) = \bigcup_{i \in I_\delta^h(U)} \left\{ x \in D : \frac{1}{\varrho_h}(x - x_i^h(\omega) \in F) \right\}.$$

By Lemma 4.3, we have that , for every $\omega \in \Omega$,

$$C_p(F_h(\omega) \cap U) \geqslant C_p(F_h'(\omega)) \geqslant (1-\delta)^p \sum_{i \in I_\delta^h(U)} C_p(F_i^h(\omega), B_{R_h}(x_i^h)) \geqslant \tag{4.12}$$

$$(1 - p\delta)(h\varrho_h^{n-p})\, C_p(F, B_{R_h/\varrho_h}(0)) \left[\frac{\# N_h(\widehat{U}_h)}{h} - \frac{\# J_h(U)}{h} - \frac{\# J_\delta^h(U)}{h} \right].$$

On the other hand, by using the elementary properties of the capacity, we immediately get that, for every $U \in \mathcal{U}$,

$$C_p(F_h(\omega) \cap U) \leqslant \sum_{i \in N_h(\widetilde{U}_h)} C_p(F_i^h(\omega), B_{R_h}(x_i^h) = \tag{4.13}$$

$$(h\varrho_h^{n-p})\, C_p(F, B_{R_h/\varrho_h}(0)) \left[\frac{\# N_h(\widetilde{U}_h)}{h} \right]$$

where we have set $\widetilde{U}_h = \{y \in D : \operatorname{dist}(x, U) < 2R_h\}$.

PROOF OF (t_1). – By Lemma 4.4 we deduce that

$$\limsup_{h \to +\infty} \frac{\boldsymbol{E}[\#(J_\delta(U))]}{h} \leqslant \frac{2^{n-p+1}}{\delta} l\, \mathcal{E}_\beta^p(U) \tag{4.14}$$

$$\limsup_{h \to +\infty} \frac{\boldsymbol{E}[\#(J_h(U))]}{h} \leqslant \frac{2^{n-p+1}}{\delta} l\, \mathcal{E}_\beta^p(U). \tag{4.15}$$

Further, it is not difficult to check that

$$\beta(U) = \begin{cases} \lim\limits_{h\to+\infty} \dfrac{\boldsymbol{E}[\#(N_h(\widehat{U}_h))]}{h} \\ \lim\limits_{h\to+\infty} \dfrac{\boldsymbol{E}[\#(N_h(\widetilde{U}_h))]}{h} \end{cases} \tag{4.16}$$

for every $U \in \mathcal{U}$ with $\beta(\partial U) = 0$.

Noticing that

$$\lim_{R\to+\infty} C_p(F,\, B_R(0)) = c\,,$$

where c is the constant defined in (3.9), we get, from (2.4), (4.13) and (4.14),

$$\alpha''_-(B) \leqslant cl\,\beta(B) \tag{4.17}$$

for every $B \in \mathcal{B}$; and from (2.4), (4.12), (4.14), (4.15) and (4.16) it follows that

$$\alpha'_-(B) \geqslant (1-\delta)^p\, cl\left[\beta(B) - \frac{2^{n-p+2}}{\delta}\, l\sigma(B\times B)\right] \tag{4.18}$$

for every $B \in \mathcal{B}$, where σ is the measure defined in Remark 2.3. From (4.17) we have

$$\nu''(B) \leqslant cl\,\beta(B) \tag{4.19}$$

for every $B \in \mathcal{B}$.

On the other hand, we also have

$$\nu'(B) \geqslant (1-\delta)^p\, cl\,\beta(B) \tag{4.20}$$

for every $B \in \mathcal{B}$. Indeed, let us fix $B \in \mathcal{B}$; for arbitrary $0<\eta<1$, take a Borel partition $(B_j)_{j\in J}$ of B with diameter of (B_j) less than η. Since ν' is superadditive, we have

$$\begin{aligned}
\nu'(B) &\geqslant \sum_{i\in J} \nu'(B_j) \geqslant (1-\delta)^p\, cl\left(\beta(B) - \frac{2^{n-p+2}}{\delta} \sum_{j\in J} \sigma(B_j\times B_j)\right) \\
&= (1-\delta)^p\, cl\left(\beta(B) - \frac{2^{n-p+2}}{\delta} \sum_{j\in J} \int\!\!\int_{B_j\times B_j} \frac{\beta(dx)\,\beta(dy)}{|x-y|^{n-p}}\right) \\
&\geqslant (1-\delta)^p\, cl\left(\beta(B) - \frac{2^{n-p+2}}{\delta} \int\!\!\int_{D_\eta} \frac{\beta(dx)\,\beta(dy)}{|x-y|^{n-p}}\right) \\
&= (1-\delta)^p\, cl\left(\beta(B) - \frac{2^{n-p+2}}{\delta}\,\sigma(D_\eta)\right),
\end{aligned}$$

where σ is the measure defined in Remark 2.3 and $D_\eta = \{(x,\,y) \in D\times D : |x -$

$y| < \eta\}$. Notice that $B_j \times B_j \subset D_\eta$, for every $j \in J$ and that the diameter of D_η is less than η. Since $\beta \in H^{-1,q}(D)$, by Remark 2.3 we find that

$$\lim_{\eta \to 0} \sigma(D_\eta) = 0$$

and we get (4.20); finally, letting $\delta \to 0$, we obtain

$$\nu'(B) \geqslant cl\,\beta(B) \geqslant \nu''(B)$$

for every $B \in \mathcal{B}$. So, (t_1) is proved, because $\nu'' \geqslant \nu'$.

PROOF OF (t_2). – We observe that, for every $U \in \mathcal{U}$, by Strong Law of Large Numbers, we have

$$\lim_{h \to +\infty} \frac{\#(N_h(\widehat{U}_h))}{h} = \beta(U) \tag{4.21}$$

for a.e. $\omega \in \Omega$, and

$$\lim_{h \to +\infty} \frac{\#(N_h(\widehat{U}_h))}{h} = \beta(U) \tag{4.22}$$

in $L^1(\Omega)$.

Since the sequence of random variables $(h^{-1}\#(N_h(\widehat{U}_h)))_{h \in \boldsymbol{N}}$ is equibounded, we also have

$$\lim_{h \to +\infty} \frac{\#(N_h(\widehat{U}_h))}{h} = \beta(U) \tag{4.23}$$

in $L^2(\Omega)$.

By (4.12), we obtain

$$\begin{aligned} &\liminf_{h \to +\infty} \boldsymbol{E}[C_p(F_h \cap U, D)\, C_p(F_h \cap V, D)] \\ &\geqslant [(1-2p\delta)\, cl]^2 \times \liminf_{h \to +\infty} \left\{ \boldsymbol{E}\left[\frac{\#(N_h(\widehat{U}_h))}{h} \, \frac{\#(N_h(\widehat{V}_h))}{h} \right] \right. \\ &- \boldsymbol{E}\left[\frac{\#(N_h(\widehat{U}_h))}{h} \, \frac{\#(J_\delta^h(\widehat{V}))}{h} \right] - \boldsymbol{E}\left[\frac{\#(N_h(\widehat{V}_h))}{h} \, \frac{\#(J_\delta^h(\widehat{U}))}{h} \right] \\ &\left. - \boldsymbol{E}\left[\frac{\#(N_h(\widehat{U}_h))}{h} \, \frac{\#(J_h(\widehat{V}))}{h} \right] - \boldsymbol{E}\left[\frac{\#(N_h(\widehat{V}_h))}{h} \, \frac{\#(J_h(\widehat{U}))}{h} \right] \right\} \end{aligned} \tag{4.24}$$

for any pair $U, V \in \mathcal{U}$ with $\overline{U} \cap \overline{V} = \emptyset$. From (4.23) we obtain

$$\lim_{h \to +\infty} \boldsymbol{E}\left[\frac{\#(N_h(\widehat{U}_h))}{h} \, \frac{\#(N_h(\widehat{V}_h))}{h} \right] = \beta(U)\beta(V) \tag{4.25}$$

moreover, by Lemma 4.4 and (4.21) we have

$$\boldsymbol{E}\left[\frac{\#(N_h(\widehat{U}_h))}{h}\,\frac{\#(J_\delta^h(\widehat{V}))}{h}\right] \leqslant \frac{2^{n-p+1}}{\delta}\,l\beta(U)\,\mathcal{E}_\beta^p(V), \tag{4.26}$$

$$\boldsymbol{E}\left[\frac{\#(N_h(\widehat{V}_h))}{h}\,\frac{\#(J_\delta^h(\widehat{U}))}{h}\right] \leqslant \frac{2^{n-p+1}}{\delta}\,l\beta(V)\,\mathcal{E}_\beta^p(U), \tag{4.27}$$

$$\boldsymbol{E}\left[\frac{\#(N_h(\widehat{U}_h))}{h}\,\frac{\#(J_h(\widehat{V}))}{h}\right] \leqslant \frac{2^{n-p+1}}{\delta}\,l\beta(U)\,\mathcal{E}_\beta^p(V), \tag{4.28}$$

$$\boldsymbol{E}\left[\frac{\#(N_h(\widehat{V}_h))}{h}\,\frac{\#(J_h(\widehat{U}))}{h}\right] \leqslant \frac{2^{n-p+1}}{\delta}\,l\beta(V)\,\mathcal{E}_\beta^p(U), \tag{4.29}$$

for any pair $U, V \in \mathcal{U}$. Then (4.24), (4.25), (4.26), (4.27), (4.28) and (4.29) give

$$\liminf_{h\to+\infty} \boldsymbol{E}\,[C_p(F_h\cap U, D)\,C_p(F_h\cap V, D)] \tag{4.30}$$

$$\geqslant [(1-p\delta)\,cl]^2\left[\beta(U)\,\beta(V) - \frac{2^{n-p+2}}{\delta}\,l\beta(U)\,\mathcal{E}_\beta^p(V) - \frac{2^{n-p+2}}{\delta}\,l\beta(V)\,\mathcal{E}_\beta^p(U)\right]$$

for any pair $U, V \in \mathcal{U}$ with $\overline{U}\cap\overline{V} = \emptyset$.

By (4.13) and (4.23) we also deduce

$$\limsup_{h\to+\infty} \boldsymbol{E}[C_p(F_h\cap U, D)\,C_p(F_h\cap V, D)] \leqslant [cl]^2\beta(U)\,\beta(V) \tag{4.31}$$

for any pair $U, V \in \mathcal{U}$ with $\beta(\partial U) = \beta(\partial V) = 0$. Estimates similar to (4.30) and (4.31) for the upper and lower limit of the sequence $(\boldsymbol{E}[C_p(F_h\cap U, D)]\boldsymbol{E}[C_p(F_h\cap V, D)])_{h\in\boldsymbol{N}}$ can be obtained in the same way. Therefore, we get for any pair $U, V \in \mathcal{U}$ with $\overline{U}\cap\overline{V} = \emptyset$

$$\limsup_{h\to+\infty} |\,\mathbf{Cov}\,[C_p(F_h\cap U, D), C_p(F_h\cap V, D)]\,|$$

$$\leqslant (cl)^2\beta(U)\,\beta(V) - [(1-p\delta)\,cl]^2\times$$

$$\times\left[\beta(U)\beta(V) - l\frac{2^{n-p+2}}{\delta}\beta(U)\,\mathcal{E}_\beta^p(V) - l\frac{2^{n-p+2}}{\delta}\beta(V)\,\mathcal{E}_\beta^p(U)\right].$$

Moreover, by i_1) of Assumptions 3.4 and by taking

$$\delta = \max\left(\sqrt{f(\operatorname{diam} U)},\ \sqrt{f(\operatorname{diam} V)}\right)$$

we obtain

$$\limsup_{h \to +\infty} |\mathbf{Cov}[C_p(F_h \cap U, D), C_p(F_h \cap V, D)]|$$

$$\leq \beta(U)\,\beta(V)(cl)^2 \left[2p\delta + \frac{2^{n-p+2}}{\delta} l(f(\operatorname{diam} U) + f(\operatorname{diam} V)) \right]$$

$$\leq (cl)^2 \beta(U)\,\beta(V) \big[2p \max(\sqrt{f(\operatorname{diam} U)}, \sqrt{f(\operatorname{diam} V)}) +$$

$$+ 2^{n-p+2} l(\sqrt{f(\operatorname{diam} U)} + \sqrt{f(\operatorname{diam} V)}) \big]$$

for every $U, V \in \mathcal{U}$ with $f(\operatorname{diam} U) < 1$ and $f(\operatorname{diam} V) < 1$. Finally, let

$$t_0 = \sup\{t \in \mathcal{O} : f(t) < 1\}.$$

So, for $\eta = f^{-1}(t_0)$,

$$\xi(x, y) = \big[2p \max(\sqrt{f(x)}, \sqrt{f(y)}) + 2^{n-p+2} l(\sqrt{f(x)} + \sqrt{f(y)} \big]$$

and $\beta_1 = cl\beta$, the assertion t_2 of Proposition 4.1 follows and the proof is accomplished.

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