Bollettino
Unione Matematica Italiana

Pirro Oppezzo, Anna Maria Rossi

Existence and uniqueness of solutions for nonlinear and non coercive problems with measure data

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **6-B** (2003), n.2, p. 415–433.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2003_8_6B_2_415_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2003.

Bollettino U. M. I.
(8) **6**-B (2003), 415-433

Existence and Uniqueness of Solutions for Nonlinear and non Coercive Problems with Measure Data.

PIRRO OPPEZZI - ANNA MARIA ROSSI

Sunto. – *Si prova l'esistenza di una soluzione rinormalizzata per un problema ellittico nonlineare noncoercivo in forma di divergenza, in presenza di termini di ordine inferiore al secondo e dato misura. In ipotesi più restrittive si ottiene anche un teorema di unicità.*

Summary. – *We prove the existence of a renormalized solution for a nonlinear non coercive divergence problem with lower order terms and measure data. In a particular case we also give a uniqueness result.*

Introduction.

In this paper we deal with a nonlinear and non coercive divergence equation containing lower order terms. Precisely we consider the following problem:

$$\text{(I)}\quad \begin{cases} -\operatorname{div}(a(\cdot, u, Du) + \Phi(\cdot, u)) + g(\cdot, u, Du) = \kappa & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\xi \in \boldsymbol{R}^n \mapsto a(x, s, \xi) \in \boldsymbol{R}^n$ is monotone, coercive and increases as $|\xi|^{p-1}$. Moreover $s \in \boldsymbol{R} \mapsto \Phi(x, s) \in \boldsymbol{R}^n$ increases as $|s|^{p-1}$, while $\xi \in \boldsymbol{R}^n \mapsto g(x, s, \xi) \in \boldsymbol{R}$ increases as $|\xi|^p$ and satisfies a sign condition with respect to s. Here κ is a Radon measure vanishing on sets with zero p-capacity.

It is well known that a solution of nonlinear Leray-Lions type equations does not lie in $\mathrm{H}_o^{1,p}(\Omega)$ when the data is a measure. Therefore problem (I) has no meaning in the usual distributional sense. So we use the framework of renormalized solutions, which seems proper for such problems with measure data (see [4], [7], [10] and others).

Denoting by $\tau_k(u)$ the truncation $\tau_k(u) = (u \vee (-k)) \wedge k$, then we look for a function $u : \Omega \to \boldsymbol{R}$ such that

$$
\text{(II)}\quad \begin{cases}
\tau_k(u) \in \mathrm{H}_o^{1,p}(\Omega) \qquad \forall k > 0 \\
\displaystyle\int_\Omega \langle a(\cdot, u, Du), D\sigma_k(u)\, v\rangle\, dx + \int_\Omega \langle \Phi(\cdot, u), D\sigma_k(u)\, v\rangle\, dx + \\
\displaystyle + \int_\Omega g(\cdot, u, Du)\, \sigma_k(u)\, v\, dx = \int_\Omega v\sigma_k(u)\, d\kappa \qquad \forall v \in \mathrm{H}_o^{1,p}(\Omega) \cap L^\infty(\Omega) \\
\displaystyle \lim_{k\to\infty} \frac{1}{k} \int_{\{k \leq |u| \leq 2k\}} |Du|^p\, dx = 0
\end{cases}
$$

where $\sigma_k(u) = ((-|u| + k + 1) \vee 0) \wedge 1$.

We get an existence result which improves the one obtained in [10] and [11], where only the lower term g was considered and the one obtained in [3], where $\Phi \equiv 0$ and stronger assumptions on g are made.

We point out that in [8], [9], [10] it was possible to get a stronger limit condition

$$
\lim_{k\to\infty} \int_{\{k \leq |u| \leq k+1\}} |Du|^p\, dx = 0\,,
$$

however also under asymptotic condition in (II), we are able, under regularity conditions for the dependence on u, to obtain an uniqueness result when g depends only on s.

A case where an operator with such a kind of lower order terms was considered is the paper [5]. In [5] the distributional formulation is possible, because g does not depend on Du and the authors assume $p > 2 - \frac{1}{n}$. Just such an assumption allows them to get a solution in a Sobolev space $W_o^{1,q}(\Omega)$, $1 < q < \frac{n}{n-1}(p-1)$.

1. – General hypotheses and definition of the problem.

Let Ω be a bounded open set in $\boldsymbol{R}^n$, $n \geq 2$. Let p be a real number such that $1 < p < n$ and let p' be its Hölder conjugate exponent $\left(\text{i.e., } p' = \frac{p}{p-1}\right)$. Moreover we denote by

$\mathcal{M}_o^p(\Omega) =$

$$
\{\mu : \mathcal{B}(\Omega) \to \boldsymbol{R} : |\mu|(\Omega) < \infty,\ |\mu|(E) = 0 \ \ \forall E \in \mathcal{B}(\Omega) \text{ such that } c_p(E) = 0\},
$$

where $\mathcal{B}(\Omega)$ is the Borel σ-algebra, $|\mu|$ is the total variation of μ and $c_p(E)$ is the usual p-capacity of a set $E \subset \Omega$ with respect to Ω.

For $k, t, \lambda > 0$ we define the functions $\tau_k, \sigma_t \colon \boldsymbol{R} \to \boldsymbol{R}$ as:

$$\tau_k(s) = (s \wedge k) \vee (-k), \qquad \sigma_t(s) = ((-|s| + t + 1) \vee 0) \wedge 1 .$$

We say that $u : \Omega \to \boldsymbol{R}$ is c_p-*quasi continuous* if for every $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset \Omega$, with $c_p(U_\varepsilon) < \varepsilon$, such that $u|_{\Omega \setminus U_\varepsilon}$ is continuous.

We shall write «c_p-a.e.» instead of «almost everywhere with respect to p-capacity».

Analogously to [1] we also define the functional class:

$$\mathcal{C}_0^{1,p}(\Omega) = \{v : \Omega \to \boldsymbol{R} : v \text{ measurable}, \ \ \tau_k(v) \in \mathrm{H}_o^{1,p}(\Omega) \text{ for every } k \in \boldsymbol{R}_+,$$

$$\exists \text{ an unique } c_p\text{-quasi continuous representative } \tilde{v} \text{ of } v\}$$

We recall that for every $u \in \mathcal{C}_0^{1,p}(\Omega)$ there exists a measurable fuction $v : \Omega \to \boldsymbol{R}^n$ such that $D\tau_k(u) = v\mathbf{1}_{\{|u| \leqslant k\}}$ a.e. in Ω for any $k > 0$ (see Lemma 2.1 in [1]). This function v, which is unique up to almost everywhere equivalence, will be denoted by Du.

Now let $a : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a Carathéodory function satisfying the following conditions:

(i) there exist $\nu \in L^1(\Omega)$, $c \in \boldsymbol{R}_+$ such that

$$\langle a(x, s, \xi), \xi \rangle \geqslant \nu(x) + c|\xi|^p$$

for a.e. $x \in \Omega$ and every $s \in \boldsymbol{R}$, $\xi \in \boldsymbol{R}^n$;

(ii) there exist $\mu \in L^{p'}(\Omega)$, $c_1, c_2 \in \boldsymbol{R}_+$ such that

(ii0) $$|a(x, s, \xi)| < \mu(x) + c_1 |\xi|^{p-1} + c_2 |s|^{p-1}$$

(ii1) $$\langle a(x, s, \xi) - a(x, s, \eta), \xi - \eta \rangle > 0$$

for a.e. $x \in \Omega$, for every $s \in \boldsymbol{R}$, $\xi, \eta \in \boldsymbol{R}^n$, $\xi \neq \eta$.
Let $g : \Omega \times \boldsymbol{R} \times \boldsymbol{R}^n \to \boldsymbol{R}$ be a Carathéodory function such that:

(iii) there exist an increasing, continuous function $\beta : [0, \infty) \to [0, \infty)$ and a nonnegative function $d \in L^1(\Omega)$ for which

$$|g(x, s, \xi)| \leqslant \beta(|s|)(|\xi|^p + d(x))$$

for a.e. $x \in \Omega$ and every $s \in \boldsymbol{R}$, $\xi \in \boldsymbol{R}^n$;

(iv) there exists $\varrho \in \boldsymbol{R}_+$ such that

$$g(x, s, \xi) \cdot s \geqslant 0$$

for a.e. $x \in \Omega$, $\xi \in \boldsymbol{R}^n$ and every $s \in \boldsymbol{R}$ such that $|s| \geqslant \varrho$.
Let $\Phi : \Omega \times \boldsymbol{R} \to \boldsymbol{R}^n$ be a Carathéodory function such that:

(v) there exists $b \in L^{\frac{n}{p-1}}(\Omega)$, for which

$$|\Phi(x, s)| \leq b(x)(1 + |s|^{p-1})$$

for a.e. $x \in \Omega$ and every $s \in \boldsymbol{R}$.

The aim of this paper is to prove the existence and, with some restrictions on the data, uniqueness of solutions for problem (I), where $\kappa \in \mathcal{M}_o^p(\Omega)$.

The notion of such a solution is determined by the following definition.

DEFINITION 1.1. – Let $\kappa \in \mathcal{M}_o^p(\Omega)$ be given. We say that $u \in \mathcal{C}_0^{1,p}(\Omega)$ is a renormalized solution of problem (I) if u satisfies

$$\text{(1.1)} \quad \int_\Omega \langle a(\cdot, u, Du), D\sigma_k(u)\, v\rangle\, dx + \int_\Omega \langle \Phi(\cdot, u), D\sigma_k(u)\, v\rangle\, dx + \int_\Omega g(\cdot, u, Du)\, \sigma_k(u)\, v\, dx = \int_\Omega v\sigma_k(u)\, d\kappa$$

for every $k \in \boldsymbol{R}_+$, $v \in \mathrm{H}_o^{1,p}(\Omega) \cap L^\infty(\Omega)$, and

$$\text{(1.2)} \quad \lim_{k \to \infty} \frac{1}{k} \int_{\{k \leq |u| \leq 2k\}} |Du|^p\, dx = 0$$

Such a definition is the same as in [7], because it is equivalent to consider $h \circ u$ in place of $\sigma_k \circ u$, with $h \in W^{1,\infty}(\boldsymbol{R})$ having compact support.

2. – Preliminaries and estimates.

For $h \in \boldsymbol{N}$, let us consider

$$\text{(2.1)} \quad \Phi_h(x, s) = \frac{\Phi(x, s)}{1 + \frac{1}{h}|\Phi(x, s)|}, \qquad g_h(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \frac{1}{h}|g(x, s, \xi)|},$$

with Φ, g like in Section 1.

We inform that from now on we shall denote by C a generic positive constant which can change from a line to another one.

THEOREM 2.1. – *Let* $f \in \mathrm{H}^{-1,p'}(\Omega)$ *and, for* $h \in \boldsymbol{N}$, Φ_h, g_h *as in* (2.1). *Then there exists* $u \in \mathrm{H}_o^{1,p}(\Omega)$, *which solves*

$$\text{(2.2)} \quad a_h(u, v) = \langle f, v\rangle \quad \textit{for any } v \in \mathrm{H}_o^{1,p}(\Omega)$$

where

$$a_h(u, v) = \int_\Omega \langle (a(\cdot, u, Du) + \Phi_h(\cdot, u)), Dv\rangle + \int_\Omega g_h(\cdot, u, Du)\, v\,.$$

PROOF. – Since g_h, Φ_h are bounded by h, we may apply Theorem 2 of [6]. ■

LEMMA 2.2. – *Let $F \in L^{p'}(\Omega)^n$, (f_h) a sequence in $H^{-1,p'}(\Omega) \cap L^1(\Omega)$ such that $\sup_h \|f_h\|_1 < \infty$ and, for any $h \in \mathbf{N}$, $u_h \in H_o^{1,p}(\Omega)$ a solution of* (2.2) *with $f_h - \operatorname{div} F$ in place of f.*

Then there exists $C_m \in \mathbf{R}_+$ such that the following estimate holds

$$\int_{\{|u_h| \leq m\}} |Du_h|^p \leq C_m \qquad \forall h \in \mathbf{N},\ \ \forall m \in \mathbf{R}_+\,.$$

PROOF. – It is not restrictive to assume $m \geq \varrho$. Let us consider $v = \varphi_\lambda(\tau_m(u_h))$ as test function in (2.2), where $\varphi_\lambda(s) = se^{\lambda s^2}$, with $\lambda = \left(\frac{\beta(\varrho)}{2c}\right)^2$.

Being $\int_{\{|u_h| \geq \varrho\}} g_h(\cdot, u_h, Du_h)\, \varphi_\lambda(\tau_m(u_h)) \geq 0$, we have:

$$\int_\Omega \langle a(\cdot, u_h, Du_h), D\tau_m(u_h)\rangle\, \varphi'_\lambda(\tau_m(u_h)) \leq$$

$$\leq \int_\Omega f_h \varphi_\lambda(\tau_m(u_h)) + \int_\Omega \langle F, D\tau_m(u_h)\rangle \varphi'_\lambda(\tau_m(u_h)) +$$

$$- \int_\Omega \langle \Phi_h(\cdot, u_h), D\tau_m(u_h)\rangle\, \varphi'_\lambda(\tau_m(u_h)) - \int_{\{|u_h| < \varrho\}} g_h(\cdot, u_h, Du_h)\, \varphi_\lambda(\tau_m(u_h)) \leq$$

$$\leq \varphi_\lambda(m) \sup_h \|f_h\|_1 + \frac{c}{8} \int_{\{|u_h| \leq m\}} |Du_h|^p + C\varphi'_\lambda(m)^{p'} \|F\|_{p'}^{p'} +$$

$$+ \int_{\{|u_h| \leq m\}} b(1 + m^{p-1})|Du_h|\varphi'_\lambda(\tau_m(u_h)) + \beta(\varrho) \int_{\{|u_h| \leq \varrho\}} (|Du_h|^p + d)|\varphi_\lambda(\tau_m(u_h))| \leq$$

$$\leq \varphi_\lambda(m) \sup_h \|f_h\|_1 + \frac{c}{4} \int_{\{|u_h| \leq m\}} |Du_h|^p + C\varphi'_\lambda(m)^{p'} \|F\|_{p'}^{p'} +$$

$$+ C(1 + m^{p-1})^{p'} \varphi'_\lambda(m)^{p'} \|b\|_{p'}^{p'} +$$

$$+ \beta(\varrho) \int_{\{|u_h| \leq m\}} \frac{1}{c} (\langle a(\cdot, u_h, Du_h), Du_h\rangle + |\nu|)\, |\varphi_\lambda(\tau_m(u_h))| + \beta(\varrho)\, \varphi_\lambda(m) \|d\|_1.$$

Using the following property of φ_λ

$$\varphi_\lambda'(s) - 2\sqrt{\lambda}\,|\varphi_\lambda(s)| \geq \frac{1}{2} \qquad \forall s \in \boldsymbol{R} \tag{2.3}$$

we get

$$\frac{1}{2}\int_{\{|u_h|\leq m\}} (\nu + c|Du_h|^p) \leq$$

$$\leq \int_{\{|u_h|\leq m\}} \langle a(\cdot, u_h, Du_h), Du_h\rangle \left[\varphi_\lambda'(\tau_m(u_h)) - \frac{\beta(\varrho)}{c}|\varphi_\lambda(\tau_m(u_h))|\right] \leq$$

$$\leq \frac{c}{4}\int_{\{|u_h|\leq m\}} |Du_h|^p + \varphi_\lambda(m)\sup_h \|f_h\|_1 + \frac{\beta(\varrho)}{c}\varphi_\lambda(m)(\|\nu\|_1 + C\|d\|_1) +$$

$$+ C[\varphi_\lambda'(m)^{p'}\|F\|_{p'}^{p'} + (1 + m^{p-1})^{p'}\varphi_\lambda'(m)^{p'}\|b\|_{p'}^{p'}].$$

Hence the boundedness of $\left(\int_{\{|u_h|\leq m\}} |Du_h|^p\right)_{h\in \boldsymbol{N}}$ follows. ∎

LEMMA 2.3. – *Let $F \in L^{p'}(\Omega)^n$, (f_h) a sequence in $H^{-1,p'}(\Omega) \cap L^1(\Omega)$ such that* $\sup_h \|f_h\|_1 < \infty$ *and, for any $h \in \boldsymbol{N}$, $u_h \in \mathrm{H}_o^{1,p}(\Omega)$ a solution of* (2.2) *with* $f_h - \operatorname{div} F$ *in place of f. Then there exists a constant $C \in \boldsymbol{R}_+$ such that:*

$$\|\ln(1 + |u_h|)\|_{\mathrm{H}_o^{1,p}(\Omega)} \leq C \qquad \forall h \in \boldsymbol{N}$$

PROOF. – Let $\phi(t) = \int_0^t \frac{1}{(1+|s|)^p}ds$. It is clear that ϕ is bounded and Lipschitz continuous, so the sequence $(\phi(u_h))_h$ is uniformly bounded and lies in $\mathrm{H}_o^{1,p}(\Omega)$. Moreover, by using $\phi(u_h)$ as test function in (2.2), remembering condition (iv) and Lemma 2.2, we have:

$$\int_\Omega c\frac{|Du_h|^p}{(1+|u_h|)^p} + \frac{\nu}{(1+|u_h|)^p} \leq \int_\Omega \langle a(\cdot, u_h, Du_h), D\phi(u_h)\rangle = a_h(u_h, \phi(u_h)) +$$

$$- \int_\Omega \left\langle \Phi_h(\cdot, u_h), \frac{Du_h}{(1+|u_h|)^p}\right\rangle - \int_\Omega g_h(\cdot, u_h, Du_h)\,\phi(u_h) \leq C\int_\Omega |f_h| +$$

$$+ \int_\Omega \left\langle F, \frac{Du_h}{(1+|u_h|)^p}\right\rangle + \int_\Omega b(1+|u_h|^{p-1})\frac{|Du_h|}{(1+|u_h|)^p} + C\int_{\{|u_h|\leq\varrho\}} \beta(\varrho)(|Du_h|^p + d) \leq$$

$$\leq C\sup_h \|f_h\|_1 + C\int_\Omega |F|^{p'} + \frac{c}{4}\int_\Omega \frac{|Du_h|^p}{(1+|u_h|)^p} + \int_\Omega b\frac{|Du_h|}{(1+|u_h|)} + C\beta(\varrho)(C_\varrho + \|d\|_1).$$

Hence by using Young's inequality too:

$$\int_{\Omega} \frac{|Du_h|^p}{(1+|u_h|)^p} \leqslant C\Big[\|\nu\|_1 + \sup_h \|f_h\|_1 + \|F\|_{p'}^{p'} + \|b\|_{p'}^{p'} + \beta(\varrho)(C_\varrho + \|d\|_1)\Big].$$

Then by Poincaré's inequality our assertion follows. ■

THEOREM 2.4. – *Let F, f_h, u_h be like in Lemma 2.2. Then there exist A, $B \in \boldsymbol{R}_+$ such that*

$$\int_{\Omega} |D\tau_k(u_h)|^p \leqslant A + kB \qquad \forall k \in \boldsymbol{R}_+,\ h \in \boldsymbol{N}. \tag{2.4}$$

PROOF. – By using the test function $v = \tau_k(u_h) - \tau_t(u_h)$ in equation (2.2), with $\varrho \leqslant t < k$, where ϱ is given in condition (iv) of Section 1, we get

$$c \int_{\{t \leqslant |u_h| \leqslant k\}} |Du_h|^p + \int_{\{t \leqslant |u_h| \leqslant k\}} \nu \leqslant k \sup_h \|f_h\|_1 + \int_{\{t \leqslant |u_h| \leqslant k\}} \langle F, Du_h \rangle +$$

$$+ \int_{\{t \leqslant |u_h| \leqslant k\}} b(1 + |u_h|^{p-1})|Du_h| \leqslant k \sup_h \|f_h\|_1 + C(\|F\|_{p'}^{p'} + \|b\|_{p'}^{p'}) +$$

$$+ \frac{c}{4} \int_{\{t \leqslant |u_h| \leqslant k\}} |Du_h|^p + C \int_{\{t \leqslant |u_h| \leqslant k\}} b^{p'} |u_h|^p \leqslant k \sup_h \|f_h\|_1 + C(\|F\|_{p'}^{p'} + \|b\|_{p'}^{p'}) +$$

$$+ \frac{c}{4} \int_{\{t \leqslant |u_h| \leqslant k\}} |Du_h|^p + C\Big(\int_{\{t \leqslant |u_h| \leqslant k\}} b^{n/(p-1)}\Big)^{p/n} \Big(\int_{\Omega} |\tau_k(u_h)|^{p^*}\Big)^{p/p^*} \leqslant$$

$$\leqslant k \sup_h \|f_h\|_1 + C(\|F\|_{p'}^{p'} + \|b\|_{p'}^{p'}) + \frac{c}{4} \int_{\{t \leqslant |u_h| \leqslant k\}} |Du_h|^p +$$

$$+ C\Big(\int_{\{t \leqslant |u_h| \leqslant k\}} b^{n/(p-1)}\Big)^{p/n} \Big(\int_{\{|u_h| \leqslant t\}} |Du_h|^p + \int_{\{|u_h| \geqslant t\}} |D\tau_k(u_h)|^p\Big)$$

By Lemma 2.3 we get $|\{|u_h| \geqslant t\}| \leqslant \dfrac{C}{(\ln(1+t))^p}$ for each $h \in \boldsymbol{N}$, $t > 0$, then we choose $t > \varrho$ such that $C\Big(\int_{\{|u_h| \geqslant t\}} b^{n/(p-1)}\Big)^{p/n} < \dfrac{c}{4}$.

Therefore, thanks also to Lemma 2.2, we obtain:

$$\int_{\{t \leqslant |u_h| \leqslant k\}} |Du_h|^p \leqslant C\Big(\int_{\Omega} |\nu| + \|F\|_{p'}^{p'} + \|b\|_{p'}^{p'} + C_t \|b\|_{\frac{n}{p-1}}^{p'}\Big) + k \sup_h \|f_h\|_1.$$

Hence Lemma 2.2 gives the assertion. ■

THEOREM 2.5. – *Let $(u_h)_{h \in \boldsymbol{Z}_+}$ be a sequence in* $\mathrm{H}_o^{1,p}(\Omega)$, *for which estimate* (2.4) *holds. Then there exist $u \in \mathcal{T}_o^{1,p}(\Omega)$, a subsequence of (u_h),* *still denoted by* (u_h), *and a function* $\omega \in L^r(\Omega)^n$ *for every* $1 \leqslant r < \dfrac{n}{n-1}$, *such that*

a) $D\tau_k \circ u_h \rightharpoonup D\tau_k \circ u$ in $L^p(\Omega)^n$ *for each* $k \in \boldsymbol{R}_+$, $u_h \to u$ *a.e. on* Ω.

b) $a(\cdot, u_h, Du_h)\, \mathbf{1}_{\{|u_h| \leqslant k\}} \rightharpoonup \omega\, \mathbf{1}_{\{|u| \leqslant k\}}$ *in* $L^{p'}(\Omega)^n$

c) $|\{x \in \Omega : |u_h| \geqslant \lambda\}| \leqslant S^{p^*}(A+B)^{\frac{p^*}{p}} \lambda^{-p^*/p'}$ *for each* $\lambda \geqslant 1$, *where* S *is the Sobolev embedding constant.*

d) *For each* $1 \leqslant r < \dfrac{n}{n-1}$ $|Du|^{p-1} \in L^r(\Omega)$, $a(\cdot, u_h, Du_h) \rightharpoonup \omega$ *in* $L^r(\Omega)(1 < p \leqslant n)$.

PROOF. – The proof is analogous to the one given in [8] (Theorem 2.2 and Proposition 2.4). ■

THEOREM 2.6. – *Let* F, f_h, u_h *like in Lemma* 2.2. *Moreover we assume* $(f_h)_h$ *to be weakly convergent in* $L^1(\Omega)$. *Then, for a suitable constant* C *and for every* $h \in \boldsymbol{N}$, *the following inequality holds:*

$$\int_{\{k \leqslant |u_h| \leqslant 2k\}} |Du_h|^p \leqslant kC\left[\int_{\{|u_h| \geqslant k\}} |f_h| + \left(\int_{\{|u_h| \geqslant k\}} b^{n/(p-1)}\right)^{p/n}\right] + \\ + C\left(\|\nu\|_1 + \|F\|_{p'}^{p'} + \|b\|_{n/(p-1)}^{p'}\right), \qquad k \geqslant \varrho . \tag{2.5}$$

Besides if $u \in \mathcal{C}_0^{1,p}(\Omega)$ *is the limit of a subsequence of* (u_h) *like in Theorem* 2.5, *it satisfies estimate* (1.2).

PROOF. – Let us consider $v = \tau_{2k}(u_h) - \tau_k(u_h)$ as test function in the equation (2.2) with $k \geqslant \varrho$. Analogously to the proof of Theorem 2.4, we have:

$$\int_{\{k \leqslant |u_h| \leqslant 2k\}} (c|Du_h|^p + \nu) \leqslant 2k \int_{\{|u_h| \geqslant k\}} |f_h| + \frac{c}{4} \int_{\{k \leqslant |u_h| \leqslant 2k\}} |Du_h|^p +$$

$$+ C \int_{\{|u_h| \geqslant k\}} (|F|^{p'} + b^{p'}) + C\left(\int_{\{|u_h| \geqslant k\}} b^{n/(p-1)}\right)^{p/n} \left(\int_\Omega |\tau_{2k}(u_h)|^{p^*}\right)^{(n-p)/n} \leqslant$$

$$\leqslant 2k \int_{\{|u_h| \geqslant k\}} |f_h| + \frac{c}{4} \int_{\{k \leqslant |u_h| \leqslant 2k\}} |Du_h|^p + C \int_{\{|u_h| \geqslant k\}} (|F|^{p'} + b^{p'}) +$$

$$+ C\left(\int_{\{|u_h| \geqslant k\}} b^{n/(p-1)}\right)^{p/n} \left(\int_{\{|u_h| \leqslant 2k\}} |Du_h|^p\right) \leqslant 2k \int_{\{|u_h| \geqslant k\}} |f_h| +$$

$$+ \frac{c}{4} \int_{\{k \leqslant |u_h| \leqslant 2k\}} |Du_h|^p + C \int_{\{|u_h| \geqslant k\}} (|F|^{p'} + b^{p'}) + C\left(\int_{\{|u_h| \geqslant k\}} b^{n/(p-1)}\right)^{p/n} (A + 2kB),$$

in virtue of Theorem 2.4.

Thanks to equi-integrability of f_h, Theorem 2.5 c) and semicontinuity of the norm, it results:

$$\frac{1}{k}\int_{\{k\leq|u_h|\leq 2k\}}|Du|^p\leq\frac{1}{k}\liminf_h\int_{\{k\leq|u_h|\leq 2k\}}|Du_h|^p\leq$$

$$\leq C\left[\varepsilon+\left(\int_{\{|u_h|\geq k\}}b^{n/(p-1)}\right)^{p/n}\right]+\frac{C}{k}\left(\|\nu\|_1+\|F\|_{p'}^{p'}+\|b\|_{p'}^{p'}+\|b\|_{n/(p-1)}^{p'}\right)$$

for each $k\geq k_\varepsilon$, $k_\varepsilon\in\boldsymbol{N}$ suitable. ■

THEOREM 2.7. – *Let $f_h\in C_o^\infty(\Omega)$, $F\in L^{p'}(\Omega)^n$, $h\in\boldsymbol{N}$, $f_h\rightharpoonup f$ in $L^1(\Omega)$. For each $h\in\boldsymbol{N}$, let $u_h\in \mathrm{H}_o^{1,p}(\Omega)$ be solution of* (2.2) *related to f_h and F. Let u be the limit of a subsequence of (u_h), still denoted by (u_h), as in theorem* 2.5. *Then for any $k>0$, it results:*

$$\text{(2.6)}\qquad \lim_{h\to\infty}\int_{\{|u_h|\leq k,|u|\leq k\}}\langle a(\cdot,u_h,Du_h)-a(\cdot,u_h,Du),D(u_h-u)\rangle\,dx=0.$$

PROOF. – For $k>0$, let us consider the function σ_k, introduced in Section 1, $\varphi_\lambda(s)=se^{\lambda s^2}$, with

$$\lambda=\frac{b(k+1)^2}{4c^2}\qquad\text{and for}\quad j\in\boldsymbol{N},\ \ \psi_j(s)=\begin{cases}1 & \text{if } |s|\leq j\\ 0 & \text{if } |s|\geq 2j\\ -|s|/j+2 & \text{if } j\leq|s|\leq 2j\,.\end{cases}$$

We choose $v_{h,j}=\varphi_\lambda(\tau_{2k+2}(u_h-u))\,\psi_j(u_h)\,\sigma_k(u)$ as test function in equation (2.2). In fact $v_{h,j}\in\mathrm{H}_o^{1,p}(\Omega)$ because $v_{h,j}=\varphi_\lambda(\tau_{2k+2}(u_h-\tau_{k+1}(u)))\,\psi_j(u_h)\,\sigma_k(u)$ a.e. in Ω, by definition of σ_k.

Moreover let $k>\varrho$, so in $\{x\in\Omega: |u_h|\geq k+1\}\cap\{x\in\Omega:\sigma_k(u)\neq 0\}$, $v_{h,j}$ has the same sign as u_h. Then, for $j>3k+3$, it results:

$$\int_\Omega\langle a(\cdot,u_h,Du_h)-a(\cdot,u_h,Du),D\tau_{2k+2}(u_h-u)\rangle\,\varphi'_\lambda(\tau_{2k+2}(u_h-u))\,\sigma_k(u)\,dx\leq$$

$$\leq\int_\Omega\langle a(\cdot,u_h,Du_h)-a(\cdot,u_h,Du),D\tau_{2k+2}(u_h-u)\rangle\,\varphi'_\lambda(\tau_{2k+2}(u_h-u))\,\psi_j(u_h)\,\sigma_k(u)\,dx+$$

$$+\int_{\{|u_h|\geq k+1\}}g_h(\cdot,u_h,Du_h)\,v_{h,j}\,dx=\int_\Omega\langle a(\cdot,u_h,Du_h),Dv_{h,j}\rangle\,dx+$$

$$-\int_\Omega\langle a(\cdot,u_h,Du_h),D\psi_j(u_h)\,\sigma_k(u)\rangle\,\varphi_\lambda(\tau_{2k+2}(u_h-u))\,dx+$$

$$- \int_\Omega \langle a(\cdot, u_h, Du), D\tau_{2k+2}(u_h - u)\rangle \varphi'_\lambda(\tau_{2k+2}(u_h - u)) \psi_j(u_h) \sigma_k(u)\, dx +$$

$$+ \int_\Omega g_h(\cdot, u_h, Du_h) v_{h,j}\, dx - \int_{\{|u_h| \le k+1\}} g_h(\cdot, u_h, Du_h) v_{h,j}\, dx = \int_\Omega f_h v_{h,j}\, dx +$$

$$+ \int_\Omega \langle F, Dv_{h,j}\rangle\, dx - \int_{\{|u_h| \le k+1\}} g_h(\cdot, u_h, Du_h) v_{h,j}\, dx +$$

$$- \int_\Omega \langle a(\cdot, u_h, Du_h), D(\psi_j(u_h) \sigma_k(u))\rangle \varphi_\lambda(\tau_{2k+2}(u_h - u))\, dx +$$

$$- \int_\Omega \langle a(\cdot, u_h, Du), D\tau_{2k+2}(u_h - u)\rangle \varphi'_\lambda(\tau_{2k+2}(u_h - u)) \psi_j(u_h) \sigma_k(u)\, dx -$$

$$- \int_\Omega \Phi_h(\cdot, u_h) Dv_{h,j}.$$

By assumptions (i) and (iii) it follows:

$$- \int_{\{|u_h| \le k+1\}} g_h(\cdot, u_h, Du_h) v_{h,j}\, dx \le \beta(k+1)\Bigg[\int_\Omega d|\varphi_\lambda(\tau_{2k+2}(u_h - u))|\, dx +$$

$$+ \int_{\{|u_h| \le k+1\}} |Du_h|^p |\varphi_\lambda(\tau_{2k+2}(u_h - u))| \sigma_k(u)\, dx\Bigg] \le$$

$$\le \beta(k+1)\Bigg[\int_\Omega d|\varphi_\lambda(\tau_{2k+2}(u_h - u))|\, dx +$$

$$+ \frac{1}{c}\int_{\{|u_h| \le k+1\}} (\langle a(\cdot, u_h, Du_h), Du_h\rangle - \nu) |\varphi_\lambda(\tau_{2k+2}(u_h - u))| \sigma_k(u)\, dx\Bigg] \le$$

$$\le \beta(k+1)\Bigg[\int_\Omega d|\varphi_\lambda(\tau_{2k+2}(u_h - u))|\, dx + \frac{1}{c}\int_\Omega |\nu \varphi_\lambda(\tau_{2k+2}(u_h - u))|\, dx +$$

$$+ \frac{1}{c}\int_{\{|u_h - u| \le 2k+2\}} \langle a(\cdot,u_h,Du_h) - a(\cdot,u_h,Du), D(u_h - u)\rangle |\varphi_\lambda(\tau_{2k+2}(u_h - u))| \sigma_k(u)\, dx +$$

$$+ \frac{1}{c}\int_{\{|u_h| \le k+1\}} \langle a(\cdot, u_h, Du_h), Du\rangle |\varphi_\lambda(\tau_{2k+2}(u_h - u))| \sigma_k(u)\, dx +$$

$$+ \frac{1}{c}\int_{\{|u_h| \le k+1\}} \langle a(\cdot, u_h, Du), D(u_h - u)\rangle |\varphi_\lambda(\tau_{2k+2}(u_h - u))| \sigma_k(u)\, dx\Bigg].$$

Let us now evaluate:

$$\int_\Omega \langle a(\cdot, u_h, Du_h), D\psi_j(u_h)\rangle\, \sigma_k(u)\, \varphi_\lambda(\tau_{2k+2}(u_h-u))\, dx \leqslant$$

$$\leqslant \varphi_\lambda(2k+2)\frac{1}{j}\int_{\{j\leqslant |u_h|\leqslant 2j\}} (\mu + c_1|Du_h|^{p-1} + c_2|u_h|^{p-1})\,|Du_h|\,dx \leqslant$$

$$\leqslant \varphi_\lambda(2k+2)\Bigg[\frac{1}{j}\int_{\{|u_h|\geqslant j\}} |\mu|^{p'}\,dx + C\frac{1}{j}\int_{\{j\leqslant|u_h|\leqslant 2j\}} |Du_h|^p\,dx +$$

$$+ c_2(2j)^{p-1}\,|\{|u_h|\geqslant j\}|^{1/p'}\,\frac{1}{j}\Bigg(\int_{\{j\leqslant|u_h|\leqslant 2j\}} |Du_h|^p\,dx\Bigg)^{1/p}\Bigg].$$

By theorem 2.5 c) $(2j)^{p-1}|\{|u_h|\geqslant j\}|^{1/p'}\left(\frac{1}{j}\right)^{1/p'}$ is bounded. Moreover by theorem 2.5 c) again, the measure of $\{|u_h|\geqslant j\}$ decreases with j uniformly with respect to h, so thanks to (2.5) and the assumptions on (f_h) and F, it is possible, for any $\varepsilon>0$, to choose $j_\varepsilon\in\mathbf{N}$ such that

$$\Big|\int_\Omega \langle a(\cdot, u_h, Du_h), D\psi_{j_\varepsilon}(u_h)\rangle\, \sigma_k(u)\, \varphi_\lambda(\tau_{2k+2}(u_h-u))\, dx\Big| \leqslant \varepsilon \quad \text{for any } h\in\mathbf{N}$$

With such a choice of j_ε and remembering the property (2.3) of φ_λ, we get:

$$\frac{1}{2}\int_\Omega \langle a(\cdot, u_h, Du_h) - a(\cdot, u_h, Du), D\tau_{2k+2}(u_h-u)\rangle\, \sigma_k(u)\, dx \leqslant$$

$$\leqslant \int_\Omega \langle a(\cdot, u_h, Du_h) - a(\cdot, u_h, Du), D\tau_{2k+2}(u_h-u)\rangle\, \sigma_k(u)\Bigg[\varphi'_\lambda(\tau_{2k+2}(u_h-u)) +$$

$$-\frac{b(k+1)}{c}\,|\varphi_\lambda(\tau_{2k+2}(u_h-u))|\Bigg]dx \leqslant \int_\Omega f_h\, v_{h,j_\varepsilon}\,dx + \int_\Omega \langle F, Dv_{h,j_\varepsilon}\rangle\,dx + \varepsilon +$$

$$-\int_\Omega \langle a(\cdot, u_h, Du_h), D\sigma_k(u)\rangle\, \psi_{j_\varepsilon}(u_h)\, \varphi_\lambda(\tau_{2k+2}(u_h-u))\,dx +$$

$$-\int_\Omega \langle a(\cdot, u_h, Du), D\tau_{2k+2}(u_h-u)\rangle\, \varphi'_\lambda(\tau_{2k+2}(u_h-u))\, \psi_{j_\varepsilon}(u_h)\, \sigma_k(u)\,dx +$$

$$+\beta(k+1)\Bigg[\int_\Omega d\,|\varphi_\lambda(\tau_{2k+2}(u_h-u))|\,dx + \frac{1}{c}\int_\Omega |\nu\varphi_\lambda(\tau_{2k+2}(u_h-u))|\,dx +$$

$$+\frac{1}{c}\int_{\{|u_h|\leqslant k+1\}}\langle a(\cdot,\,u_h,\,Du_h),\,Du\rangle\,|\varphi_\lambda(\tau_{2k+2}(u_h-u))\,|\sigma_k(u)\,dx+$$

$$\left.+\frac{1}{c}\int_{\{|u_h|\leqslant k+1\}}\langle a(\cdot,u_h,Du),D(\tau_{k+1}(u_h)-\tau_{k+1}(u))\rangle\,|\varphi_\lambda(\tau_{2k+2}(u_h-u))\,|\sigma_k(u)\,dx\right]+$$

$$-\int_{\{|u_h-u|\leqslant 2k+2\}}\langle \Phi_h(\cdot,\,u_h),\,D(u_h-u)\rangle\,\varphi'_\lambda(\tau_{2k+2}(u_h-u))\,\psi_{j_\varepsilon}(u_h)\,\sigma_k(u)\,dx+$$

$$-\int_\Omega\langle \Phi_h(\cdot,\,u_h),\,D\psi_{j_\varepsilon}(u_h)\,\sigma_k(u)\rangle\,\varphi_\lambda(\tau_{2k+2}(u_h-u))\,dx\,.$$

Clearly $v_{h,j_\varepsilon}\to 0$ a.e. in Ω and $(v_{h,j_\varepsilon})_h$ is uniformly bounded; moreover $Dv_{h,j_\varepsilon}\rightharpoonup 0$ in $L^p(\Omega)^n$, because

$$Dv_{h,j_\varepsilon}=\varphi'_\lambda(\tau_{2k+2}(u_h-u))(D\tau_{3k+3}(u_h)-D\tau_{k+1}(u))\,\mathbf{1}_{\{|u_h-u|\leqslant 2k+2\}}\psi_{j_\varepsilon}(u_h)\,\sigma_k(u)+$$

$$-\varphi_\lambda(\tau_{2k+2}(u_h-u))\,D(\tau_{2j_\varepsilon}(u_h)-\tau_{j_\varepsilon}(u_h))\,\frac{\operatorname{sign}(u_h)}{j_\varepsilon}\sigma_k(u)+$$

$$+\varphi_\lambda(\tau_{2k+2}(u_h-u))\,\psi_{j_\varepsilon}(u_h)\,D\sigma_k(u)\,,$$

so it results

$$\lim_h\int_\Omega f_h\,v_{h,j_\varepsilon}\,dx=0,\qquad \lim_h\int_\Omega\langle F,\,Dv_{h,j_\varepsilon}\rangle\,dx=0\,.$$

Besides $a(\cdot,\,u_h,\,Du_h)\,\psi_{j_\varepsilon}(u_h)\rightharpoonup\omega\psi_{j_\varepsilon}(u)$ in the $L^{p'}$-norm by Theorem 2.5 b) and $(D\sigma_k(u))\,\varphi_\lambda(\tau_{2k+2}(u_h-u))\to 0$ in $L^p(\Omega)^n$, so, as $h\to\infty$,

$$\int_\Omega\langle a(\cdot,\,u_h,\,Du_h),\,D\sigma_k(u)\rangle\,\psi_{j_\varepsilon}(u_h)\,\varphi_\lambda(\tau_{2k+2}(u_h-u))\,dx\to 0\,.$$

The fourth integral in the last inequality goes to zero, because

$$D\tau_{2k+2}(u_h-u)\varphi'_\lambda(\tau_{2k+2}(u_h-u))\,\sigma_k(u)\rightharpoonup 0\ \text{ in } L^p(\Omega)^n,\ \text{ while}$$

$$a(\cdot,\,u_h,\,Du)\,\psi_{j_\varepsilon}(u_h)\to a(\cdot,\,u,\,Du)\,\psi_{j_\varepsilon}(u)\ \text{ in } L^{p'}(\Omega)^n.$$

By dominated convergence and reasons similar to previous ones also integrals between square brackets go to zero as $h\to\infty$.

Now we observe that $|\Phi_h(\cdot,\,u_h)\,\psi_{j_\varepsilon}(u_h)|=\dfrac{|\Phi(\cdot,\,u_h)|}{1+(1/h)\,|\Phi(\cdot,\,u_h)|}\psi_{j_\varepsilon}(u_h)\leqslant b(1+(2j_\varepsilon)^{p-1})$ and $\dfrac{1}{h}|\Phi(\cdot,\,u_h)|\leqslant\dfrac{1}{h}b(1+(2j_\varepsilon)^{p-1})\to 0$. So, being Φ a Caratheodory's function, and $b\in L^{n/(p-1)}(\Omega)$, we have $\Phi_h(\cdot,\,u_h)\,\psi_{j_\varepsilon}(u_h)\to\Phi(\cdot,\,u)\,\psi_{j_\varepsilon}(u)$ a.e. in Ω and also in the $L^{n/(p-1)}$-norm. On the other hand

$(D\tau_{2k+2}(u_h-u))\,\sigma_k(u) \rightharpoonup 0$ in $L^p(\Omega)^n$, hence

$$\int_{\{|u_h-u|\leq 2k+2\}} \langle \Phi_h(\cdot, u_h), D(u_h-u)\rangle\, \varphi'_\lambda(\tau_{2k+2}(u_h-u))\, \psi_{j_\varepsilon}(u_h)\, \sigma_k(u)\, dx \to 0\,.$$

Moreover $\Phi_h(\cdot, u_h)\, \varphi_\lambda(\tau_{2k+2}(u_h-u))\, \mathbf{1}_{\{|u_h|\leq 2j_\varepsilon\}} \to 0$ in $L^{n/(p-1)}(\Omega)^n$, while $D\psi_{j_\varepsilon}(u_h)\,\sigma_k(u)+\psi_{j_\varepsilon}(u_h)\,D\sigma_k(u) \rightharpoonup D\psi_{j_\varepsilon}(u)\,\sigma_k(u)+\psi_{j_\varepsilon}(u)\,D\sigma_k(u)$ in $L^p(\Omega)^n$. This implies the convergence to 0 for the last integral.

This concludes the proof. ■

THEOREM 2.8. – *Let $u_h, u \in \mathcal{C}_0^{1,p}(\Omega)$ be such that, for any $k>0$, $\tau_k(u_h) \rightharpoonup \tau_k(u)$ in $H_o^{1,p}(\Omega)$ and a.e. in Ω. If moreover for any $k>0$* (2.6) *holds, then, by passing to a subsequence, if necessary, it results:*

$$Du_h \to Du \text{ a.e. in } \Omega,\ \mathbf{1}_{\{|u|\leq t\}} D\tau_k(u_h) \to \mathbf{1}_{\{|u|\leq t\}} D\tau_k(u)$$
$$\text{strongly in } L^p(\Omega)^n,\ \forall k, t>0\,.$$

PROOF. – See [10] Theorem 3.3. ■

3. – Existence and uniqueness theorems.

THEOREM 3.1. – *Under hypotheses in Section* 1, *let $\kappa \in \mathcal{M}_o^p(\Omega)$. Then there exists a renormalized solution $u \in \mathcal{C}_0^{1,p}(\Omega)$ of problem* (I) *in the sense of Definition* 1.1.

PROOF. – We recall that by Theorem 2.1 of [2] there exist $f\in L^1(\Omega)$ and $F\in L^{p'}(\Omega)^n$ such that $\kappa = f - \operatorname{div} F$. Now let (f_h) be a sequence of $C_o^\infty(\Omega)$ such that $f_h \to f$ in L^1. For every $h\in \boldsymbol{N}$ let $u_h \in H_o^{1,p}(\Omega)$ be a solution of Problem (I) related to $f_h - \operatorname{div} F$. Let u the limit of a subsequence of (u_h) like in Theorem 2.5. Let us prove that u is the desired solution.

Let $v\in H_o^{1,p}(\Omega)\cap L^\infty(\Omega)$, we choose $v\sigma_k(u)\,\sigma_{k+1}(u_h)$ as test function in (2.2). We obtain

$$\int_\Omega \langle a(\cdot, u_h, Du_h), D\sigma_{k+1}(u_h)\,\sigma_k(u)\,v\rangle\, dx + \int_\Omega \langle \Phi_h(\cdot, u_h), D\sigma_{k+1}(u_h)\,\sigma_k(u)\,v\rangle\, dx +$$
$$+\int_\Omega g_h(\cdot, u_h, Du_h)\,\sigma_{k+1}(u_h)\,\sigma_k(u)\,v\,dx =$$
$$=\int_\Omega f_h v\sigma_k(u)\,\sigma_{k+1}(u_h)\,dx + \int_\Omega \langle F, D\sigma_{k+1}(u_h)\,\sigma_k(u)\,v\rangle\,dx\,.$$

We prove that by passing to the limit as $h\to\infty$ in such an equation we get (1.1).

Thanks to the Theorem 2.5 b) and Theorem 2.8, it results

$$a(\cdot, u_h, Du_h)\mathbf{1}_{\{|u_h|\leq k+2\}} \rightharpoonup a(\cdot, u, Du)\,\mathbf{1}_{\{|u|\leq k+2\}} \qquad \text{in } L^{p'}(\Omega)^n$$

while $D(\sigma_k(u)\,\sigma_{k+1}(u_h)\,v) \to D(\sigma_k(u)\,v)$ in $L^p(\Omega)^n$. Therefore

$$\lim_{h\to\infty}\int_\Omega a(\cdot, u_h, Du_h), D\sigma_{k+1}(u_h)\,\sigma_k(u)\,v\rangle\,dx = \int_\Omega \langle a(\cdot, u, Du), D\sigma_k(u)\,v\rangle\,dx\,.$$

In virtue of the estimates:

$$|\mathbf{1}_{\{|u_h|\leq k+2\}}\Phi_h(\cdot, u_h)| \leq b(1+(k+2)^{p-1}) \in L^{n/(p-1)}(\Omega) \text{ and}$$

$$\lim_{h\to\infty}\left|\frac{1}{h}\Phi_h(\cdot, u_h)\,\mathbf{1}_{\{|u_h|\leq k+2\}}\right| \leq \lim_{h\to\infty}\frac{1}{h}b(1+(k+2)^{p-1}) = 0$$

it follows that $\Phi_h(\cdot, u_h)\,\mathbf{1}_{\{|u_h|\leq k+2\}} \to \Phi(\cdot, u)\,\mathbf{1}_{\{|u|\leq k+2\}}$ a.e. and in $L^{n/(p-1)}(\Omega)^n$. Hence:

$$\lim_{h\to\infty}\int_\Omega \langle\Phi_h(\cdot, u_h), D\sigma_{k+1}(u_h)\,\sigma_k(u)\,v\rangle\,dx = \int_\Omega \langle\Phi(\cdot, u), D\sigma_k(u)\,v\rangle\,dx\,.$$

Thanks to the growth of g, assumed in (iii) of Section 1, it results:

$$\mathbf{1}_{\{|u_h|\leq k+2\}}\mathbf{1}_{\{|u|\leq k+1\}}\,|g_h(\cdot, u_h, Du_h)| \leq \beta(k+2)(d+|D\tau_{k+2}(u_h)|^p)\,\mathbf{1}_{\{|u|\leq k+1\}}\,.$$

The right hand side of the previous inequality is strongly convergent in $L^1(\Omega)$, so, remembering that g is a Caratheodory function, we have

$$g_h(\cdot, u_h, Du_h)\,\mathbf{1}_{\{|u_h|\leq k+2\}}\mathbf{1}_{\{|u|\leq k+1\}} \to g(\cdot, u, Du)\,\mathbf{1}_{\{|u|\leq k+1\}} \text{ a.e. and in } L^1(\Omega)\,.$$

This clearly implies:

$$\lim_{h\to\infty}\int_\Omega g_h(\cdot, u_h, Du_h)\,\sigma_{k+1}(u_h)\,\sigma_k(u)\,v\,dx = \int_\Omega g(\cdot, u, Du)\,\sigma_k(u)\,v\,dx\,.$$

Finally the convergence of

$$\int_\Omega f_h v\sigma_k(u)\,\sigma_{k+1}(u_h)\,dx + \int_\Omega \langle F, D\sigma_{k+1}(u_h)\,\sigma_k(u)\,v\rangle\,dx$$

to $\int_\Omega fv\sigma_k(u)\,dx + \int_\Omega \langle F, D\sigma_k(u)\,v\rangle\,dx$ is clear and our assertion is proved.

Estimate (1.2) follows from Theorem 2.6. ■

REMARK 3.2. – If $\Phi = 0$ it is possible to obtain in place of (1.2), the stronger condition

$$\lim_{k\to\infty}\int_{\{k\leq|u|\leq k+1\}} |Du|^p\,dx = 0\,.$$

In [9], in the case where g, Φ, H are equal to zero and a doesn't depend on u, is proved the uniqueness of the renormalized solution for a unilateral problem too.

In the next Theorem we give an uniqueness result in the actual context.

THEOREM 3.3. – *In addition to the hypotheses given in Section* 1, *let us assume on* a, Φ *and* g *the following ones:*

(vii) *there exists* $\mu_1 \in L^{p'}(\Omega)$ *and for each* $k > 0$ *there exists* $c(k) > 0$ *such that:*

$$|a(x, s, \xi) - a(x, s', \xi)| \leq c(k)|s - s'|(\mu_1 + |\xi|^{p-1})$$

for each $s, s' \in [-k, k]$, $\xi \in \mathbf{R}^n$, *and a.e.* x *in* Ω.

$$|\Phi(x, s) - \Phi(x, s')| \leq c(k)\, b|s - s'|$$

for each $s, s' \in [-k, k]$, *and a.e.* x *in* Ω, *where* $b \in L^{n/(p-1)}(\Omega)$ *is the same as in* (vi) *of Section* 1.

(viii) $g : \Omega \times \mathbf{R} \to \mathbf{R}$ *is such that*

$$|g(x, s)| \leq \beta(s)\, d \quad \textit{for a.e. } x \textit{ in } \Omega \textit{ and } \forall s \in \mathbf{R}$$

$$s \in \mathbf{R} \mapsto g(x, s) \in \mathbf{R} \quad \textit{is strictly increasing}.$$

where β, d *are the same of* (iii) *in Section* 1.

Then the renormalized solution of problem (I) *is unique.*

PROOF. – Let $u, w \in \mathcal{C}_0^{1,p}(\Omega)$ be solutions of problem (I).

If $\varepsilon \in \left]0, \frac{1}{3}\right[$ we let $\lambda_\varepsilon(s) = \begin{cases} \dfrac{\ln(s/\varepsilon)}{\ln(1/\varepsilon)} & \text{if } s \geq \varepsilon \\ 0 & \text{if } s < \varepsilon . \end{cases}$

We observe that $\lim_{\varepsilon \to 0} \lambda_\varepsilon(s) = \mathbf{1}_{]0, \infty[}(s)$. Moreover, for $k \geq 1$, $\lambda_\varepsilon(s) \leq 1 + \ln k$ for $s \leq k$. Let ψ_j be defined as in the proof of Theorem 2.7. As test function in (1.1) we choose $v_{\varepsilon,j} = \psi_j(u)\, \psi_j(w)\, \lambda_\varepsilon(\tau_\alpha(u - w))$, with $\alpha > 2j$. It results $v_{\varepsilon,j} \in \mathrm{H}_o^{1,p}(\Omega)$, because $v_{\varepsilon,j} = \psi_j(u)\, \psi_j(w)\, \lambda_\varepsilon(\tau_\alpha(\tau_{2j}(u) - \tau_{2j}(w)))$. Now we choose the same test function $v_{\varepsilon,j}$ in the equation relative to w and we subtract it from the one relative to u. We obtain

$$\int_\Omega \langle a(\cdot, u, Du) - a(\cdot, w, Dw), Dv_{\varepsilon,j}\rangle +$$

$$+ \int_\Omega \langle \Phi(\cdot, u) - \Phi(\cdot, w), Dv_{\varepsilon,j}\rangle + \int_\Omega (g(\cdot, u) - g(\cdot, w))\, v_{\varepsilon,j} = 0$$

We decompose the first integral in the following way

$$\int_\Omega \langle a(\cdot, u, Du) - a(\cdot, u, Dw), D\lambda_\varepsilon(\tau_\alpha(u-w))\rangle\, \psi_j(u)\, \psi_j(w) +$$

$$+\int_\Omega \langle a(\cdot, u, Dw) - a(\cdot, w, Dw), D\lambda_\varepsilon(\tau_\alpha(u-w))\rangle\, \psi_j(u)\, \psi_j(w) +$$

$$+\int_\Omega \langle a(\cdot, u, Du) - a(\cdot, w, Dw), D\psi_j(u)\, \psi_j(w)\rangle\, \lambda_\varepsilon(\tau_\alpha(u-w)).$$

The first integral of such a decomposition is non-negative, so

$$\limsup_{\varepsilon\to 0} \int_\Omega (g(\cdot, u) - g(\cdot, w))\, v_{\varepsilon, j} \leqslant \limsup_{\varepsilon\to 0}\Big[-\int_\Omega \langle \Phi(\cdot, u) - \Phi(\cdot, w), Dv_{\varepsilon, j}\rangle +$$

$$-\int_\Omega \langle a(\cdot, u, Dw) - a(\cdot, w, Dw), D\lambda_\varepsilon(\tau_\alpha(u-w))\rangle\, \psi_j(u)\, \psi_j(w) +$$

$$-\int_\Omega \langle a(\cdot, u, Du) - a(\cdot, w, Dw), D\psi_j(u)\psi_j(w)\rangle\, \lambda_\varepsilon(\tau_\alpha(u-w))\Big].$$

Now we evaluate

$$\limsup_{\varepsilon\to 0} -\int_\Omega \langle \Phi(\cdot, u) - \Phi(\cdot, w), Dv_{\varepsilon, j}\rangle =$$

$$=\limsup_{\varepsilon\to 0}\Big[-\int_{\{\varepsilon < u-w \leqslant \alpha\}} \langle \Phi(\cdot, u) - \Phi(\cdot, w), D\tau_\alpha(u-w)\rangle \frac{1}{\tau_\alpha(u-w)}\, \frac{1}{\ln(1/\varepsilon)}\, \psi_j(u)\, \psi_j(w) +$$

$$-\int_\Omega \langle \Phi(\cdot, u) - \Phi(\cdot, w), D\psi_j(u)\, \psi_j(w)\rangle\, \lambda_\varepsilon(\tau_\alpha(u-w))\Big] \leqslant$$

$$\leqslant \limsup_{\varepsilon\to 0}\Big[c(2j) \int_{\{\varepsilon < u-w \leqslant \alpha\}} b\,|u-w|\,|D\tau_\alpha(u-w)|\, \frac{1}{|\tau_\alpha(u-w)|}\, \frac{1}{\ln(1/\varepsilon)}\, \psi_j(u)\, \psi_j(w) +$$

$$-\int_\Omega \langle \Phi(\cdot, u) - \Phi(\cdot, w), D\psi_j(u)\psi_j(w)\rangle\, \lambda_\varepsilon(\tau_\alpha(u-w))\Big] =$$

$$= -\int_{\{0 < u-w\}} \langle \Phi(\cdot, u) - \Phi(\cdot, w), D\psi_j(u)\, \psi_j(w)\rangle$$

because $b(D\tau_\alpha(u-w))\,\psi_j(u)\,\psi_j(w)\in L^1(\Omega)$ and $\langle \Phi(\cdot,u)-\Phi(\cdot,w),$ $D\psi_j(u)\,\psi_j(w)\rangle\in L^1(\Omega)$. Thanks to assumption (vii) on a analogously we obtain

$$\limsup_{\varepsilon\to 0} -\int_{\{\varepsilon<u-w\leqslant\alpha\}} \langle a(\cdot,u,Dw)-a(\cdot,w,Dw), D\tau_\alpha(u-w)\rangle\cdot$$

$$\cdot\frac{1}{\tau_\alpha(u-w)}\,\frac{1}{\log(1/\varepsilon)}\,\psi_j(u)\,\psi_j(w)=0\,.$$

Moreover it is clear that

$$\limsup_{\varepsilon\to 0} -\int_\Omega \langle a(\cdot,u,Du)-a(\cdot,w,Dw), D\psi_j(u)\,\psi_j(w)\rangle\,\lambda_\varepsilon(\tau_\alpha(u-w))=$$

$$=-\int_{\{0<u-w\}} \langle a(\cdot,u,Du)-a(\cdot,w,Dw), D\psi_j(u)\,\psi_j(w)\rangle\,.$$

Then

$$\limsup_{\varepsilon\to 0}\int_\Omega (g(\cdot,u)-g(\cdot,w))\,v_{\varepsilon,j}=\int_{\{0<u-w\}}(g(\cdot,u)-g(\cdot,w))\,\psi_j(u)\,\psi_j(w)\leqslant$$

$$\leqslant-\int_{\{0<u-w\}}\langle \Phi(\cdot,u)-\Phi(\cdot,w)+a(\cdot,u,Du)-a(\cdot,w,Dw), D\psi_j(u)\,\psi_j(w)\rangle\,.$$

By passing to the limit as $j\to\infty$ and using (1.2) we have

$$\limsup_{j} -\int_{\{0<u-w\}}\langle \Phi(\cdot,u), D\psi_j(w)\rangle\,\psi_j(u)\leqslant$$

$$\leqslant\limsup_{j}\frac{1}{j}\int_{\{j\leqslant|w|\leqslant 2j\}} b(1+|u|^{p-1})\,|Dw|\,\psi_j(u)\leqslant$$

$$\leqslant\limsup_{j}\Bigg[\frac{1}{j}\,\|b\|_{p'}\Bigg(\int_{\{j\leqslant|w|\leqslant 2j\}}|Dw|^p\Bigg)^{1/p}+$$

$$+\left(\frac{1}{j}\right)^{1/p'}\Bigg(\frac{1}{j}\int_{\{j\leqslant|w|\leqslant 2j\}}|Dw|^p\Bigg)^{1/p}(2j)^{p/p'}\Bigg(\Bigg(\int_{\{j\leqslant|w|)}b^{n/p-1}\Bigg)^{p/n}|\{j\leqslant|w|\}|^{1-p/n}\Bigg)^{1/p'}\Bigg]=0\,.$$

In fact, in virtue of estimate (c) of Theorem 2.5 it results

$$|\{j \leqslant |w|\}|^{(1-p/n)1/p'} \leqslant j^{-(p^*/p')(1-p/n)1/p'} \quad \text{and} \quad -\frac{1}{p'} + \frac{p}{p'} - \frac{p^*}{p'}(1-\frac{p}{n})\frac{1}{p'} = 0 .$$

With analogous computations for the other terms of the last side in (3.1), thanks to the corresponding growth conditions, we get

$$\limsup_j \int_{\{0<u-w\}} \langle \Phi(\cdot,u) - \Phi(\cdot,w) + a(\cdot,u,Du) - a(\cdot,w,Dw), D\psi_j(u)\,\psi_j(w)\rangle = 0 .$$

Therefore, by (3.1)

$$\limsup_j \int_{\{0<u-w\}} (g(\cdot,\, u) - g(\cdot,\, w))\, \psi_j(u)\, \psi_j(w) \leqslant 0 .$$

Finally by Fatou's lemma it follows that $\int_{\{0<u-w\}} (g(\cdot,\, u) - g(\cdot,\, w)) \leqslant 0$, so the increasing property of $g(x, \cdot)$ ensures $u \leqslant w$. The roles of u and w may be exchanged, so that $u = w$.

REFERENCES

[1] P. Bènilan - L. Boccardo - T. Gallouët - R. Gariepy - M. Pierre - J. L. Vazquez, *An L^1-Theory of Existence and Uniqueness of Solutions of Nonlinear Elliptic Equations*, Ann. Sc. Norm. Sup. Pisa (4), **22**, no. 2 (1995), 241-273.

[2] L. Boccardo - T. Gallouët - L. Orsina - *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, Ann. Inst. H. Poincarè Anal. Non Linèaire, **13**, no. 5 (1996), 539-551.

[3] L. Boccardo - T. Gallouët - L. Orsina, *Existence and nonexistence of solutions for some nonlinear elliptic equations*, J. Anal. Math., **73** (1997), 203-223.

[4] G. Dal Maso - F. Murat - L. Orsina - A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **28**, no. 4 (1999), 741-808.

[5] T. Del Vecchio - M. R. Posteraro, *Existence and regularity results for nonlinear elliptic equations with measure data*, Adv. Differential Equations, **1**, no. 5 (1996), 899-917.

[6] J. Leray - J. L. Lions, *Quelques résultats de Višik sur les problèmes elliptiques semilinéaires par les mèthodes de Minty et Browder*, Bull. Soc. Math. France, **93** (1965), 97-107.

[7] F. Murat, *Soluciones renormalizadas de EDP elipticas no lineales*, Publications du Laboratoire d'Analyse Numerique n. 93023, Université Pierre et Marie Curie, Paris (1993).

[8] P. Oppezzi - A. M. Rossi, *Renormalized Solutions for Divergence Problems with L^1 Data*, Atti Sem. Mat. Fis. Univ. Modena Suppl. Vol., **46** (1998), 889-914.

[9] P. Oppezzi - A. M. Rossi, *Unilateral problems with measure data: links and convergence*, Differential Integral Equations, **14**, no. 9 (2001), 1051-1076.
[10] P. Oppezzi - A. M. Rossi, *Renormalized Solutions for Equations with Lower Order Terms and Measure Data*, preprint n. 411, DIMA, Università di Genova, April 2000.
[11] A. Porretta, *Existence for elliptic equations in L^1 having lower order terms with natural growth*, Portugal. Math., **57**, no. 2 (2000), 179-190.

Pirro Oppezzi: D. I. M. A, Università di Genova
Via Dodecaneso 35, 16146 Genova, Italy
e-mail: oppezzi@dima.unige.it

Anna Maria Rossi: D. I. M. E. T, Università di Genova
Piazzale Kennedy, Pad. D, 16129 Genova, Italy
e-mail: rossia@dima.unige.it

Pervenuta in Redazione
il 23 ottobre 2001