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Γ-Convergence of Constrained Dirichlet Functionals.

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Sunto. – *Dato $\Omega \subset \boldsymbol{R}^n$ aperto, limitato e connesso, con frontiera Lipschitziana e volume $|\Omega|$, si prova che la successione $\mathcal{F}_k$ di funzionali di Dirichlet definiti in $H^1(\Omega; \boldsymbol{R}^d)$, con vincoli di volume $\boldsymbol{v}^k$ su $m \geq 2$ insiemi di livello prescritti, tali che $\sum_{i=1}^m v_i^k < |\Omega|$ per ogni k, Γ-converge, quando $\boldsymbol{v}^k \to \boldsymbol{v}$ con $\sum_{i=1}^m v_i = |\Omega|$, al quadrato della variazione totale in $BV(\Omega; \boldsymbol{R}^d)$, con vincoli di volume $\boldsymbol{v}$ sui medesimi insiemi di livello.*

Summary. – *Given an open, bounded and connected set $\Omega \subset \boldsymbol{R}^n$ with Lipschitz boundary and volume $|\Omega|$, we prove that the sequence $\mathcal{F}_k$ of Dirichlet functionals defined on $H^1(\Omega; \boldsymbol{R}^d)$, with volume constraints $\boldsymbol{v}^k$ on $m \geq 2$ fixed level-sets, and such that $\sum_{i=1}^m v_i^k < |\Omega|$ for all k, Γ-converges, as $\boldsymbol{v}^k \to \boldsymbol{v}$ with $\sum_{i=1}^m v_i = |\Omega|$, to the squared total variation on $BV(\Omega; \boldsymbol{R}^d)$, with $\boldsymbol{v}$ as volume constraint on the same level-sets.*

1. – Introduction.

Ambrosio et al. considered in [3] a class of variational problems (which we shall refer to as «Dirichlet-type problems with volume constraints on level-sets») motivated by recent studies on models for immiscible fluids systems (see [11]), and also related to heat flow in materials with two or more phases (see [1]). More precisely, they were concerned with the following (type of) problem: given an open, bounded and connected Lipschitz domain $\Omega \subset \boldsymbol{R}^n$, m vectors $z_1, \ldots, z_m \in \boldsymbol{R}^d$ and m positive real numbers $v_1, \ldots, v_m$ with the property $\sum v_i < |\Omega|$ (here, $|\cdot|$ denotes the Lebesgue measure in $\boldsymbol{R}^n$), minimize

$$\int_\Omega |\nabla u|^2 \, dx \tag{1.1}$$

among $u \in W^{1,2}(\Omega; \boldsymbol{R}^d)$ satisfying the constraints $|\{u = z_i\}| = v_i$ for all $i = 1, \ldots, m$. Therefore, the unknowns of the problem are essentially the level-sets $\{v = z_i\}$ of a solution v, since v has to be harmonic on the complement of the level-sets (at least in a weak sense).

Existence results in the vector case have been established in [3], even though a quite restrictive hypothesis is assumed on $z_1, \ldots, z_m$ (i.e., they need to be extremal points of their convex hull), so that in the scalar-valued case ($d = 1$) only 2 levels are allowed. However, more general existence and regularity results have been later obtained by Tilli [15] and by Mosconi-Tilli [13] for the scalar case, while new results for the general vector-valued case, that do not require the extremality of the prescribed levels, have been recently found by Tilli and the author. In [3] it is also proved that a sequence of (suitably rescaled) functionals of type (1.1), in the scalar case with only two prescribed levels and with the measures of level-sets tending to some limit values v_1, v_2, for which $v_1 + v_2 = |\Omega|$, Γ-converges to the squared total variation in the space $BV_{z,v}(\Omega)$ of piecewise-constant BV-functions $u : \Omega \to \{z_1, z_2\}$ satisfying the constraints $|\{u = z_1\}| = v_1$ and $|\{u = z_2\}| = v_2$ (see also [14]).

The Γ-convergence has not yet been considered in its full generality, i.e., in the vector case with m generic levels. The aim of this paper is, therefore, to prove such a general result, which is done in Section 3 (Theorem 3.1) after some preliminary lemmas. Actually, the most remarkable part is the proof of the lim sup inequality (see Section 2 for the definition of Γ-convergence), while the lim inf inequality follows easily by the same argument used in [3].

The technique used to prove the lim sup inequality consists of several steps. First of all, we make use of an approximation result due to Baldo (see Lemma 2.1 and [5]) to restrict the verification of the lim sup inequality to functions $u \in BV_{z,v}(\Omega)$ such that the underlying partition of finite perimeter $\mathcal{P}$, whose components are precisely the m level-sets of u, is «polyhedral» and «transverse» to $\partial\Omega$ (see Remark 3.2). To prove the inequality in the polyhedral case, we need to define the approximating sequence $(u_k)_k$ in such a way that u_k belongs to $W^{1,2}(\Omega)$, it has an «almost constant» slope (varying as k varies) near the interfaces of $\mathcal{P}$ (see Lemma 3.4), and moreover it verifies the volume constraints, which is indeed the most delicate part of the proof (see, in particular, Lemma 3.6). We defer the precise statement and the proof of Γ-convergence after some basic definitions and results, collected in the following section.

2. – Preliminary definitions and results.

By $\boldsymbol{R}^n$ we denote the real Euclidean space of dimension $n \geq 2$. $B_r(x)$ denotes the open Euclidean n-ball centered at $x \in \boldsymbol{R}^n$ with radius $r > 0$; we use B_r in place of $B_r(0)$. We write ω_n for the Lebesgue measure $\mathcal{L}^n$ of the unit ball of $\boldsymbol{R}^n$. Then, the volume of B_r is $|B_r| = \omega_n r^n$ (we use $|A|$ instead of $\mathcal{L}^n(A)$). We also denote by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\boldsymbol{R}^n$ (see, e.g., [12] and [7] for definition and properties). We then denote by $[S]_\varepsilon$ the *ε-tubular neighbourhood* of a set $S \subset \boldsymbol{R}^t$, with $\varepsilon > 0$, and by $\mathcal{M}^{n-1}(S)$ the

$(n-1)$-dimensional Minkowski content in $\boldsymbol{R}^n$. One has the following property (for a proof, see [4], Theorem 2.106): if S is a compact subset of $\boldsymbol{R}^{n-1}$ and $f: \boldsymbol{R}^{n-1} \to \boldsymbol{R}^n$ is a Lipschitz mapping, then

$$\mathcal{M}^{n-1}(f(S)) = \mathcal{H}^{n-1}(f(S)). \tag{2.1}$$

The Sobolev and BV spaces are denoted, respectively, by $W^{1,p}(\Omega)$ and $BV(\Omega)$. We also consider the vector-valued spaces $W^{1,p}(\Omega; \boldsymbol{R}^d)$ and $BV(\Omega; \boldsymbol{R}^d)$, defined by taking the d-power of the corresponding scalar spaces (see, for instance, [9] and [4]).

Given a Borel set E, its characteristic function is $\chi_E(x)$. We define the *perimeter* of E in Ω as

$$P(E, \Omega) := \sup \left\{ \int_{E \cap \Omega} div\, g(x)\, dx : g \in C_0^1(\Omega; \boldsymbol{R}^n),\ |g(x)| \leq 1 \right\}.$$

We say that E has finite perimeter (or, is a set of finite perimeter) in Ω if $P(E, \Omega) < +\infty$. In particular, if $\chi_E \in L^1(\Omega)$ then E has finite perimeter in Ω if and only if $\chi_E \in BV(\Omega)$. Moreover, this notion of perimeter agrees with the $(n-1)$-dimensional area of the boundary of E, when ∂E is of class C^1 or Lipschitz. As for the notation, we write $P(E)$ instead of $P(E, \boldsymbol{R}^n)$ when $\Omega = \boldsymbol{R}^n$. For further details about BV-spaces and perimeter, see, e.g., [2], [4], [7], and [10].

The notion of *partition* of finite perimeter used here is that of a finite collection $\mathcal{F} = (F_i)_{i=1}^m$ of Borel subsets of Ω, such that

(i) F_i has locally finite perimeter in Ω;

(ii) $|F_i \cap F_j| = 0$ whenever $i \neq j$;

(iii) $\left| \Omega \setminus \bigcup_{i=1}^m F_i \right| = 0$.

Given $\mathcal{F}$ as above, we define the perimeter of $\mathcal{F}$ in Ω as

$$P(\mathcal{F}, \Omega) = \frac{1}{2} \sum_i P(F_i, \Omega).$$

A class of partitions of finite perimeter which is relevant for our purposes is that of *polyhedral* partitions $\mathcal{P}$, whose components P_i are such that $\partial P_i \cap \Omega = N_i \cap \Omega$, where N_i is a $(n-1)$-dimensional polyhedron (PL manifold) in $\boldsymbol{R}^n$, for all $i = 1, \ldots, m$. For such $\mathcal{P}$ we denote by S_{ij} the $(n-1)$-dimensional *regular interface* between P_i and P_j within Ω, that is,

$$S_{ij} := N_i^* \cap N_j^* \cap \Omega$$

(here, N^* is the set of points of the $(n-1)$-dimensional polyhedron N that do not belong to the $(n-2)$-skeleton of N). Then, we conveniently define the *sin-*

gular set Σ as the following subset of $\overline{\Omega}$:

$$\Sigma := \left\{x \in \overline{\Omega} : x \in \overline{S}_{ij} \text{ for some } i<j\right\} \setminus \bigcup_{i<j} S_{ij}.$$

Hence, the perimeter of a polyhedral partition is precisely the sum of the area of the regular interfaces S_{ij}. On the other hand, Σ contains, for example, corner points or multiple points where three or more interfaces meet inside Ω, plus the set $\Sigma^* = \Sigma \cap \partial\Omega$ of «singular boundary points», which is a compact subset of $\partial\Omega$, for which $\mathcal{M}^{n-1}(\Sigma^*) = \mathcal{H}^{n-1}(\Sigma^*)$ holds as a consequence of (2.1). Finally, we will say that $\mathcal{P}$ is *transverse* to $\partial\Omega$ if

$$\mathcal{M}^{n-1}(\Sigma^*) = \mathcal{H}^{n-1}(\Sigma^*) = 0\,,$$

and in this case it turns out also that

$$\mathcal{M}^{n-1}(\Sigma) = \mathcal{H}^{n-1}(\Sigma) = 0\,. \tag{2.2}$$

We now state an approximation result of a generic partition of finite perimeter in Ω by means of a sequence of polyhedral, transverse partitions (see [5], Lemma 3.1).

LEMMA 2.1. – *Let $\Omega \subset \boldsymbol{R}^n$ be open, bounded and Lipschitz, and let $\mathcal{F}$ be a partition of finite perimeter in Ω. Then there exists a sequence $(\mathcal{P}^h)_h$ of polyhedral, transverse partitions of finite perimeter in Ω, such that $|P_i^h| = |F_i|$ for all i, h, $\chi_{P_i^h}$ converges to χ_{F_i} in $L^1(\Omega)$ for all i, and $P(\mathcal{P}^h, \Omega)$ converges to $P(\mathcal{F}, \Omega)$, for $h \to +\infty$.*

Given m vectors $z^1, \dots, z^m \in \boldsymbol{R}^d$ and a vector $\boldsymbol{v} \in \boldsymbol{R}^m$, whose components are m positive numbers $v_1, \dots, v_m$, we define

$$H^1_{z,\boldsymbol{v}}(\Omega) := \left\{u \in W^{1,2}(\Omega; \boldsymbol{R}^d) : |\{u = z^i\}| = v_i \text{ for all } i = 1, \dots, m\right\},$$

and $BV_{z,\boldsymbol{v}}(\Omega)$ is defined in a similar way. Of course, $H^1_{z,\boldsymbol{v}}(\Omega)$ is non-empty if and only if $\sum_i v_i < |\Omega|$, while $BV_{z,\boldsymbol{v}}(\Omega)$ is significative also in the case $\sum_i v_i = |\Omega|$, where it coincides with the space of piecewise-constant, vector-valued BV-functions that can be represented as $\sum_i z^i \chi_{U_i}$, where $\mathcal{U} = (U_i)_{i=1}^m$ is a partition with finite perimeter, verifying the volume constraints $|U_i| = v_i$ for all i. In this case, one can show that

$$\int_\Omega |Du| = \sum_{i<j} |z^i - z^j|\, \mathcal{H}^{n-1}(S_{ij}) \tag{2.3}$$

(see [4], Theorem 3.78).

We recall the definition of *Γ-convergence* (see [6]), with respect to the L^1-topology: let F, F_k be functionals defined on $L^1(\Omega)$ with values in $\boldsymbol{R} \cup \{+\infty\}$, for all $k \in \mathbf{N}$. Then, we say that $\mathcal{F}_k$ Γ-converges to $\mathcal{F}$ as $k \to +\infty$ if and only if, for all $u \in L^1(\Omega)$, the following relations hold:

$$\forall u_k \to u \text{ in } L^1(\Omega) \quad \text{we have} \quad \mathcal{F}(u) \leq \liminf_k \mathcal{F}_k(u_k), \tag{2.4}$$

$$\exists u_k \to u \text{ in } L^1(\Omega) \quad \text{such that} \quad \mathcal{F}(u) \geq \limsup_k \mathcal{F}_k(u_k). \tag{2.5}$$

We finally conclude this section with some basic definitions and notations about networks. A network is a pair (G, A), where G is a finite set of *nodes* $\{1, \ldots, m\}$, that are «connected» by some oriented arcs (elements of A). The arc $a \in A$ going from node i to node j (which i and j «belong to») is represented by $i \to j$, or even by $a_- \to a_+$, with the obvious meaning for a_- and a_+. The set A of all arcs of G is, in fact, a subset of $G \times G$. A *path* of length s in a network (G, A) is an s-tuple of nodes $\sigma = (i_1, \ldots, i_s)$, such that i_r and i_{r+1} are connected by some arc $a \in A$ (regardless of orientation): more precisely, we will write $a \in \sigma$ and $-a \in \sigma$ if, respectively, $a = i_r \to i_{r+1}$ and $a = i_{r+1} \to i_r$. A network (G, A) is said to be *connected* if any pair of nodes can be joined by a path, while it is called *simple* if any two nodes are connected by at most one arc, and if there are no arcs of type $i \to i$. For more details on the subject, see, for instance, [8].

3. – The Γ-convergence result.

Let us take an open, bounded and connected set $\Omega \subset \mathbf{R}^n$ with Lipschitz boundary, a set of $m \geq 2$ vectors $\mathbf{z}^1, \ldots, \mathbf{z}^m \in \mathbf{R}^d$, and a vector $\mathbf{v}$ with m real components $v_1, \ldots, v_m > 0$, such that $\sum_i v_i = |\Omega|$. Let $(\mathbf{v}^k)_{k \in \mathbf{N}}$ be a sequence of vectors in $\mathbf{R}^m$ verifying

(i) for all $k \in \mathbf{N}$, $\sum_{s=1}^{m} v_s^k < |\Omega|$ and $v_i^k > 0$ for all $i = 1, \ldots, m$;

(ii) $\lim_k v_i^k = v_i$ for all $i = 1, \ldots, m$.

We can now state the following theorem:

THEOREM 3.1 (Γ-convergence). – *Under the previous hypotheses, we consider the following sequence of functionals: for $u \in L^1(\Omega)$ and $k \in \mathbf{N}$,*

$$\mathcal{F}_k(u) = \begin{cases} \left(|\Omega| - \sum_i v_i^k\right) \int_\Omega |\nabla u(x)|^2 \, dx & \text{if } u \in H^1_{z, v^k}(\Omega), \\ +\infty & \text{otherwise}. \end{cases}$$

Then, the sequence $\mathcal{F}_k$ Γ-converges to the functional

$$\mathcal{F}(u) = \begin{cases} \left(\int_\Omega |Du| \right)^2 & \text{if } u \in BV_{z,v}(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

with respect to the L^1-topology. Moreover, any sequence $(u_k)_k \subset L^1(\Omega)$, such that $\mathcal{F}_k(u_k) \leqslant C < +\infty$, is relatively compact in $L^1(\Omega)$.

Here comes the proof of Theorem 3.1. Compactness of sequences that are bounded in energy, together with lim inf inequality (2.4), are shown hereafter by following the same argument of [3], while the lim sup inequality (2.5) will be proved after some preliminary lemmas.

Proof of compactness and lim inf *inequality.*

We fix a function $u \in L^1(\Omega)$ and a sequence $(u_k)_k$, such that $u_k \in H^1_{z,v^k}(\Omega)$ (all these spaces of functions are vector-valued, but we adopt a shorter notation), and suppose that there exists a constant $C > 0$ such that

$$\mathcal{F}_k(u_k) \leqslant C \qquad \forall k .$$

By Schwartz's inequality, we get

$$\left(\int_\Omega |\nabla u_k| \right)^2 = \left(\int_{\Omega \setminus \bigcup_i \{u_k = z^i\}} |\nabla u_k| \right)^2 \leqslant \mathcal{F}_k(u_k) \leqslant C . \tag{3.1}$$

Moreover, we know that $|\{u_k = z^1\}| \geqslant v_1/2$ for k sufficiently large, hence by Poincaré's inequality we also get that $(u_k)_k$ is bounded in $L^1(\Omega)$. Therefore, $(u_k)_k$ is bounded in $W^{1,1}(\Omega)$, hence compact in $L^1(\Omega)$, which proves the last statement of Theorem 3.1. In particular, there exists a subsequence u_{k_s} converging in $L^1(\Omega)$ to a BV-function v. At this point, to prove (2.4) we suppose that $u_k \to u$ in $L^1(\Omega)$, hence $u = v$ a.e. on Ω, that is, $u \in BV(\Omega)$. Moreover, by the semicontinuity property of the BV-norm and by (3.1), we obtain:

$$\liminf_k \mathcal{F}_k(u_k) \geqslant \liminf_k \left(\int_\Omega |\nabla u_k| \right)^2 \geqslant \left(\int_\Omega |Du| \right)^2 .$$

Finally, the fact that u belongs to $BV_{z,v}(\Omega)$ follows by noticing that, up to subsequences, $u_k(x) \to u(x)$ for almost all $x \in \Omega$, hence, by Fatou's inequality, we get

$$v_i = \limsup_k |\{u_k = z^i\}| \leqslant |\{u = z^i\}| \quad \forall i = 1, \ldots, m,$$

but this is possible only if $|\{u = z^i\}| = v_i$ for all i. Therefore, u satisfies the level-set constraints, as wanted. ∎

Proof of limsup *inequality.*

The proof will proceed by several steps. The first one is a remark pointing out a relevant consequence of the density of polyhedral partitions, in the sense of Lemma 2.1.

REMARK 3.2. – It is well-known that the inequality (2.5) needs only to be proved for a subset of functions that is dense in energy. Hence, thanks to Lemma 2.1, we can confine ourselves to the case of piecewise-constant BV-functions whose underlying partition is polyhedral and transverse.

Therefore, it is sufficient to prove the following theorem:

THEOREM 3.3 (Polyhedral case). – *Let $u \in BV_{z,v}(\Omega)$ have an associated polyhedral and transverse partition $\mathcal{P}$. Then there exists a sequence $(u_k)_k$ such that $u_k \in H^1_{z,v^k}(\Omega)$ for all k, u_k converges to u in $L^1(\Omega)$ and*

$$\limsup_k \mathcal{F}_k(u_k) \leq \mathcal{F}(u).$$

To prove this theorem we need some preliminary lemmas.

LEMMA 3.4 (The function ψ_ε). – *Let u and $\mathcal{P}$ be as in Theorem* 3.3. *Then, for a given $\varepsilon > 0$, there exists a function $\psi_\varepsilon(x)$ and a constant $K > 0$ depending only on $\{z^1, \ldots, z^m\}$, such that*

(i) $\psi_\varepsilon \in W^{1,\infty}(\Omega)$ *and* $\|\nabla \psi_\varepsilon\|_\infty \leq K/\varepsilon$;

(ii) ψ_ε *takes the constant value z^i on $P_i \setminus \bigcup_{i'<j'} [S_{i'j'}]_{\varepsilon_{i'j'}}$, where $\varepsilon_{i'j'} = |z^{i'} - z^{j'}|\varepsilon$;*

(iii) *there exist $c, \varepsilon_0 > 0$ depending only on $\mathcal{P}$ and $\{z^i\}$, such that for $0 < \varepsilon < \varepsilon_0$ we have $|\nabla \psi_\varepsilon(x)| = (2\varepsilon)^{-1}$ for almost all $x \in \Omega \cap \left(\bigcup_{i'<j'} [S_{i'j'}]_{\varepsilon_{i'j'}} \setminus [\Sigma]_{c\varepsilon}\right)$. Here, Σ is the singular set defined in Section* 2.

PROOF. – The function ψ_ε can be constructed in various ways. Our choice here is to define ψ_ε by means of a combination of distance functions from the interfaces, which is quite simple from a computational point of view and requires, at the same time, neither «a priori» localization (i.e., choice of ε small enough), nor the use of any further Lipschitz extension near singular points. First of all, we define $d_{ij}(x) = dist(x, S_{ij})$ and $\varepsilon_{ij} = \varepsilon |z^i - z^j|$, then put $h_{ij}(x) = \max(1 - \varepsilon_{ij}^{-1} d_{ij}(x), 0)$. On each (open) P_i, we consider the function

$\phi_i : P_i \to \boldsymbol{R}^d$ defined as

$$\phi_i(x) = z^i \cdot \prod_{j \neq i} (1 - h_{ij}(x)) + \sum_{i' < j'} \frac{z^{i'} + z^{j'}}{2} h_{i'j'}(x),$$

and finally construct, for $x \in \Omega$,

$$\psi_\varepsilon(x) = \begin{cases} \phi_i(x) & \text{if } x \in P_i \\ \sum_{i' < j'} \frac{z^{i'} + z^{j'}}{2} h_{i'j'}(x) & \text{otherwise}. \end{cases}$$

We claim that ψ_ε satisfies our requirements.

(i) By definition, ψ_ε is a K/ε-Lipschitz function on each P_i (for a suitable K depending only on $\{z_1, \ldots, z_m\}$, hence it only remains to observe that it is continuous along every interface S_{ij}, which is true since

$$\phi_{i|S_{ij}} = \sum_{i' < j'} \frac{z^{i'} + z^{j'}}{2} h_{i'j'} = \phi_{j|S_{ij}}.$$

(ii) This follows easily by the definition.

(iii) This can be easily seen by the following argument: first, we can take $0 < \varepsilon_0 < \dfrac{d}{2 \max_{i,j} |l_i - l_j|}$, where d denotes the minimum distance between pairs of disjoint, closed $(n-1)$-dimensional faces of the polyhedral interface set (which, of course, contains only a finite number of such faces). If $\varepsilon < \varepsilon_0$, then one can easily prove that

$$x \in [S_{ij}]_{\varepsilon_{ij}} \cap [S_{hk}]_{\varepsilon_{hk}} \quad \text{for } (i, j) \neq (h, k),$$

implies

$$dist(x, \Sigma) < c\varepsilon ,$$

where $c > 0$ is a constant depending only on the data of the problem and on the partition $\mathcal{P}$ (in particular, on the fact that all possible dihedral angles, formed by each pair of faces meeting at some point of Σ, are non-zero and hence cannot be smaller than some positive lower bound). Hence, if $x \in (P_i \cap [S_{ij}]_{\varepsilon_{ij}}) \setminus [\Sigma]_{c\varepsilon}$, then

$$\psi_\varepsilon(x) = z^i (1 - h_{ij}(x)) + \frac{z^i + z^j}{2} h_{ij}(x),$$

and the claim follows from the definition of h_{ij} and from the fact that $|\nabla d_{ij}(x)| = 1$ almost everywhere. ■

At this point, we need to choose $\varepsilon_k > 0$ such that

$$\left| \bigcup_i \{ \psi_{\varepsilon_k}(x) = z^i \} \right| = \sum_i v_i^k . \tag{3.2}$$

This can be done, thanks to the continuity of $|[S_{ij}]_{\varepsilon_{ij}} \cap \Omega|$ with respect to ε, and up to the elimination of possibly existing portions of level-sets contained in the intersection of two or more tubular neighbourhoods, which can be accomplished, for example, by adding a suitable Lipschitz function whose support contains the undesired pieces of level-sets and whose slope is larger than K/ε. Anyway, volume adjustments will be, in general, necessary to let each level-set $\{\psi_{\varepsilon_k} = z^k\}$ satisfy the volume constraints v_i^k. This is actually a crucial point of the proof, and indeed one can argue the need of choosing «good» adjustments, i.e., adjustments with a low energy cost that will vanish in the limit. Therefore, we are going to prove that this kind of adjustments can be performed by slightly perturbing ψ_{ε_k} near the interfaces of the partition $\mathcal{P}$ associated to the function u (see Lemma 3.6).

The following lemma, whose proof is omitted, is a standard flow-type result on networks (see [8] and Section 2 for related definitions) and allows us to perform «virtual» adjustments of volumes.

LEMMA 3.5 (Tuning flow). – *Let (G, A) be a connected and simple graph, with $m \geq 2$ nodes, and let $f_0, f_1 : G \to (0, \infty)$ be such that*

$$\sum_{i \in G} (f_0(i) - f_1(i)) = 0 . \tag{3.3}$$

Now, define $\delta = \sum_{i \in G} |f_0(i) - f_1(i)|$. Then there exists a «tuning flow» $T : A \to$ $\boldsymbol{R}$ such that $\max\{|T(a)| : a \in A\} \leq \delta/2$ and «$f_1 = f_0 + T$», i.e.,

$$f_1(i) = f_0(i) + \sum_{a : a_+ = i} T(a) - \sum_{a : a_- = i} T(a) \quad \forall i \in G .$$

LEMMA 3.6 (Volume adjustments). – *Under the hypotheses of Theorem 3.3, let $u \in BV_{z,v}(\Omega)$ have an associated polyhedral, transverse partition $\mathcal{P}$, and let $k \in \boldsymbol{N}$, ε_k be as in (3.8), and ψ_{ε_k} be as in Lemma 3.4. Then, for k large enough, there exists $\widehat{\psi}_k$ such that*

(a) *$\widehat{\psi}_k$ satisfies the volume constraints on the level-sets;*

(b) *$\widehat{\psi}_k$ tends to u in $L^1(\Omega)$ as $k \to \infty$;*

(c) *the difference between the (rescaled) Dirichlet energies of ψ_{ε_k} and $\widehat{\psi}_k$ tends to 0 as $k \to \infty$.*

PROOF. – Let us start by choosing points $x_{ij} \in \Omega$ and $r > 0$, such that every x_{ij} lies on the polyhedral interface S_{ij} of $\mathcal{P}$, that the ball $B_{4r}(x_{ij})$ is contained in

$\Omega \setminus \bigcup_{(h,k) \neq (i,j)} [S_{hk}]_{\varepsilon_{hk}}$ and cuts a flat $(n-1)$-dimensional disc $4D_{ij}$ out of S_{ij}. We stress that such x_{ij} and r exist for any polyhedral partition and depend only upon $\mathcal{P}$, $\{z_i\}$ and u; moreover, for $\varepsilon = \varepsilon_k$ sufficiently small (or, equivalently, k large enough) they can be fixed once for all.

Now, we focus on each ball $B_{2r}(x_{ij})$ and choose k so large that

$$\varepsilon < \frac{r}{\max_{i<j} |z^i - z^j|}.$$

This implies that the portion of the boundary of $[S_{ij}]_{\varepsilon_{ij}}$ cut by $B_{2r}(x_{ij})$ is the union of two flat $(n-1)$-dimensional discs parallel to $4D_{ij}$, lying on opposite sides at a distance less than r from S_{ij}, and whose radius is greater than r. Let us denote by π_{ij} the orthogonal projection of $\boldsymbol{R}^n$ onto the hyperplane γ containing D_{ij} (the disc of radius r which is concentric to $4D_{ij}$) and by ν_{ij} the corresponding normal, pointing from P_i toward P_j. Therefore, any $x \in \boldsymbol{R}^n$ can be represented with a suitable coordinate system obtained by adding ν_{ij} to an orthogonal $(n-1)$-dimensional frame having center x_{ij} and axes lying on γ: we will henceforth write $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$ represents $\pi_{ij}(x)$ on the hyperplane γ and x_n is the component of $x - \pi_{ij}(x)$ with respect to ν_{ij}. Now, we denote by B_r' the ball of radius r contained in $\boldsymbol{R}^{n-1}$ and centered in 0, then take $\beta \in C_0^\infty(B_r'; [0,1])$ not identically zero, and write b for the L^1-norm of β. For $t_{ij} \in (-r/2, r/2)$ we clearly have that the graphs of the two functions $t_{ij}\beta(x') - \varepsilon_{ij}$, $t_{ij}\beta(x') + \varepsilon_{ij}$ are contained in B_{2r}.

Therefore, given $\boldsymbol{t} = (t_{ij})_{ij}$ as above, we define $\widehat{\psi}_k^{\boldsymbol{t}}$ as follows:

$$\widehat{\psi}_k^{\boldsymbol{t}}(x) = \begin{cases} \psi_{\varepsilon_k}(x - t_{ij}\beta(x')\nu_{ij}) & \text{if } x' \in B_r' \text{ and } x_n \in (m_{\boldsymbol{t}}(x'), M_{\boldsymbol{t}}(x')), \\ z^i & \text{if } x' \in B_r' \text{ and } x_n \leqslant m_{\boldsymbol{t}}(x'), \\ z^j & \text{if } x' \in B_r' \text{ and } x_n \geqslant M_{\boldsymbol{t}}(x'), \\ \psi_{\varepsilon_k}(x) & \text{otherwise}, \end{cases}$$

where $x' = \pi_{ij}(x)$ and, with a little abuse of notation, $m_{\boldsymbol{t}}(x') = t_{ij}\beta(x') - \varepsilon_{ij}$ and $M_{\boldsymbol{t}}(x') = t_{ij}\beta(x') + \varepsilon_{ij}$.

Now, we can compute $\nabla\widehat{\psi}_k^{\boldsymbol{t}}(x)$ with respect to the local coordinate system. Indeed, by the chain rule, one gets

$$\nabla\widehat{\psi}_k^{\boldsymbol{t}}(x) = \nabla\psi_\varepsilon(x', x_n - t_{ij}\beta(x')) + \frac{\partial\psi_\varepsilon}{\partial x_n} \otimes (-t_{ij}\nabla\beta(x'), 0)$$

for $x' \in B_r(x_{ij})$ and $x_n \in (m_{\boldsymbol{t}}(x'), M_{\boldsymbol{t}}(x'))$, with $u \otimes v$ being, as usual, the $(m \times n)$ matrix defined by $(u \otimes v)(w) = \langle v, w \rangle u$ as a linear operator from $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$.

Hence, by recalling the definition of $\psi_\varepsilon(x)$, we observe that, for such x,

$$\nabla \widehat{\psi}_k^t(x) = \left(\bar{0}, \dots, \bar{0}, \frac{\partial \psi_\varepsilon}{\partial x_n}\right) + \frac{\partial \psi_\varepsilon}{\partial x_n} \otimes (-t_{ij} \nabla \beta(x'), 0)$$

$$= \frac{\partial \psi_\varepsilon}{\partial x_n}(x', x_n - t_{ij}\beta(x')) \otimes (-t_{ij} \nabla \beta(x'), 1),$$

where $\bar{0}$ denotes the null (column) vector in $\boldsymbol{R}^m$. Finally, by noting that $|u \otimes v| = |u||v|$, by integrating over Ω and by using a change of variables together with Fubini's theorem, we deduce that

$$(3.4) \qquad \int_\Omega |\nabla \psi_\varepsilon(x)|^2 dx \leq \int_\Omega |\nabla \widehat{\psi}_k^t(x)|^2 dx \leq (1 + Q^2 \max_{i,j} t_{ij}^2) \int_\Omega |\nabla \psi_\varepsilon(x)|^2 dx,$$

where $Q = \max_{x' \in B_r'} |\nabla \beta(x')|$. At this point, we can immediately see that, since $\varepsilon = \varepsilon_k$ is infinitesimal as $k \to \infty$, the difference Δ^k between the vector of prescribed volumes $\boldsymbol{v}^k$ and the vector of actual volumes of the level-sets of ψ_ε is infinitesimal, too. A consequence of this fact is that we can choose k sufficiently large, such that $|\Delta_i^k| < rb/2$ for all $i = 1, \dots, m$ (we recall that $b = \|\beta\|_{L^1(B')}$), and hence a vector $(b\tau_{ij})_{ij}$ can be determined, with $\tau_{ij} \in (-r/2, r/2)$, to perform the adjustment of volumes. Indeed, we can consider a simple network ($G = \{1, \dots, m\}, A$), whose nodes represent the level-sets, i.e., the components of $\mathcal{P}$, and whose arcs represent the non-negligible interfaces between level-sets. Thanks to the hypotheses on Ω and $\boldsymbol{v}$, one can easily prove that the network associated to u is connected: indeed, suppose by contradiction that $N_1 \cup N_2 = \{1, \dots, m\}$ and $N_1 \cap N_2 = \emptyset$ with the property that, for all $i \in N_1$ and all $j \in N_2$, neither $i \to j$ nor $j \to i$ belong to A; consequently, the pair $\left\{\bigcup_{i \in N_1} P_i, \bigcup_{j \in N_2} P_j\right\}$ is a bipartition of Ω with null perimeter, but this contradicts the relative isoperimetric inequality on Ω (we recall that Ω is a bounded, connected Lipschitz domain), since both components of the bipartition have positive volume.

We also define two functions f_0, f_1 on G as $f_0(i) = |\{\psi_\varepsilon = z_i\}|$ and $f_1(i) = v_i^k$. By Lemma 3.5 we find a flow T whose value on the arc $i \to j$ corresponds exactly to $b\tau_{ij}$, that is, we can find τ_{ij}. Then, we set $\widehat{\psi}_k = \widehat{\psi}_k^\tau$ (where $\tau = (\tau_{ij})_{ij}$) and immediately get (a). Finally, τ tends to 0 as $\Delta^k \to 0$ (that is, as $k \to \infty$), therefore (b) holds (the sequence $(\psi_{\varepsilon_k})_k$ is uniformly bounded in $L^\infty(\Omega)$ – see the proof of Lemma 3.4 – and converges to u almost everywhere), while (c) is a consequence of (3.4). ■

REMARK 3.7. – The previous lemma says that we can, in some sense, forget about the volume adjustments and perform our energy computations directly on ψ_ε.

PROOF OF THEOREM 3.3. – Let k_0 be large enough, so that for all $k \geq k_0$ the corresponding $\varepsilon = \varepsilon_k > 0$ satisfying (3.2) is small and the conclusions of Lemma 3.6 hold. Taking into account Remark 3.7, we only need to estimate

$$\lim_{k \to \infty} \Big(|\Omega| - \sum_i v_i^k \Big) \int_\Omega |\nabla u_k|^2,$$

where $u_k := \psi_{\varepsilon_k}$. We define $S_{ij}^\varepsilon = [S_{ij}]_{\varepsilon_{ij}} \setminus [\Sigma]_{c\varepsilon}$ and get

$$|S_{ij}^\varepsilon| \leq |[S_{ij}]_{\varepsilon_{ij}}| = 2\varepsilon_{ij} \mathcal{H}^{n-1}(S_{ij}) + o(\varepsilon_{ij}) \tag{3.5}$$

thanks to (2.1), where $o(\varepsilon_{ij})$ denotes an infinitesimal of higher order than ε_{ij}. By taking into account (2.2), we also have

$$|[\Sigma]_{c\varepsilon}| = o(\varepsilon), \tag{3.6}$$

hence if we define $\varrho_k = |\Omega| - \sum v_i^k$ and split the Dirichlet integral into a sum of integrals over S_{ij}^ε and $[\Sigma]_{c\varepsilon}$, by (3.5), (3.6), (2.3), Lemma 3.4, and by noting that $\varrho_k \leq \sum_{i<j} |[S_{ij}]_{\varepsilon_{ij}}|$, we obtain

$$\begin{aligned}
\varrho_k \int_\Omega |\nabla u_k(x)|^2 dx &\leq \varrho_k \sum_{i<j} \int_{S_{ij}^\varepsilon} |\nabla \psi_\varepsilon(x)|^2 dx + \varrho_k \int_{[\Sigma]_{c\varepsilon} \cap \Omega} |\nabla \psi_\varepsilon(x)|^2 dx \\
&\leq (2\varepsilon)^{-2} \varrho_k \Big(\sum_{i<j} [2\varepsilon_{ij} \mathcal{H}^{n-1}(S_{ij}) + o(\varepsilon_{ij})] + o(\varepsilon) \Big) \\
&\leq \Big(\sum_{i<j} (|z^i - z^j| \mathcal{H}^{n-1}(S_{ij}) + o(1)) \Big)^2 + o(1) \\
&\leq \Big(\int_\Omega |Du| \Big)^2 + o(1),
\end{aligned}$$

and this concludes the proof. ■

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