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Homogenization of a One-Dimensional Model for Compressible Miscible Flow in Porous Media.

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Sunto. – *Si considera un modello unidimensionale di flusso in un mezzo poroso eterogeneo di due fluidi miscibili e compressibili. Si studia l'omogeneizzazione del sistema parabolico che governa tale flusso, e si dimostra la stabilità della derivazione al livello macroscopico.*

Summary. – *We discuss the homogenization of a one-dimensional model problem describing the motion of a compressible miscible flow in porous media. The flow is governed by a nonlinear system of parabolic type coupling the pressure and the concentration. Using the technique of renormalized solutions for parabolic equations and a compensated compactness argument, we prove the stability of the homogenization process.*

1. – Introduction and main result.

We consider a compressible miscible flow in the simple physical setting, a one-dimensional porous medium. We assume that the flow occurs during the time interval $(0, T)$, $T > 0$, in $\Omega = (0, 1)$. Let $\Omega_T = \Omega \times (0, T)$. We denote by u the concentration of mass of one of the two fluids of the mixture, and by p the pressure. The equations of the flow are given in Douglas and Roberts [5], Peaceman [10], Scheideger [11]. The pressure $p(x, t)$ verifies the equation

$$\phi(x)\, a(u)\, \partial_t p + \partial_x q = 0 \quad \text{in } \Omega_T, \tag{1.1}$$

where the rate of flow $q(x, t)$ is given by the Darcy law

$$q = -\frac{k(x)}{\mu(u)} \partial_x p \quad \text{in } \Omega_T, \tag{1.2}$$

where $\phi(x)$ and $k(x)$ are the rock porosity and permeability and $\mu(u)$ is the viscosity of the mixture. We neglect here the gravitational term. The concentration $u(x, t)$ is such that

$$\phi(x)\,\partial_t u + q\,\partial_x u + \phi(x)\,b(u)\,\partial_t p - \partial_x(\phi(x)(d_m + d_p\,|q|)\,\partial_x u) = 0 \quad \text{in } \Omega_T, \tag{1.3}$$

where d_m and d_p are respectively the molecular diffusion constant and the dispersion constant. The functions a and b are defined on the interval $(0, 1)$ by

$$a(u) = (z_1 - z_2)u + z_2, \qquad b(u) = (z_1 - z_2)\,u(1-u),$$

where the nonnegative numbers z_1 and z_2 are the compressibility factors of each component of the mixture.

We assume

$$\phi \in L^\infty(\Omega), \quad 0 < \phi^- \leqslant \phi(x) \leqslant \phi^+ \quad \text{a.e. in } \Omega, \tag{1.4}$$

$$k \in L^\infty(\Omega), \quad 0 < k^- \leqslant k(x) \leqslant k^+ \quad \text{a.e. in } \Omega. \tag{1.5}$$

We consider an extension of μ to $\mathbb{R}$ such that

$$\mu \in W^{1,\infty}(\mathbb{R}), \quad 0 < \mu^- \leqslant \mu(u) \leqslant \mu^+ \quad \forall u \in \mathbb{R}. \tag{1.6}$$

For instance, in the Koval model (cf. Ref. [6]), μ is defined on the interval $(0, 1)$ by

$$\mu(u) = \mu(0)(1 + (M^{1/4} - 1)\,u)^{-4},$$

where $M = \mu(0)/\mu(1)$ is the mobility ratio. The molecular diffusion and the dispersion are assumed such that

$$d_m > 0, \qquad d_p > 0. \tag{1.7}$$

The equations (1.1)-(1.3) are provided with the initial and boundary conditions:

$$q(0, t) = q(1, t) = 0, \qquad p(x, 0) = p_0(x), \tag{1.8}$$

$$\partial_x u(0, t) = \partial_x u(1, t) = 0, \quad u(x, 0) = u_0(x), \tag{1.9}$$

for $x \in \Omega$ and $t \in (0, T)$. For sake of simplicity, we impose here no-flow boundary conditions. But our results remain true for other conditions (see Remark 1 below). We assume that the initial conditions verify

$$p_0 \in H^1(\Omega), \quad u_0 \in H^1(\Omega), \quad 0 \leqslant u_0(x) \leqslant 1 \quad \text{a.e. in } \Omega. \tag{1.10}$$

The existence of a solution (p, u) is proved by Amirat and Ziani [3], using a semi-Galerkin approach and the technique of renormalized solutions for parabolic equations. More precisely, the following result has been established.

THEOREM 1. – *Suppose that assumptions* (1.4)-(1.7), *and* (1.10) *hold. Then, Problem* (1.1)-(1.3), *provided with the boundary conditions* (1.8)-(1.9), *admits a weak solution* (p, u) *in the following sense:*

i) $p \in L^\infty(0, T; W^{1,1}(\Omega)) \cap W^{1,\theta}(\Omega_T)$, *for* $\theta \in (1, 3/2)$, *and is solution of Problem* (1.1), (1.8) *verified in* $L^\theta(\Omega_T)$;

ii) $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, *with* $0 \leq u(x, t) \leq 1$ *for almost every* $(x, t) \in \Omega_T$, *and is a weak solution of* (1.3), (1.9), *that is u satisfies the integral identity*

$$\int_{\Omega_T} \phi u\, \partial_t g\, dx\, dt + \int_{\Omega_T} (q\partial_x u + \phi b(u)\, \partial_t p)\, g\, dx\, dt + \int_{\Omega_T} \phi(d_m + d_p\, |q|)\, \partial_x u \partial_x g\, dx\, dt = \int_{\Omega} \phi(x)\, u_0(x)\, g(x, 0)\, dx$$

with $q = -k(x)\, \partial_x p/\mu(u)$, *and for any testing function g in* $C^1(\overline{\Omega}_T)$ *with support contained in* $\overline{\Omega} \times [0, T[$. *Moreover, the function* $|q|^{1/2}\partial_x u$ *belongs to* $L^2(\Omega_T)$, *and the function* $|q|\partial_x u$ *to* $L^{2s/(s+1)}(\Omega_T)$, *with* $s < 2\theta$.

In this paper, we investigate the homogenization of Problem (1.1)-(1.3), (1.8)-(1.9) when porosity ϕ and permeability k are highly oscillating. Let ε belongs to a sequence of positive real numbers which converges to zero. The porosity and the permeability are now denoted by ϕ^ε and k^ε and are highly oscillating with respect to ε. We assume that ϕ^ε and k^ε are mesurable functions satisfying

$$0 < \phi^- \leq \phi^\varepsilon(x) \leq \phi^+ \quad \text{a.e. in } \Omega,$$

$$0 < k^- \leq k^\varepsilon(x) \leq k^+ \quad \text{a.e. in } \Omega,$$

and we define the functions ϕ^*, ϕ_{-1} and k_{-1} by

$$\phi^\varepsilon \rightharpoonup \phi^*, \ \frac{1}{\phi^\varepsilon} \rightharpoonup \frac{1}{\phi_{-1}} \quad \text{weakly* in } L^\infty(\Omega_T), \tag{1.11}$$

$$\frac{1}{k^\varepsilon}\, \frac{1}{k_{-1}} \quad \text{weakly* in } L^\infty(\Omega_T). \tag{1.12}$$

Let $(p_\varepsilon, u_\varepsilon)$ be a weak solution of

$$\phi^\varepsilon(x)\, a(u_\varepsilon)\, \partial_t p_\varepsilon + \partial_x q_\varepsilon = 0 \quad \text{in } \Omega_T, \tag{1.13}$$

$$q_\varepsilon = -\frac{k^\varepsilon(x)}{\mu(u_\varepsilon)}\, \partial_x p_\varepsilon \quad \text{in } \Omega_T, \tag{1.14}$$

$$\phi^\varepsilon(x)\, \partial_t u_\varepsilon + q_\varepsilon\, \partial_x u_\varepsilon + \phi^\varepsilon(x)\, b(u_\varepsilon)\, \partial_t p_\varepsilon - \partial_x(\phi^\varepsilon(x)(d_m + d_p\, |q_\varepsilon|)\, \partial_x u_\varepsilon) = 0 \quad \text{in } \Omega_T, \tag{1.15}$$

$$(1.16)\qquad q_\varepsilon(0,t)=q_\varepsilon(1,t)=0 \ \text{ for } t\in(0,T),\ p_\varepsilon(x,0)=p_0(x) \ \text{ for } x\in\Omega,$$

$$(1.17)\qquad \partial_x u_\varepsilon(0,t)=\partial_x u_\varepsilon(1,t)=0 \ \text{ for } t\in(0,T),\ u_\varepsilon(x,0)=u_0(x) \ \text{ for } x\in\Omega\,.$$

We want to describe the limit (p, q, u) of the sequence $(p_\varepsilon, q_\varepsilon, u_\varepsilon)$ as $\varepsilon \to 0$. The main result of this paper is the following.

THEOREM 2. – *There exists a subsequence of* $(p_\varepsilon, u_\varepsilon)$ *which weakly converges as* $\varepsilon \to 0$ *to* (p, u), *where* (p, u) *is a weak solution of the homogenized problem*

$$(1.18)\qquad \phi^* a(u)\,\partial_t p + \partial_x q = 0 \quad \text{in } \Omega_T,$$

$$(1.19)\qquad q = -\frac{k_{-1}(x)}{\mu(u)}\,\partial_x p \quad \text{in } \Omega_T,$$

$$(1.20)\qquad \phi^*\,\partial_t u + q\,\partial_x u + \phi^*\, b(u)\,\partial_t p - \partial_x(\phi_{-1}(d_m + d_p\,|q|)\,\partial_x u) = 0 \quad \text{in } \Omega_T,$$

$$(1.21)\qquad q(0,t)=q(1,t)=0 \quad \text{for } t\in(0,T),\ p(x,0)=p_0(x) \ \text{ for } x\in\Omega\,,$$

$$(1.22)\qquad \partial_x u(0,t)=\partial_x u(1,t)=0 \ \text{ for } t\in(0,T),\ u(x,0)=u_0(x) \ \text{ for } x\in\Omega\,.$$

Let us mention some previous papers dealing with the homogenization. For the model without molecular diffusion and dispersion, see Refs. [1]-[2]. The immiscible case was treated in [4].

Our aim is now to establish Theorem 2. The rest of the paper is organized as follows. In Section 2, we establish some estimates on the concentration and the pressure, and we use the technique of renormalized solutions as in Ref. [9] to obtain estimates on the Darcy velocity. In Section 3 we pass to the limit on oscillating solutions through a compensated compactness argument.

2. – Preliminary estimates.

We recall the following result of existence for the system (1.13)-(1.17) (cf. [3]).

THEOREM 3. – *There exists a weak solution* $(p_\varepsilon, u_\varepsilon)$, *with* p_ε *in the space* $L^\infty(0,T; W^{1,1}(\Omega)) \cap W^{1,\theta}(\Omega_T)$, *with* $\theta\in(1, 3/2)$, *and the function* u_ε *in the space* $L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$, *to the nonlinear system* (1.13)-(1.15) *provided with the boundary and initial conditions* (1.16)-(1.17). *Moreover we have*

$$0 \leqslant u_\varepsilon(x,t) \leqslant 1$$

for almost every $(x, t) \in \Omega_T$, *and the function* $|q_\varepsilon|^{1/2} \partial_x u_\varepsilon$ *belongs to* $L^2(\Omega_T)$.

The techniques applied in this section are used in Ref. [3] to prove the existence of a solution, but we recall most of the proofs to clearly specify the dependence of the oscillating solutions on the parameter ε. We begin by the following properties of the rate of flow q_ε and of the concentration u_ε.

LEMMA 1. – i) *The Darcy velocity* q_ε *is uniformly bounded in* $L^\infty(0, T; L^1(\Omega))$. *More precisely, for all* $\varepsilon > 0$, *we have the estimate*

$$\int_\Omega |q_\varepsilon(x, t)|\,dx \leqslant \frac{k^+}{\mu^-} \int_\Omega |p_0'(x)|\,dx \quad \text{a.e. in } (0, T). \tag{2.1}$$

ii) *The sequence* (u_ε) *is uniformly bounded in* $L^2(0, T; H^1(\Omega))$, *and the sequence* $(|q_\varepsilon|^{1/2} \partial_x u_\varepsilon)$ *is uniformly bounded in* $L^2(\Omega_T)$.

PROOF. – The first point comes directly from [3]. For the second one, we multiply Eq. (1.15) by u_ε and integrate over Ω. We obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi^\varepsilon |u_\varepsilon(.\,, t)|^2\,dx + \int_\Omega \phi^\varepsilon (d_m + d_p |q_\varepsilon|)|\partial_x u_\varepsilon|^2\,dx + \int_\Omega q_\varepsilon \partial_x u_\varepsilon u_\varepsilon\,dx + \int_\Omega \phi^\varepsilon \partial_t p_\varepsilon b(u_\varepsilon)\,u_\varepsilon\,dx = 0\,.$$

We introduce the functions $g, h : \mathbb{R} \to \mathbb{R}$, defined as $g(s) = sb(s) - a(s)$ and $h(s) = 1 + g(s)/a(s)$. We can write

$$\begin{aligned}\int_\Omega \phi^\varepsilon b(u_\varepsilon)\,\partial_t p_\varepsilon u_\varepsilon\,dx &= \int_\Omega \phi^\varepsilon a(u_\varepsilon)\,\partial_t p_\varepsilon\,dx + \int_\Omega \phi^\varepsilon g(u_\varepsilon)\,\partial_t p_\varepsilon\,dx \\ &= -\int_\Omega h(u_\varepsilon)\,\partial_x q_\varepsilon\,dx = \int_\Omega h'(u_\varepsilon)\,\partial_x u_\varepsilon q_\varepsilon\,dx\,.\end{aligned}$$

Since $h \in W^{1,\infty}(\mathbb{R})$, and $|u_\varepsilon(x, t)| \leqslant 1$ a.e. in Ω_T, using the Cauchy-Schwarz inequality and the inequality $ab \leqslant \delta a^2 + (1/4\delta)\,b^2$ for any real $\delta > 0$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \phi^\varepsilon |u_\varepsilon(.\,, t)|^2\,dx &+ \int_\Omega \phi^\varepsilon (d_m + d_p |q_\varepsilon|)|\partial_x u_\varepsilon|^2\,dx \\ &\leqslant \int_\Omega |h'(u_\varepsilon)|\,|\partial_x u_\varepsilon|\,|q_\varepsilon|\,dx + \int_\Omega |q_\varepsilon|\,|\partial_x u_\varepsilon|\,|u_\varepsilon|\,dx\end{aligned}$$

$$\leq C_1 \int_\Omega |q_\varepsilon| |\partial_x u_\varepsilon| dx + C_2 \int_\Omega |q_\varepsilon| |\partial_x u_\varepsilon| dx$$

$$\leq \delta \int_\Omega |q_\varepsilon| |\partial_x u_\varepsilon|^2 dx + \frac{C}{4\delta} \int_\Omega |q_\varepsilon| dx ,$$

for any $\delta > 0$. Since the sequence (q_ε) is bounded in $L^\infty(0, T; L^1(\Omega))$, we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \phi^\varepsilon |u_\varepsilon(. , t)|^2 dx + \int_\Omega (d_m \phi^- + (d_p \phi^- - \delta) |q_\varepsilon|) |\partial_x u_\varepsilon|^2 dx \leq \frac{C}{4\delta} ,$$

where C is a constant which does not depend of ε. Taking $0 < \delta < d_p \phi^-$ and using the Gronwall lemma, we obtain the result of the lemma.

We now use the technique of renormalized solutions to estimate the flux function q_ε. We claim that the following result holds true.

LEMMA 2. – *The sequence (q_ε) is bounded in $L^\theta(0, T; W^{1,\theta}(\Omega)) \cap L^s(\Omega_T)$ for any reals $\theta \in (1, 3/2)$ and $s < 2\theta$. Moreover the sequence (p_ε) is bounded in $W^{1,\theta}(\Omega_T)$.*

PROOF. – Note that the function q_ε is a solution of

$$\partial_t \left(\frac{\mu(u_\varepsilon)}{k^\varepsilon} q_\varepsilon \right) - \partial_x \left(\frac{1}{\phi^\varepsilon a(u_\varepsilon)} \partial_x q_\varepsilon \right) = 0 \quad \text{in } \Omega_T, \tag{2.2}$$

$$q_\varepsilon(0, t) = q_\varepsilon(1, t) = 0 \quad \text{for } t \in (0, T), \tag{2.3}$$

$$\left(\frac{\mu(u_\varepsilon)}{k^\varepsilon} q_\varepsilon \right)(x, 0) = -p_0'(x) \quad \text{for } x \in \Omega . \tag{2.4}$$

Let then $m \geq 0$ be an integer. Throughout the sequel we denote by C a generic constant independant of ε and m. We define the odd function S_m on $\mathbb{R}$ by

$$S_m(s) = \begin{cases} 0 & \text{if } 0 \leq s < 2^m, \\ s - 2^m & \text{if } 2^m \leq s < 2^{m+1}, \\ 2^m & \text{if } s \geq 2^{m+1}, \end{cases}$$

and the set B_m by

$$B_m = \{(x, t) \in \Omega_T; 2^m \leq |\mu(u_\varepsilon) q_\varepsilon|(x, t) \leq 2^{m+1}\} .$$

We multiply eq. (2.2) by $S_m(\mu(u_\varepsilon)\, q_\varepsilon)$ and integrate over Ω to obtain

$$\int_\Omega \partial_t\left(\frac{\mu(u_\varepsilon)}{k^\varepsilon} q_\varepsilon\right) S_m(\mu(u_\varepsilon)\, q_\varepsilon)\, dx + \int_\Omega \frac{\mu(u_\varepsilon)}{\phi^\varepsilon a(u_\varepsilon)} S_m'(\mu(u_\varepsilon)\, q_\varepsilon)\, |\partial_x q_\varepsilon|^2\, dx =$$

$$-\int_\Omega \frac{\mu'(u_\varepsilon)}{\phi^\varepsilon a(u_\varepsilon)} S_m'(\mu(u_\varepsilon)\, q_\varepsilon)\, q_\varepsilon\, \partial_x q_\varepsilon\, \partial_x u_\varepsilon\, dx \quad \text{in } (0, T).$$

We introduce the function $\mathcal{S}_m \colon \mathbb{R} \to \mathbb{R}$, defined as $\mathcal{S}_m(s) = \int_0^s S_m(z)\, dz$. Integrating the latter relation on $(0, T)$, we obtain

$$\int_\Omega \frac{1}{k^\varepsilon} \mathcal{S}_m(\mu(u_\varepsilon)\, q_\varepsilon)(x, T)\, dx - \int_\Omega \frac{1}{k^\varepsilon} \mathcal{S}_m(\mu(u_\varepsilon)\, q_\varepsilon)(x, 0)\, dx +$$

$$\int_{B_m} \frac{\mu(u_\varepsilon)}{\phi^\varepsilon a(u_\varepsilon)} S_m'(\mu(u_\varepsilon)\, q_\varepsilon)\, |\partial_x q_\varepsilon|^2\, dx\, dt =$$

$$-\int_{B_m} \frac{\mu'(u_\varepsilon)}{\phi^\varepsilon a(u_\varepsilon)} S_m'(\mu(u_\varepsilon)\, q_\varepsilon)\, q_\varepsilon\, \partial_x q_\varepsilon\, \partial_x u_\varepsilon\, dx\, dt\, .$$

Since $(\mu(u_\varepsilon)\, q_\varepsilon)(x, 0) = -k^\varepsilon(x)\, p_0'(x)$ in Ω and $|\mathcal{S}_m(s)| \leqslant 2^m |s|$ for any $s \in \mathbb{R}$, and using the properties of S_m, k, ϕ, μ and a, we have

$$\int_\Omega \frac{1}{k^\varepsilon} \mathcal{S}_m(\mu(u_\varepsilon)\, q_\varepsilon)(x, T)\, dx + \int_{B_m} \frac{\mu(u_\varepsilon)}{\phi^\varepsilon a(u_\varepsilon)} S_m'(\mu(u_\varepsilon)\, q_\varepsilon)\, |\partial_x q_\varepsilon|^2\, dx\, dt \leqslant$$

$$2^m \frac{k^+}{\mu^-} \int_\Omega |p_0'(x)|\, dx + C \int_{B_m} |q_\varepsilon|\, |\partial_x q_\varepsilon|\, |\partial_x u_\varepsilon|\, dx\, dt \leqslant$$

$$2^m C \int_\Omega |p_0'(x)|\, dx + \delta \int_{B_m} |\partial_x q_\varepsilon|^2\, dx\, dt + \frac{C}{4\delta} \int_{B_m} |q_\varepsilon|^2\, |\partial_x u_\varepsilon|^2\, dx\, dt \leqslant$$

$$2^m C \int_\Omega |p_0'(x)|\, dx + \delta \int_{B_m} |\partial_x q_\varepsilon|^2\, dx\, dt + 2^m \frac{C}{2\delta} \int_{B_m} |q_\varepsilon|\, |\partial_x u_\varepsilon|^2\, dx\, dt\, ,$$

for any real $\delta > 0$. By hypothesis on p_0 and Lemma 1-ii), it implies

$$\frac{1}{2^m} \int_{B_m} |\partial_x q_\varepsilon|^2\, dx\, dt \leqslant C\, . \tag{2.5}$$

Now let θ be a real number, $1 < \theta < 2$. Then using the Hölder inequality, we get

$$\int_{B_m} |\partial_x q_\varepsilon|^\theta \,dx\,dt \leqslant \left(\int_{B_m} |\partial_x q_\varepsilon|^2 \,dx\,dt\right)^{\theta/2} |B_m|^{1-\theta/2},$$

where $|B_m|$ denotes the measure of B_m. We note that on B_m, we have $|q_\varepsilon| \geqslant C2^m$ where the constant C depends only on variations of the function μ. Therefore,

$$|B_m| \leqslant \frac{C}{2^m} \int_{B_m} |q_\varepsilon(x,t)| \,dx\,dt.$$

By the Hölder inequality, we also have

$$\int_{B_m} |q_\varepsilon(x,t)| \,dx\,dt \leqslant \left(\int_{B_m} |q_\varepsilon(x,t)|^s \,dx\,dt\right)^{1/s} |B_m|^{1/s'},$$

for any $s, s' > 1$ with $1/s + 1/s' = 1$. This produces

$$|B_m| \leqslant \frac{C}{2^{ms}} \left(\int_{B_m} |q_\varepsilon(x,t)|^s \,dx\,dt\right).$$

Thus, in view of inequality (2.5),

$$\int_{B_m} |\partial_x q_\varepsilon(x,t)|^\theta \,dx\,dt \leqslant \frac{C}{2^{m(s(1-\theta/2)-\theta/2)}} \left(\int_{B_m} |q_\varepsilon(x,t)|^s \,dx\,dt\right)^{1-\theta/2}$$

and then

$$\text{(2.6)} \qquad \sum_{m\geqslant 0} \int_{B_m} |\partial_x q_\varepsilon(x,t)|^\theta \,dx\,dt \leqslant \sum_{m\geqslant 0} \frac{C}{2^{m(s(1-\theta/2)-\theta/2)}} \left(\int_{B_m} |q_\varepsilon(x,t)|^s \,dx\,dt\right)^{1-\theta/2}.$$

We now choose $s > \theta/(2-\theta)$. Using the discrete Hölder inequality, the right-hand side of inequality (2.6) is majorized as

$$\sum_{m\geqslant 0} \frac{1}{2^{m(s(1-\theta/2)-\theta/2)}} \left(\int_{B_m} |q_\varepsilon|^s \,dx\,dt\right)^{1-\theta/2} \leqslant \left(\sum_{m\geqslant 0} \frac{1}{2^{mr(s(1-\theta/2)-\theta/2)}}\right)^{1/r} \left(\sum_{m\geqslant 0} \left(\int_{B_m} |q_\varepsilon|^s \,dx\,dt\right)^{(1-\theta/2)\,r'}\right)^{1/r'}$$

with $1/r + 1/r' = 1$. Choosing $r' = 2/(2-\theta)$, we infer from (2.6)

$$\sum_{m \geq 0} \int_{B_m} |\partial_x q_\varepsilon|^\theta \, dx \, dt \leq C \left(\sum_{m \geq 0} \int_{B_m} |q_\varepsilon|^s \, dx \, dt \right)^{1-\theta/2} \leq C \left(\int_{\Omega_T} |q_\varepsilon|^s \, dx \, dt \right)^{1-\theta/2}.$$

Now we define B as the set

$$B = \{(x, t) \in \Omega_T ; \, 0 \leq |\mu(u_\varepsilon) \, q_\varepsilon| (x, t) \leq 1\},$$

so that $\Omega_T = B \cup \left(\bigcup_{m \geq 0} B_m \right)$. So we have to estimate $\int_B |\partial_x q_\varepsilon|^\theta \, dx \, dt$. Let $R, \mathcal{R} \colon \mathbb{R} \to \mathbb{R}$ defined as

$$R(s) = \begin{cases} 1 & \text{if } s \geq 1, \\ s & \text{if } -1 \leq s \leq 1, \\ -1 & \text{if } s \geq -1, \end{cases}$$

$$\mathcal{R}(s) = \int_0^s R(z) \, dz \, .$$

We multiply Eq. (2.2) by $R(\mu(u_\varepsilon) \, q_\varepsilon)$ and integrate over Ω_T. Similarly to the first part of the proof, we get

$$\int_\Omega \frac{1}{k^\varepsilon} \mathcal{R}(\mu(u_\varepsilon) \, q_\varepsilon)(x, T) \, dx - \int_\Omega \frac{1}{k^\varepsilon} \mathcal{R}(\mu(u_\varepsilon) \, q_\varepsilon)(x, 0) \, dx +$$

$$\int_B \frac{\mu(u_\varepsilon)}{\phi^\varepsilon a(u_\varepsilon)} R'(\mu(u_\varepsilon) \, q_\varepsilon) \, |\partial_x q_\varepsilon|^2 \, dx \, dt =$$

$$- \int_B \frac{\mu'(u_\varepsilon)}{\phi^\varepsilon a(u_\varepsilon)} R'(\mu(u_\varepsilon) \, q_\varepsilon) \, q_\varepsilon \, \partial_x q_\varepsilon \, \partial_x u_\varepsilon \, dx \, dt \, .$$

We have

$$\left| \int_\Omega \frac{1}{k^\varepsilon} \mathcal{R}(\mu(u_\varepsilon) \, q_\varepsilon)(x, 0) \, dx \right| \leq C \int_\Omega |p_0'(x)| \, dx \, .$$

Then, using the properties of $\mathcal{R}$, ϕ, μ and a, it follows

$$\int_\Omega \frac{1}{k^\varepsilon} \mathcal{R}(\mu(u_\varepsilon) \, q_\varepsilon)(x, T) \, dx + \int_B \frac{\mu(u_\varepsilon)}{\phi^\varepsilon a(u_\varepsilon)} R'(\mu(u_\varepsilon) \, q_\varepsilon) \, |\partial_x q_\varepsilon|^2 \, dx \, dt \leq$$

$$C \int_\Omega |p_0'(x)| \, dx + \delta \int_B |\partial_x q_\varepsilon|^2 \, dx \quad dt + \frac{C}{4\delta} \int_B |q_\varepsilon| \, |\partial_x u_\varepsilon|^2 \, dx \, dt \, ,$$

for any real $\delta > 0$. This implies

$$\int_B |\partial_x q_\varepsilon|^2 \, dx \, dt \leqslant C \, .$$

Finally we have established the estimate

$$\int_{\Omega_T} |\partial_x q_\varepsilon|^\theta \, dx \, dt \leqslant C\left(1 + \left(\int_{\Omega_T} |q_\varepsilon|^s \, dx \, dt\right)^{1-\theta/2}\right),$$

for any $s > 2/(2-\theta)$.

We now use the Gagliardo-Nirenberg multiplicative embedding inequality for the flux function $q_\varepsilon(\cdot, t)$, which satisfies $q_\varepsilon(0, t) = q_\varepsilon(1, t) = 0$ for almost every $t \in (0, T)$. We have

$$\left(\int_\Omega |q_\varepsilon(x, t)|^s \, dx\right)^{1/s} \leqslant C\left(\int_\Omega |\partial_x q_\varepsilon(x, t)|^\theta \, dx\right)^{\lambda/\theta} \left(\int_\Omega |q_\varepsilon(x, t)|^r \, dx\right)^{(1-\lambda)/r},$$

with $r \geqslant 1$, $0 \leqslant \lambda \leqslant 1$, and such that

$$\lambda = \left(\frac{1}{r} - \frac{1}{s}\right)\left(1 - \frac{1}{\theta} + \frac{1}{r}\right)^{-1}.$$

We take $r = 1$, and this gives

$$\lambda = \left(1 - \frac{1}{s}\right)\left(2 - \frac{1}{\theta}\right)^{-1}.$$

Then, since $\|q_\varepsilon\|_{L^\infty(0, T; L^1(\Omega))} \leqslant C$,

$$\int_{\Omega_T} |q_\varepsilon(x, t)|^s \, dx \, dt \leqslant C \int_0^T \left(\int_\Omega |\partial_x q_\varepsilon(x, t)|^\theta \, dx\right)^{(s-1)/(2\theta-1)} dt \, .$$

Applying the Hölder inequality to the right-hand side yields for $s < 2\theta$

$$\int_0^T \left(\int_\Omega |\partial_x q_\varepsilon(x, t)|^\theta \, dx\right)^{(s-1)/(2\theta-1)} dt \leqslant C\left(\int_{\Omega_T} |\partial_x q_\varepsilon(x, t)|^\theta \, dx \, dt\right)^{(s-1)/(2\theta-1)}.$$

Therefore,

$$\int_{\Omega_T} |\partial_x q_\varepsilon|^\theta \, dx \, dt \leqslant C\left(1 + \left(\int_{\Omega_T} |\partial_x q_\varepsilon|^\theta \, dx \, dt\right)^{\theta^*}\right),$$

with $\theta^* = \dfrac{(2-\theta)(s-1)}{2(2\theta-1)}$, for $\theta/(2-\theta) < s < 2\theta$. We note that $0 < \theta^* < 1$. This implies

$$\|\partial_x q_\varepsilon\|_{L^\theta(\Omega_T)} \leq C,$$

$$\|q_\varepsilon\|_{L^s(\Omega_T)} \leq C,$$

for any real $\theta < 3/2$. And this ends the proof of the lemma.

We can now state and prove the following result.

LEMMA 3. – *The sequences* (u_ε) *and* (q_ε) *are sequentially compact in* $L^2(\Omega_T)$.

PROOF. – In view of Lemma 2, the sequence (q_ε) is bounded in $L^\theta(0, T; W^{1,\theta}(\Omega))$. Moreover, it follows from Eq. (2.2) that $\left(\partial_t\left(\dfrac{\mu(u_\varepsilon)}{k^\varepsilon} q_\varepsilon\right)\right)$ is uniformly bounded in $L^\theta(0, T; W^{-1,\theta}(\Omega))$. Let the real θ' be defined by $1/\theta + 1/\theta' = 1$. Since $\left(\dfrac{\mu(u_\varepsilon)}{k^\varepsilon} q_\varepsilon\right)$ is bounded in $L^s(\Omega_T)$, and since the imbedding $L^s(\Omega) \subset W^{-1,\theta'}(\Omega)$ is compact, a compactness argument of Aubin (cf Ref. [7]) implies that $\left(\dfrac{\mu(u_\varepsilon)}{k^\varepsilon} q_\varepsilon\right)$ is sequentially compact in $L^{\theta'}(0, T; W^{-1,\theta'}(\Omega))$.

Now let (w_ε) be the sequence defined by

$$w_\varepsilon = \mu(u_\varepsilon)\, q_\varepsilon.$$

Since (w_ε) is bounded in $L^\theta(0, T; W^{1,\theta}(\Omega))$ (cf. lemmas 1-ii) and 2), we can define a function $w \in L^\theta(0, T; W^{1,\theta}(\Omega))$ such that (up to a subsequence, not relabeled for convenience)

$$w_\varepsilon \rightharpoonup w \quad \text{weakly in } L^\theta(0, T; W^{1,\theta}(\Omega)).$$

We then assert that

$$\frac{w_\varepsilon}{k^\varepsilon} \rightharpoonup \frac{w}{k_{-1}} \quad \text{weakly in } L^\theta(\Omega_T).$$

Indeed, taking a test function $\psi \in \mathcal{D}(\Omega_T)$, we study the limit as $\varepsilon \to 0$ of

$$\int_{\Omega_T} \frac{w_\varepsilon(x,t)}{k^\varepsilon(x)} \psi(x,t)\, dx\, dt = \int_\Omega \frac{1}{k^\varepsilon(x)} \left(\int_0^T w_\varepsilon(x,t)\, \psi(x,t)\, dt \right) dx\,.$$

Let $v_\varepsilon \in L^\theta(\Omega)$ be defined by

$$v_\varepsilon(x) = \int_0^T w_\varepsilon(x,t)\, \psi(x,t)\, dt\,.$$

We have $\partial_x v_\varepsilon \in L^\theta(\Omega)$, and so the sequence (v_ε) is compact in $L^\theta(\Omega)$. We denote by v its strong limit, then

$$v_\varepsilon \to v \quad \text{in } L^\theta(\Omega),$$

and v satisfies

$$v(x) = \int_0^T w(x, t)\, \psi(x, t)\, dt .$$

We then have as $\varepsilon \to 0$

$$\int_{\Omega_T} \frac{w_\varepsilon(x, t)}{k^\varepsilon(x)} \psi(x, t)\, dx\, dt \to \int_\Omega \frac{v(x)}{k_{-1}(x)} dx =$$

$$\int_\Omega \frac{1}{k_{-1}(x)} \left(\int_0^T w(x, t)\, \psi(x, t)\, dt \right) dx = \int_{\Omega_T} \frac{w(x, t)}{k_{-1}(x)} \psi(x, t)\, dx\, dt ,$$

and our assertion is proved.

We then obtain the sequential compactness of (w_ε) in $L^2(\Omega_T)$ by using the following decomposition

$$\int_{\Omega_T} \frac{1}{k^\varepsilon} (w_\varepsilon - w)^2\, dx\, dt = \left\langle \frac{1}{k^\varepsilon} w_\varepsilon,\, w_\varepsilon - w \right]_{L^{\theta\subset}(0, T; W^{-1, \theta'}(\Omega)) \times L^\theta(0, T; W_0^{1, \theta}(\Omega))} -$$

$$\int_{\Omega_T} \frac{w_\varepsilon}{k^\varepsilon} w\, dx\, dt + \int_{\Omega_T} \frac{1}{k^\varepsilon} w^2\, dx\, dt .$$

Clearly, the right-hand side tends to zero with ε. And since $k^\varepsilon(x) > 0$ a.e. in Ω, we have shown

$$w_\varepsilon \to w \quad \text{in } L^2(\Omega_T).$$

Furthermore, in view of lemmas 1 and 2, and of eq. (1.15), the sequence $(\phi^\varepsilon \partial_t u_\varepsilon)$ is bounded in $L^\theta(0, T; W^{-1, \theta}(\Omega))$. Since (u_ε) is bounded in $L^\infty(\Omega_T)$ and then $(\phi^\varepsilon u_\varepsilon)$ in $L^{\theta'}(\Omega_T)$, a compactness argument of Aubin implies that $(\phi^\varepsilon u_\varepsilon)$ is sequentially compact in $L^{\theta'}(0, T; W^{-1, \theta'}(\Omega))$. On the other hand, we can assert the existence of a function $u \in L^2(0, T; H^1(\Omega))$ such that, up to a subsequence,

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(0, T; H^1(\Omega)).$$

Following the lines used to prove the weak convergence of the sequence

$(w_\varepsilon/k^\varepsilon)$ to w/k_{-1} in $L^\theta(\Omega_T)$, we can state that

$$\phi^\varepsilon u_\varepsilon \rightharpoonup \phi^* u \quad \text{weakly in } L^2(\Omega_T).$$

We then consider the decomposition

$$\int_{\Omega_T} \phi^\varepsilon (u_\varepsilon - u)^2 \, dx \, dt = \langle \phi^\varepsilon u_\varepsilon, u_\varepsilon - u \rangle_{L^{\theta'}(0,T;W^{-1,\theta'}(\Omega)) \times L^\theta(0,T;W_0^{1,\theta}(\Omega))} - \int_{\Omega_T} \phi^\varepsilon u_\varepsilon u \, dx \, dt + \int_{\Omega_T} \phi^\varepsilon u^2 \, dx \, dt \, .$$

Since its right-hand side tends to zero with ε, and since $\phi^\varepsilon(x) \geq \phi^- > 0$ a.e. in Ω, we conclude that the sequence (u_ε) converges strongly to u in $L^2(\Omega_T)$.

Finally, since the sequences $(w_\varepsilon = \mu(u_\varepsilon)\, q_\varepsilon)$ and (u_ε) are compact in $L^2(\Omega_T)$, we conclude that (q_ε) is sequentially compact in $L^2(\Omega_T)$.

3. – Proof of Theorem 2.

The results of the previous section allow us to assert the existence of functions $u \in L^2(0, T; H^1(\Omega))$, $q \in L^s(\Omega_T) \cap L^\theta(0, T; W^{1,\theta}(\Omega))$, $p \in W^{1,\theta}(\Omega_T)$ such that, up to subsequences not relabeled for convenience, we have the following convergences

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \text{ and a.e. in } \Omega_T,$$

$$q_\varepsilon \rightharpoonup q \quad \text{weakly in } L^\theta(0, T; W^{1,\theta}(\Omega)), \text{ and a.e. in } \Omega_T,$$

$$p_\varepsilon \rightharpoonup p \quad \text{weakly in } W^{1,\theta}(\Omega_T), \text{ and a.e. in } \Omega_T.$$

Moreover, we recall that we have shown in the proof of Lemma 3 that

$$\frac{\mu(u_\varepsilon)\, q_\varepsilon}{k^\varepsilon} \rightharpoonup \frac{\mu(u)\, q}{k_{-1}} \quad \text{weakly in } L^\theta(\Omega_T),$$

and then

$$q = -\frac{k_{-1}(x)}{\mu(u)} \partial_x p \, .$$

On the other hand, we can assert

$$\frac{1}{a(u_\varepsilon)} \partial_x q_\varepsilon \rightharpoonup \frac{1}{a(u)} \partial_x q \quad \text{weakly in } L^\theta(\Omega_T).$$

We are now going to determine the limit of the sequence $\left(\phi^\varepsilon \partial_t p_\varepsilon = -\frac{\partial_x q_\varepsilon}{a(u_\varepsilon)} \right)$

by using the Div-Curl lemma of Murat and Tartar (cf. Ref. [8]) as follows. We define two sequences of vector fields (A^{ε}) and (B^{ε}) in $\mathbb{R}^2$ by

$$A^{\varepsilon} = (\phi^{\varepsilon}, 0),$$

$$B^{\varepsilon} = \left(\frac{\partial_x q_\varepsilon}{a(u_\varepsilon)\, \phi^{\varepsilon}}, \frac{\mu(u_\varepsilon)}{k^{\varepsilon}} q_\varepsilon \right).$$

We first note that the sequence (A^{ε}) (respectively (B^{ε})) is bounded in $(L^{\theta'}(\Omega_T))^2$ (respectively in $(L^{\theta}(\Omega_T))^2$), where the real θ' is such that $1/\theta + 1/\theta' = 1$. Moreover,

$$\mathrm{div}_{t,x}(A^{\varepsilon}) = \mathrm{curl}_{t,x}(B^{\varepsilon}) = 0\,.$$

Therefore,

$$\langle A^{\varepsilon}, B^{\varepsilon}\rangle = -\phi^{\varepsilon}\,\partial_t p_\varepsilon \rightharpoonup \langle A, B\rangle = -\phi^*\,\partial_t p \text{ in the sense of distributions in } \Omega_T.$$

On the other hand, $\langle A^{\varepsilon}, B^{\varepsilon}\rangle = \frac{\partial_x q_\varepsilon}{a(u_\varepsilon)} \rightharpoonup \frac{\partial_x q}{a(u)}$ weakly in $L^{\theta}(\Omega_T)$. The identification gives

$$-\phi^*\,\partial_t p = \frac{\partial_x q}{a(u)}\,.$$

Then, we deduce that q and p are solutions of the following system

$$\phi^*\, a(u)\,\partial_t p + \partial_x q = 0 \quad \text{in } \Omega_T, \tag{3.7}$$

$$q = -\frac{k_{-1}(x)}{\mu(u)}\,\partial_x p \quad \text{in } \Omega_T, \tag{3.8}$$

$$q(0,t) = q(1,t) = 0, \quad p(x,0) = p_0(x)\,. \tag{3.9}$$

We study now the limit behaviour of the concentration equation (1.15):

$$\phi^{\varepsilon}\,\partial_t u_\varepsilon + q_\varepsilon\,\partial_x u_\varepsilon + \phi^{\varepsilon}\, b(u_\varepsilon)\,\partial_t p_\varepsilon - \partial_x(\phi^{\varepsilon}(d_m + d_p\,|q_\varepsilon|)\,\partial_x u_\varepsilon) = 0\,.$$

Our previous results, and in particular the strong convergence of the sequences (q_ε) and (u_ε), allow us to pass to the limit as $\varepsilon \to 0$ and to obtain u solution of

$$\phi^*\,\partial_t u + q\,\partial_x u + \phi^*\, b(u)\,\partial_t p - \partial_x(\overline{\phi^{\varepsilon}(d_m + d_p\,|q_\varepsilon|)\,\partial_x u_\varepsilon}) = 0 \quad \text{in } \Omega_T, \tag{3.10}$$

where $(\overline{\phi^{\varepsilon}(d_m + d_p\,|q_\varepsilon|)\,\partial_x u_\varepsilon})$ denotes at this step the weak $L^{\theta}(\Omega_T)$-limit of the sequence $(\phi^{\varepsilon}(d_m + d_p\,|q_\varepsilon|)\,\partial_x u_\varepsilon)$. Let us specify this limit. To this aim, we use once again the Div-Curl lemma. We define the vectors fields C^{ε} and D^{ε} in

$\mathbb{R}^2$ by

$$C^{\varepsilon} = \left(\frac{1}{\phi^{\varepsilon}}, 0\right),$$

$$D^{\varepsilon} = (\phi^{\varepsilon}(d_m + d_p |q_\varepsilon|)\, \partial_x u_\varepsilon, \phi^{\varepsilon} u_\varepsilon).$$

The sequence (C^{ε}) (respectively (D^{ε})) is uniformly bounded in $(L^{\theta'}(\Omega_T))^2$ (respectively in $(L^{\theta}(\Omega_T))^2$). We have $\mathrm{div}_{t,x}(C^{\varepsilon}) = 0$. And by Eq. (1.15), $\mathrm{curl}_{t,x}(D^{\varepsilon}) = q_\varepsilon \partial_x u_\varepsilon + \phi^{\varepsilon} b(u_\varepsilon)\, \partial_t p_\varepsilon$ is also bounded in $L^{\theta}(\Omega_T)$. Therefore,

$$\langle C^{\varepsilon}, D^{\varepsilon}\rangle \rightharpoonup \frac{1}{\phi_{-1}} \overline{(\phi^{\varepsilon}(d_m + d_p |q_\varepsilon|)\, \partial_x u_\varepsilon)}$$

in the sense of distributions in Ω_T. On the other hand,

$$\langle C^{\varepsilon}, D^{\varepsilon}\rangle = (d_m + d_p |q_\varepsilon|)\, \partial_x u_\varepsilon \rightharpoonup (d_m + d_p |q|)\, \partial_x u$$

weakly in $L^{\theta}(\Omega_T)$. By the unicity of the limit, we conclude that

$$\overline{(\phi^{\varepsilon}(d_m + d_p |q_\varepsilon|)\, \partial_x u_\varepsilon)} = \phi_{-1}(d_m + d_p |q|)\, \partial_x u.$$

The function u is therefore solution of

$$\phi^* \partial_t u + q\, \partial_x u + \phi^* b(u)\, \partial_t p - \partial_x(\phi_{-1}(d_m + d_p |q|)\, \partial_x u) = 0 \quad \text{in } \Omega_T. \tag{3.11}$$

This completes the proof of Theorem 2.

REMARK 1. – *This result remains true for other choices of boundary conditions in* (1.8)-(1.9) *and* (1.16)-(1.17). *Let us consider for instance the case of a given pressure drop: we replace* (1.16)-(1.17) *by*

$$p_\varepsilon(0, t) = p_1(t), \quad p_\varepsilon(1, t) = p_2(t), \quad p_\varepsilon(x, 0) = p_0(x),$$

$$u_\varepsilon(0, t) = u_1(t), \quad u_\varepsilon(1, t) = u_2(t), \quad u_\varepsilon(x, 0) = p_0(x),$$

for $t \in (0, T)$ *and* $x \in \Omega$. *We assume* $p_i \in H^{3/4}(]0, T[)$, $u_i \in L^{\infty}(]0, T[)$ *and* $0 \leqslant u_i(t) \leqslant 1$ *a.e. in* $(0, T)$, *for* $i = 1, 2$. *Then the function* q_ε *is a solution of*

$$\partial_t\left(\frac{\mu(u_\varepsilon)}{k^{\varepsilon}} q_\varepsilon\right) - \partial_x\left(\frac{1}{\phi^{\varepsilon} a(u_\varepsilon)} \partial_x q_\varepsilon\right) = 0 \quad \text{in } \Omega_T,$$

$$\partial_x q_\varepsilon(0, t) = -\phi^{\varepsilon} a(u_1)\, p_1'(t), \ \partial_x q_\varepsilon(1, t) = -\phi^{\varepsilon} a(u_2)\, p_2'(t) \quad \text{for } t \in (0, T),$$

$$\left(\frac{\mu(u_\varepsilon)}{k^{\varepsilon}} q_\varepsilon\right)(x, 0) = -p_0'(x) \quad \text{for } x \in \Omega.$$

Thus, the tools of the proof of Lemma 2 *can be used in this case, with minor changes due to the Neumann boundary conditions. We could also choose* *mixed boundary conditions for the pressure, that is*

$$p_\varepsilon(0, t) = p_1(t), \quad q_\varepsilon(1, t) = q_1(t) \quad \text{for } t \in (0, T),$$

with $q_1 \in H^{1/2}(]0, T[)$.

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