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On Existence of Equilibria of Set-Valued Maps.

GRZEGORZ GABOR - MARC QUINCAMPOIX

Sunto. – *L'articolo fornisce delle condizioni sufficienti per l'esistenza di punti di equilibrio di applicazioni multivoche Lipschitziane in assegnati sottoinsiemi di spazi finito-dimensionali. Il principale contributo del presente articolo consiste nel fatto che non si danno condizioni di regolarità sulla frontiera degli insiemi considerati. L'approccio è basato sullo studio del comportamento delle traiettorie della corrispondente inclusione differenziale.*

Summary. – *The present paper is devoted to sufficient conditions for existence of equilibria of Lipschitz multivalued maps in prescribed subsets of finite-dimensional spaces. The main improvement of the present study lies in the fact that we do not suppose any regular assumptions on the boundary of the subset. Our approach is based on behaviour of trajectories to the corresponding differential inclusion.*

1. – Introduction.

The problem of finding «equilibria» of single- or multivalued maps, i.e. points with $0 \in F(x)$, is very important in many branches of nonlinear analysis. One can mention e.g. the calculus of variation or control theory. The central question is to find an equilibrium in a prescribed subset K of a space. Sufficient conditions are described by various assumptions on behaviour of a map on the boundary ∂K of the set K, namely, positive distance from zero on ∂K (in Brouwer type results), antipodal-preservation (in Borsuk type theorems) or several tangency conditions.

There are many papers studying the problem of existence of equilibria using analytical or topological methods. After some pioneering results of Fan [17, 18] and Browder [6], Cornet in [12] obtained an equilibrium theorem for upper semicontinuous maps with nonempty closed convex values defined on a compact convex subset K of a normed space.

For a nonconvex set K, as far as we know, the most general results have been obtained by Ben-El-Mechaiekh and Kryszewski [3, 4] and by Ćwiszewski and Kryszewski [14]. In [4] the authors have proved:

PROPOSITION 1.1 ([4], Theorem 5.3). – *Let K be a compact $\mathcal{L}$-retract* (1) *in a normed space E with the Euler characteristic $\chi(K) \neq 0$. If $\Phi : K \multimap E$ is an upper semicontinuous (in short u.s.c.) map with closed convex values satisfying*

$$\Phi(x) \cap C_K(x) \neq \emptyset, \qquad \text{for every } x \in K, \tag{1}$$

then Φ has an equilibrium.

Here and throughout the paper we will denote respectively by $T_K(x)$, $C_K(x)$, $N_K(x) := C_K(x)^-$, the Bouligand cone, the Clarke tangent cone and the normal cone to K in a point $x \in K$ (cf. [2]).

The results in [14] imply the following one which explains a sufficient condition of existence of equilibria written in terms of normal cones (see [14], Corollary 2.2, comp. [4], Theorem 3.8).

PROPOSITION 1.2. – *An upper semicontinuous map $\Phi : K \multimap \mathbb{R}^n$ with compact convex values defined on a compact $\mathcal{L}$-retract $K \subset \mathbb{R}^n$ with $\chi(K) \neq 0$ possesses an equilibrium in $Int\, K$ provided*

$$\Phi(x) \cap N_K(x) = \emptyset, \qquad \text{for every } x \in \partial K. \tag{2}$$

The results above generalize the one obtained by Plaskacz [22] (on compact proximal retracts), an extension of the Haddad-Lasry Theorem which reduces to a fixed point theorem when Φ does not depend on t (cf. [19] and also Theorem 5.3.4 p. 237 in [1]). We also refer to papers by Clarke *et al.* [10], and by Cornet and Czarnecki [13] (on compact epi-Lipschitz sets).

There are two main kinds of assumptions used to solve the problem of finding equilibria in K, namely:

- regularity assumption on K,
- tangency condition on the boundary ∂K.

In the present paper, we consider a tangency condition which allows us to enlarge the class of sets considered in above cited references *in the context of finite-dimensional space*. This condition is, in some sense, the opposite one to (1). It can be expressed as an outwardness of a map F with respect to trajectories of the following differential inclusion

$$\dot{x}(t) \in F(x(t)) \quad \text{for almost all } t \geqslant 0. \tag{3}$$

Namely we shall prove that, when no trajectory starting from a point x_0 of the boundary of the set K reaches $Int\, K$, then an equilibrium of F exists in K. This result will be stated later on for a rather large class of subsets $K \subset \mathbb{R}^n$.

(1) See definition 2.3 in Section 2.

2. – Preliminaries.

2.1. *Notations and assumptions.*

Throughout the paper we shall investigate existence of equilibria of a given multivalued map $F : \mathbb{R}^n \multimap \mathbb{R}^n$ on a closed set $K \subset \mathbb{R}^n$.

We say that $F : X \multimap \mathbb{R}^n$ is a *Marchaud* map if it is u.s.c. with compact convex values and there is a constant $c > 0$ such that $|F(x)| = \sup\{|y| \mid y \in F(x)\} \leq c(1 + |x|)$, for every $x \in X$.

We denote by $S_F(x_0)$ the set of all absolutely continuous solutions of the following Cauchy problem

$$\begin{cases} \dot{x}(t) \in F(x(t)) & \text{for a.a. } t \geq 0, \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{4}$$

DEFINITION 2.1. – **1** For $x_0 \in \mathbb{R}^n$, we say that a trajectory $x \in S_F(x_0)$ is *viable in* K, when $x(t) \in K$, for all $t \geq 0$.

2 A set K is said to be *viable under* F, if for each $x_0 \in K$ there is at least one solution to (4) which is viable in K.

3 The *viability kernel of* K *for* F (written $Viab_F(K)$) is the largest closed subset of K viable under F (possibly empty, in general). Equivalently (see [2], Theorem 4.1.2), $Viab_F(K)$ is the subset of all initial states such that at least one solution starting from them is viable in K.

We shall need also some notions from topological analysis.

As a homology functor we will mean here the Čech functor (denoted simply by H) with compact carriers (see e.g. [16]).

We say that the set K is of *finite type*, if the graded vector space $\{H_q(K)\}_{q \geq 0}$ is of finite type, i.e. if $H_q(K) = 0$, for almost all $q \geq 0$, and $\dim H_q(K) < \infty$, for all $q \geq 0$. Then the *Euler characteristic* $\chi(K) := \sum_{q=0}^{\infty} (-1)^q \dim H_q(K)$ is defined (see e.g. [7]). It is known (see also [7]) that $\chi(K)$ is equal to the Lefschetz number of the identity map $id_K : K \to K$. Since the Lefschetz number is a homotopy invariant, we obtain the well-known

PROPOSITION 2.2. – *If K is of finite type and $A \subset K$ is a strong deformation retract* (2) *of K, then $\chi(K) = \chi(A)$.*

Following [4] we have:

(2) We say that $A \subset K$ is a *strong deformation retract* of K, if there is a homotopy $h : K \times [0, 1] \to K$ such that $h(x, 0) = x$, $h(x, 1) \in A$, for every $x \in K$, and $h(x, t) = x$, for each $x \in A$ and $t \in [0, 1]$.

DEFINITION 2.3. – A subset A of a metric space X is an $\mathcal{L}$*-retract* (of X), if there are an open neighbourhood U of A in X, a retraction $r : U \to A$ and a constant $L > 0$ such that $d(r(x), x) \leqslant L\, dist(x, A)$, for every $x \in U$.

One can easily prove (see [4], Example 4.4) that each Lipschitz retraction is an $\mathcal{L}$-retraction.

2.2. *Differential Inclusions with constraints.*

Let us recall that, when F is a Marchaud map, a set K is viable under F if and only if

$$F(x) \cap T_K(x) \neq \emptyset, \qquad \text{for every } x \in \partial K, \tag{5}$$

where

$$T_K(x) := \{v \in \mathbb{R}^n \mid \liminf_{h \to 0^+} dist(x + hv, K)/h = 0\}$$

stands for the Bouligand contingent cone to K in x (cf. [2], Theorem 3.3.2).

It is well known that the Clarke cone $C_K(x)$ is contained in $T_K(x)$ and, in general, it may occur that $C_K(x)$ is essentially smaller. The equality holds for e.g. sleek sets, where the map $T_K(\cdot)$ is lower semicontinuous (in short: l.s.c.). Moreover, the following result is true.

PROPOSITION 2.4 (see e.g. [2], Theorem 5.1.5). – *Let K be a closed subset of $\mathbb{R}^n$ and $F : K \multimap \mathbb{R}^n$ be l.s.c. at $x \in K$, and*

$$\exists \delta > 0 \;\; \forall z \in B_K(x, \delta) : F(z) \subset T_K(z).$$

Then $F(x) \subset C_K(x)$.

For further considerations, we need to define a subset of the boundary of K:

$$K_s := \{x_0 \in \partial K \mid \text{every solution to the Cauchy problem for } F \text{ starting from } x_0 \text{ leaves } K \text{ immediately}\}$$

By saying that a solution x leaves K immediately we mean that for every $\varepsilon > 0$ there exists some $0 < t < \varepsilon$ such that $x(t) \notin K$.

The set K_s, which is used in our main theorem, can be characterized by some tangential conditions in the following way.

Denote (comp. [23])

$$K_{\Rightarrow} := \{x \in \partial K \mid F(x) \cap T_K(x) = \emptyset\},$$

One can check (see [8], Lemma 3.1) that

$$K_{\Rightarrow} \subset K_s \subset \overline{K_{\Rightarrow}}.$$

In [8] the author has given (Proposition 3.1) the following characterization of the set K_s.

Let K be closed and F a Marchaud map locally Lipschitz around $x \in \overline{K_{\Rightarrow}} \setminus K_{\Rightarrow}$.

If

$$F(x) \cap ((\mathbb{R}^n \setminus T_{\mathbb{R}^n \setminus K}(x)) \cup T_{\partial K \setminus K_{\Rightarrow}}(x)) = \emptyset ,$$

then $x \in K_s$.

If

$$F(x) \cap (\mathbb{R}^n \setminus T_{\mathbb{R}^n \setminus K}(x)) \neq \emptyset \qquad \text{or} \qquad F(x) \cap T_{K_{\Rightarrow}}(x) = \emptyset ,$$

then $x \notin K_s$.

We shall also add some informations concerning viability kernels.

In [8] and [9], some sufficient conditions of nonemptiness of the viability kernels are discussed together with the following useful

LEMMA 2.5 ([9], Proposition 3.1, [8]). – *If K is compact and F is a Marchaud map, $Viab_F(K)$ is nonempty if and only $Viab_{-F}(K)$ is nonempty.*

Here $Viab_{-F}(K)$ is the viability kernel for the following backward differential inclusion

$$\dot{x}(t) = -F(x(t)) .$$

In the proof of our main theorem we will also use the following result describing a behaviour of a map on a boundary of the viability kernel (see [8], Lemma 5.1).

LEMMA 2.6. – *Let F be a Lipschitz map and $x \in \partial Viab_F(K) \cap Int\, K$. Then*

$$Int F(x) \cap T_{Viab_F(K)}(x) = \emptyset .$$

Now, assume that we have a sequence $\{F_m : \mathbb{R}^n \multimap \mathbb{R}^n\}$ of u.s.c. maps with the same at most linear growth and satisfying

$$Graph(F) = \bigcap_{m=1}^{\infty} Graph(F_m) .$$

Then, using the Stability Theorem (see [2], Theorem 3.6.4), one obtains

PROPOSITION 2.7. – *If subsets X_m are viable under F_m and $X = \bigcap_{m=1}^{\infty} X_m$, then X is viable under F. In particular,*

$$Viab_F(K) = \bigcap_{m=1}^{\infty} Viab_{F_m}(K).$$

3. – Main result.

Denote $\widehat{K} = \overline{\mathbb{R}^n \setminus K}$ the closure of the complement of$K$.

THEOREM 3.1. – *Let $K = \overline{IntK} \subset \mathbb{R}^n$ be a compact subset of finite type with $\chi(K) \neq 0$, $\Omega \subset \mathbb{R}^n$ an open neighbourhood of K and $F : \Omega \multimap \mathbb{R}^n$ a Marchaud map satisfying*

(6) *for every $\varepsilon > 0$ there exists a Lipschitz ε-approximation* (3) *f of F such that $f(x) \in T_{\widehat{K}}(x)$ for every $x \in \partial K$, and $Viab_f(K) \cap \partial K = \emptyset$.*

Then F has an equilibrium in K.

REMARK 3.2. – The condition (6) is clearly fulfilled if

$$\forall x \in \partial K, \quad F(x) \cap T_K(x) = \emptyset$$

or, more generally, if $K_s = \partial K$.

COROLLARY 3.3. – *Let $K = \overline{Int\,K} \subset \mathbb{R}^n$ be a compact subset of finite type with $\chi(K) \neq 0$, $\Omega \subset \mathbb{R}^n$ an open neighbourhood of K and $F : \Omega \multimap \mathbb{R}^n$ a Lipschitz map with compact convex nonempty values, satisfying*

(7) $$F(x) \subset T_{\widehat{K}}(x), \text{ for every } x \in \partial K, \text{ and } Viab_F(K) \cap \partial K = \emptyset .$$

Then F has an equilibrium in $Int\,K$.

Indeed, every Lipschitz selection f of F (which exists by e.g. Theorem 1.9.1 in [1]) satisfies (6).

PROOF OF THEOREM 3.1. – We divide this proof into some steps.

STEP 1. Let $\varepsilon > 0$ and let f be an arbitrary Lipschitz ε-approximation of F satisfying (3.7). We show that K is invariant under $-f$, i.e. that each trajectory for $= -f$ starting in K remains in K forever.

Indeed, let x be a solution for $-f$ starting from an interior point of K. Supposing, on the contrary, that x leaves K, it behaves at some $s > 0$. Then $y(\cdot) :=$

(3) We say that $f : X \to \mathbb{R}^n$ is an ε-approximation of $F : X \multimap \mathbb{R}^n$ if $f(x) \in F(B(x, \varepsilon)) + B(0, \varepsilon)$ for every $x \in X$.

$x(s-\cdot)$ is a solution for f starting from $x(s)\in\partial K$ and reaching the interior point of K. But this is impossible (see assumption (6)). By continuity of the solution map S_f, the set K is invariant under $-f$.

By Lemma 2.5 and assumption (6) it follows that $X := Viab_f(K)\subset Int\,K$. By the definition, X is closed in K and hence, compact. Therefore, $dist\,(X, \partial K) = \eta > 0$.

STEP 2. Define the family of Lipschitz maps $F_m\colon \Omega \to \mathbb{R}^n$,

$$F_m(x) = f(x) + cl\,B\left(0, \frac{1}{m}\right), \qquad \text{for every } x\in\Omega .$$

Since $f(x)\in F_{m+1}(x)\subset F_m(x)$, for every $m\geqslant 1$, it is easily seen (see Proposition 2.7) that $Viab_f(K)$ is an intersection of a decreasing sequence of the sets $X_m = Viab_{F_m}(K)$. Without any loss of generality we can assume that $X_m\subset Int\,K$, since there is $m_0\geqslant 1$ such that $X_m\subset X+\eta B(0, 1)$, for every $m\geqslant m_0$.

We show that $X_{m+1}\subset Int\,X_m$. Indeed, if we supposed that there isa point $x_0\in X_{m+1}\cap\partial X_m$, then we would find a viable solution for F_{m+1} in X_m, since $X_{m+1}\subset X_m$. But, on the other hand, from Lemma 2.6 it follows that $F_{m+1}(x_0)\cap T_{X_m}(x_0) = \emptyset$; a contradiction.

Before going further we shall state a result which proof is postponed at the end of the present section.

LEMMA 3.4. – *Let f be a Lipschitz selection of F. Consider a closed set $A = \overline{Int\,(A)}$ such that*

$$X = Viab_F(K)\subset Int\,(A)\subset A\subset K . \tag{8}$$

Then the entry function

$$x_0 \mapsto e_A(x_0) := \inf\{t>0,\ y(t)\in A \ \textit{where}\ y = S_{-f}(x_0)\}$$

is lower semicontinuous and takes finite values on K.

Furthermore, if for some $\alpha>0$,

$$\forall x\in\partial A,\ (f(x)+B(0, \alpha))\cap T_A(x) = \emptyset , \tag{9}$$

then e_A is Lipschitz on K.

STEP 3. For every $m\geqslant 1$ denote $Y_m = \overline{Int\,X_m}$ and fix some $m\geqslant 1$.

We define the entry function $e_m\colon K\to[0, \infty)$,

$$e_m(x) = \inf\{t>0 \mid S_{-f}(x)(t)\in Y_m\} .$$

Because $f(x)\in f(x)+B(0, 1/m)$ and by Lemma 2.6, each trajectory starting from ∂Y_m goes immediately into $Int Y_m$. By Lemma 3, the function e_m is Lipschitz. Because $K\backslash Int\,Y_m$ is compact, there exists $T>0$ such that every trajectory for $-f$ starting from K attains Y_m before T.

This allows us to define the homotopy $h_m\colon K \times [0,1] \to K$,

$$h_m(x,t) = \begin{cases} S_{-f}(x)(tT), & \text{for } tT < e_m(x), \\ S_{-f}(x)(e_m(x)), & \text{for } tT \geqslant e_m(x). \end{cases}$$

One can see that h_m is a strong deformation of K onto Y_m. From Proposition 2.2 it follows that $\chi(Y_m) \neq 0$.

STEP 4. Analogously as in the first step of the proof, we show that each Y_m is invariant under $-f$. Thus, by the Invariance Theorem (see e.g. [2], Theorem 5.3.4), $-f(x) \in T_{Y_m}(x)$, for every $x \in Y_m$=2E From Proposition 2.4 it follows that $-f(x) \in C_{Y_m}(x)$.

STEP 5. Denote $r_m := h_m(\cdot, 1)\,|_{Int\, Y_{m-1}}\colon Int\, Y_{m-1} \to Y_m$, which is a retraction. We show that it is an $\mathcal{L}$-retraction.

Let x_0 and x_1 be two arbitrary points in $Int\, Y_{m-1}$ and assume that $e_m(x_0) \leqslant e_m(x_1)$. Then

$$|r_m(x_0) - r_m(x_1)| =$$

$$\left| x_0 + \int_0^{e_m(x_0)} (-f(S_{-f}(x_0)(s))\, ds - x_1 - \int_0^{e_m(x_1)} (-f(S_{-f}(x_1)(s))\, ds \right| \leqslant$$

$$|x_0 - x_1| + \int_0^{e_m(x_0)} |f(S_{-f}(x_0)(s) - f(S_{-f}(x_1)(s)|\, ds + \int_{e_m(x_0)}^{e_m(x_1)} |f(S_{-f}(x_1)(s)|\, ds \leqslant$$

$$|x_0 - x_1| + LTe^{LT}|x_0 - x_1| + M|e_m(x_1) - e_m(x_0)|,$$

where L is a Lipschitz constant for f and M is a constant which bounds values of f on K. Since e_m is Lipschitz, the map r_m so. Therefore, it is an $\mathcal{L}$-retraction (comp. Definition 2.3).

STEP 6. We use Proposition 1.1 for $-f$ and Y_m to obtain an equilibrium $z \in Y_m$, i.e. $-f(z) = 0 = 3Df(z)$. Since ε was arbitrary, we can consider a sequence $\varepsilon_k := \frac{1}{k}$ and find a sequence $\{f_k\}$ of Lipschitz $\frac{1}{k}$-approximations of F and a sequence $\{z_k\} \subset K$ such that $f_k(z_k) = 0$.

Since $0 = f_k(z_k) \in F\left(B\left(z_k, \frac{1}{k}\right)\right) + B\left(0, \frac{1}{k}\right)$, there are $x_k \in B\left(z_k, \frac{1}{k}\right)$ and $y_k \in F(x_k)$ with $|y_k| < \frac{1}{k}$. By compactness of K, we can assume that $z_k \to x \in K$. Therefore $x_k \to x$, $y_k \to 0$ and, since F has a closed graph, $0 \in F(x)$, which ends the proof. ∎

PROOF OF LEMMA 3.4. – At first, supposing (8), from Proposition 4.2.4 in [2], e_A is lower semicontinuous. Assume, by contradiction, that for some $x_0 \in K$ we have $e_A(x_0) = +\infty$. Then the closure of the ω-limit set $cl\left(\bigcap_{t>0} \times[t, +\infty)\right)$ of $x = S_{-f}(x_0)$ is a nonempty viable set (cf. Theorem 3.7.2 in [2]) contained in $Viab_{-F}(K \backslash Int(A))$. So, by Lemma 2.5, $Viab_F(K \backslash Int(A)) \neq \emptyset$; a contradiction.

Assume that (9) is satisfied. By similar arguments that discussed in Step 1 of the proof of our main theorem, one can deduce that A is invariant under the following differential inclusion

$$\dot{x}(t) \in -f(x(t)) + cl(B(0, \alpha)).$$

Theorem 4.3.8 in [11] yields

$$\forall x \in \partial A, \ \ \forall p \in NP_A(x), \ \ \forall b \in cl B(0, 1), \ \langle -f(x), p + \alpha b \rangle \leqslant 0,$$

where $NP_A(x)$ denotes the set of proximal normals (4) to A at x. So

$$\forall x \in \partial A, \ \ \forall p \in NP_A(x), \ \langle -f(x), p \rangle \leqslant -\alpha |p|.$$

This implies that e_A is Lipschitz on K from [26]. ■

4. – Concluding remarks.

For more regular sets the problem of finding equilibria can be simplified, as one can see in the following:

COROLLARY 4.1. – *Let $K = \overline{Int K} \subset \mathbb{R}^n$ be a compact $\mathcal{L}$-retract with $\chi(K) \neq 0$, $\Omega \subset \mathbb{R}^n$ an open neighbourhood of K and $F : \Omega \multimap \mathbb{R}^n$ a Marchaud map satisfying* (6). *Then F has an equilibrium in $Int K$.*

PROOF. – Since, as in the above proof, we can show that K is invariant under $-f$, Proposition 2.4 implies that $-f(x) \in C_K(x)$, for every $x \in K$. Thus the existence of an equilibrium follows from Proposition 1.1 and arguments from the last step of the proof of Theorem 3.1. ■

Below we give examples showing that there are natural situations where the set K is not an $\mathcal{L}$-retract and a map F satisfies assumptions of Theorem 3.1.

(4) Firstly introduced in [5]:

$$NP_A(x) := \{p \in \mathbb{R}^n, \exists a > 0, d_A(x + ap) = |p|\}.$$

EXAMPLE 4.2. – Let

$$A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x < 1 \text{ and } y > \sqrt{1-(x-1)^2} + \sqrt{3} - 1\right\} \cup$$

$$\cup \left\{(x, y) \in \mathbb{R}^2 \mid -1 < x < 0 \text{ and } y > \sqrt{1-(x+1)^2} + \sqrt{3} - 1\right\}$$

and $K = cl\,B((0, 0), 2) \backslash A$. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x, y)$.

Then K is not an $\mathcal{L}$-retract, but it is a contractible set with $\chi(K) = 1$, and f is a simple example of a Lipschitz map with $K_s = \partial K$. Notice that $v = 3D(0, \sqrt{3} - 1)$ is a point in ∂K with $f(v) \in T_K(v)$ and there is no trajectory starting from v and remainingin K. Nevertheless, f has an equilibrium in K.

The second observation is that the opposite (in some sense) tangential condition to (1):

$$F(x) \cap T_K(x) = \emptyset, \qquad \text{for every } x \in \partial K, \tag{10}$$

considered in [8] seems to be too strong, because in the point v we have $T_K(v) = \mathbb{R}^2$ and assumption (10) is impossible to be satisfied.

EXAMPLE 4.3. – Consider the set $K := K_1 \cup K_2 \cup K_3 \cup K_4$, where

$$K_1 := \{(x, y) \in cl\,B((0, 0), 3) \mid x \leqslant 0\},$$

$$K_2 := \{(x, y) \in cl\,B((1, 0), 3) \mid x \geqslant 1\},$$

$$K_3 := \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, 0 < x < 1,\ -3 \leqslant y \leqslant \sin\frac{\pi}{2x} + 2\right\},$$

$$K_4 := \{0\} \times [1, 3]$$

and the map $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x, y) := \begin{cases} (x, y), & \text{for } x \leqslant 0, \\ (0, y), & \text{for } 0 < x < 1, \\ (x-1, y), & \text{for } x \geqslant 1. \end{cases}$$

The set K is not even an absolute neighbourhood retract but (6) is still fulfilled. Notice that this stronger assumption $K_s = \partial K$ is not satisfied.

The problem of finding equilibria is a special case of the one of studying behaviour of invariant sets of flows generated by differential equations. Notice that our assumptions imply the existence of a Lipschitz approximation f of F which generates a continuous flow. Moreover, behaviour of f on ∂K guarantees that $(K, \partial K)$ is a compact index pair (5) in the sense of Conley index theory (see e.g. [20] for more details) for the largest invariant set S in K, namely $S = Viab_f(K) \cap Viab_{-f}(K) \subset Int\, K$.

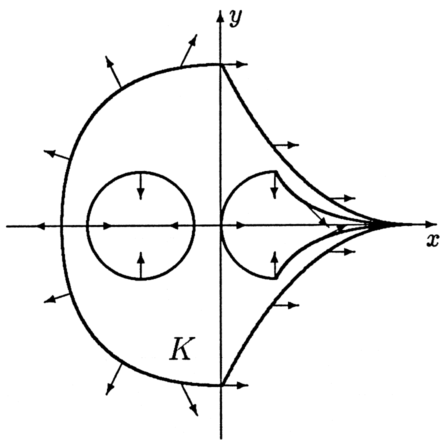
Therefore, in some situations we can find an equilibrium of f (and hence for F) using properties of the Conley index (see [21], [25]). Namely, we can obtain

PROPOSITION 4.4. – *Let $K = \overline{Int\, K}$ be a compact set, such that $(K, \partial K)$ forms an ENR pair. Assume that $\Omega \subset \mathbb{R}^n$ is an open neighbourhood of K and $F : \Omega \multimap \mathbb{R}^n$ is a Marchaud map satisfying assumption* (6).

If $\chi(K, \partial K) = \chi(K) - \chi(\partial K) \neq 0$, then there is an equilibrium of F in $Int\, K$.

In Theorem 3.1 we do not assume that $(K, \partial K)$ is a pair of *ENR*-spaces. Moreover, assumption $\chi(K) \neq 0$ does not imply that $\chi(K, \partial K) \neq 0$, as we can observe in the following example.

EXAMPLE 4.5. – Consider the set $K \subset \mathbb{R}^2$ as below, and a Lipschitz map $F : \mathbb{R}^2 \multimap \mathbb{R}^2$ being single-valued on ∂K with directions illustrated by arrows.



It is seen that $K_s = \partial K$. The set K is homotopically equivalent to

$$K_1 = cl\, B((0,0),4) \setminus (B((0,0),1) \cup B((3,0),1)).$$

It is easy to compute $\chi(K) = -1$, $\chi(\partial K) = -1$ and hence, $\chi(K, \partial K) = 0$.

By Theorem 3.1, there is an equilibrium of F in $Int\,(K)$.

(5) We say that (N_1, N_0) is a compact *index pair* for an invariant set S, if

(i) S is the largest invariant subset of $\overline{N_1 \setminus N_0}$ (i.e. $\overline{N_1 \setminus N_0}$ is an isolating neighbourhood of S);

(ii) if $x_0 \in N_0$, $t > 0$ and $S_f(x_0)([0, t]) \subset N_1$, then $S_f(x_0)([0, t]) \subset N_0$ (i.e. N_0 is *positively invariant in* N_1);

(iii) each trajectory leaving N_1, leaves it through N_0 (i.e. N_0 isan *exit set* for N_1).

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