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Martin Oxenham, Rey Casse

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Towards the Determination of the Regular n-Covers of $PG(3, q)$.

MARTIN OXENHAM (*) - REY CASSE

Sunto. – *Si dice che un insieme S di rette di $PG(3, q)$* copre *n* volte *un punto P di $PG(3, q)$, se esistono esattamente n rette di S incidenti P. Un insieme di rette di $PG(3, q)$ che copre n volte ogni punto di $PG(3, q)$ si dice n*-cover. *In questa nota, dopo una descrizione degli esempi noti di n-cover e delle rispettive proprietà, viene mostrato come gli n-cover di $PG(3, q)$ possono essere utilizzati per la costruzione di classi di disegni di Sperner quasi-n-multipli. Infine, allo scopo di ottenere nuovi esempi di tali disegni mediante la derivazione di quelli esistenti, si introduce la nozione di n-cover* regolare. *I risultati principali sono: la dimostrazione della non esistenza di un* 2*-cover regolare di $PG(3, q)$ per $q > 2$ e quella della non esistenza di un n-cover regolare $(n \geq 3)$ per $q \geq n+2$.*

Summary. – *A set of lines S of $PG(3, q)$ is said to* cover *a point P of $PG(3, q)$ n times if there are exactly n lines of S incident with P. An n*-cover *of $PG(3, q)$ is a set of lines of $PG(3, q)$ which covers each point of $PG(3, q)$ n times. In this paper, the properties and known examples of n-covers are reviewed and it is demonstrated how n-covers of $PG(3, q)$ can be used to construct classes of quasi-n-multiple Sperner designs. Finally, motivated by the problem of deriving these designs to arrive at new examples, the notion of* regular *n-covers of $PG(3, q)$ is introduced. The main results of the paper are that no regular 2-covers of $PG(3, q)$ exist for $q > 2$ and that no regular n-covers $(n \geq 3)$ exist whenever $q \geq n+2$.*

1. – Introduction.

A set of lines S of $PG(3, q)$ is said to *cover* a point P of $PG(3, q)$ n times if there are exactly n lines of S incident with P. An *n-cover* $(n \geq 0)$ of $PG(3, q)$ is a set of lines $\mathcal{C}_n$ of $PG(3, q)$ which covers each point of $PG(3, q)$ n times. By counting the number of flags in the set

$$\{(P, l) \mid P \in \mathcal{P},\ l \in \mathcal{C}_n\}$$

(*) This work was done while the first author was visiting the Department of Pure Mathematics at the University of Adelaide during recreation leave in October, 1999.

in two different ways where $\mathcal{P}$ if the pointset of $PG(3, q)$, it is immediate that $\mathcal{C}_n$ has cardinality $n(q^2+1)$. In particular, a 1-cover of $PG(3, q)$ is a set of $q^2 + 1$ lines covering the pointset of $PG(3, q)$ once, that is a spread of $PG(3, q)$. It is also clear that the union of k pairwise disjoint n_i-covers for $i = 1, \ldots, k$ is an $(n_1 + n_2 + \ldots + n_k)$-cover. Similarly, if $\mathcal{C}_{n_1}$ is an n_1-cover embedded in an n_2-cover $\mathcal{C}_{n_2}$, then $\mathcal{C}_{n_2} \setminus \mathcal{C}_{n_1}$ is an $(n_2 - n_1)$-cover.

The literature abounds with papers on spreads of $PG(3, q)$. In contrast, only several authors have investigated n-covers of $PG(3, q)$ with $n > 1$. In [43], Rao discusses k-fold spreads of m-spaces in $PG(n, q)$ which are defined to be sets of m-spaces such that each point of $PG(n, q)$ lies in exactly k of the m-spaces (see also [23], p. 83). In particular, an n-fold spread of 1-spaces in $PG(3, q)$ is precisely an n-cover of $PG(3, q)$. Beutelspacher in [5] refers to n-covers of $PG(m, q)$, but his definition bears little resemblance to the definition given herein and so his results are not relevant to the current discussion. In [15], Ebert considers n-covers in relation to the problem of extending partial packings of $PG(3, q)$ (ie sets of pairwise disjoint spreads) to complete packings of $PG(3, q)$ (ie maximal partial packings). Since the complement of the union of the spreads in a partial packing of size k is a $(q^2 + q + 1 - k)$-cover, the problem can be cast in terms of determining if such a cover can be partitioned into $(q^2 + q + 1 - k)$ spreads. Accordingly Ebert defines a *proper n-cover* to be an n-cover which cannot be partitioned into n-spreads. However, in [36] an alternative of a proper n-cover is given. Therein, an n-cover is said to be *proper* if it cannot be partitioned into an n_1-cover and an n_2-cover with n_1 and $n_2 \geqslant 1$. This is the definition adopted in this paper. It proves to be more useful for discussing the structure of n-covers because, with respect to this definition, the proper covers are the elemental sets from which all other covers can be constructed. It is also more suitable for discussing the issue of irreducibility of a class of quasi-n-multiple Sperner designs which can be constructed from n-covers of $PG(3, q)$ (see [36]). This construction is presented in section 3.

To develop further the ideas introduced above, the following additional definitions and results are required.

DEFINITION 1.1. – *Let $\mathcal{C}_n$ be a n-cover of $PG(3, q)$. Then $\mathcal{C}_n$ is said to be* symplectic *if it can be embedded in a general linear complex of $PG(3, q)$.*

It is noted that, since each plane of $PG(3, q)$ contains $q + 1$ lines of a general linear complex and these lines form a planar pencil, a symplectic n-cover $\mathcal{C}_n$ of $PG(3, q)$ with $2 \leqslant n \leqslant q + 1$ uniquely determines the general linear complex in which it is embedded because the $n \geqslant 2$ lines of $\mathcal{C}_n$ through each point P of $PG(3, q)$ are sufficient to reconstruct the planar pencil with vertex P.

DEFINITION 1.2. – [36] *Let $\mathcal{C}_n$ be a n-cover of $PG(3, q)$. An m-lateral in $\mathcal{C}_n$ is a sequence $(l_i)_{i=1}^{m}$ of m distinct lines of $\mathcal{C}_n$ such that l_i intersects l_{i+1} for each $i = 1, \ldots, m-1$ and l_m intersects l_1.*

THEOREM 1.1. – [15] *Let $\mathcal{C}_2$ be a 2-cover of $PG(3, q)$. If $\mathcal{C}_2$ contains a proper $(2m+1)$-lateral $(l_i)_{i=1}^{2m+1}$ for some $m \geqslant 1$, then $\mathcal{C}_2$ is a proper 2-cover.*

PROOF. – Assume that $\mathcal{C}_2$ is not a proper 2-cover of $PG(3, q)$. Then $\mathcal{C}_2$ is the union of two disjoint spreads $\mathcal{S}_1$ and $\mathcal{S}_2$. Without loss of generality, let l_1 lie in $\mathcal{S}_1$. Then, since l_1 intersects l_2, it follows that l_2 lies in $\mathcal{S}_2$. By continuing to argue along these lines, it can be shown that all odd-numbered lines lie in $\mathcal{S}_1$ and all even-numbered lines lie in $\mathcal{S}_2$. In particular, l_{2m+1} lies in $\mathcal{S}_1$. However, l_{2m+1} intersects l_1, therefore l_{2m+1} also lies in $\mathcal{S}_2$. This contradicts the fact that the two spreads are disjoint. Hence, $\mathcal{C}_2$ is proper.

DEFINITION 1.3. – [36] *Let $\mathcal{C}_n$ be an n-cover of $PG(3, q)$. Then $\mathcal{C}_n$ is said to be a* dual *n-cover if each plane of $PG(3, q)$ contains exactly n lines of $\mathcal{C}_n$.*

The duality of n-covers of $PG(3, q)$ is an extension of the duality of spreads which is discussed by Bruen and Fisher in [10]. There they prove that all t-spreads of $PG(2t+1, q)$ are dual. The following result establishes that all n-covers of $PG(3, q)$ are also dual, so in the particular case that $t = n = 1$, the result of Bruen and Fisher is recovered. However the reasoning here is different to that given in [10].

THEOREM 1.2. – [36] *Let $\mathcal{C}_n$ be an n-cover of $PG(3, q)$. Then $\mathcal{C}_n$ is dual.*

PROOF. – Let N be the number of lines of $\mathcal{C}_n$ in an arbitrary plane π of $PG(3, q)$ and for each point P in π let $\tau(P)$ be the number of lines of $\mathcal{C}_n$ in π which are incident with P. Then by counting the number of flags in the set

$$\{(P, l) \mid P \text{ is a point of } \pi, \ l \text{ is a line of } \mathcal{C}_n \text{ lying in } \pi\}$$

in two different ways, it follows that

$$N(q+1) = \sum_{P \in \pi} \tau(P).$$

As stated in the opening paragraph of the introduction, $\mathcal{C}_n$ has $n(q^2+1)$ lines. Hence the number of lines of $\mathcal{C}_n$ not in π is

$$\begin{aligned} n(q^2+1) - N &= \sum_{P \in \pi} (n - \tau(P)) \\ &= n(q^2+q+1) - \sum_{P \in \pi} \tau(P) \\ &= n(q^2+q+1) - N(q+1). \end{aligned}$$

On rearranging the terms in this equation, it follows that

$$N = n$$

and so $\mathcal{C}_n$ is dual.

DEFINITION 1.4. – [36] *Let* $\mathcal{C}_n$ *be an* n*-cover of* $PG(3, q)$. *The* spectrum *of* $\mathcal{C}_n$ (*denoted by* SPEC $(\mathcal{C}_n)$) *is the set of all non-negative integers* $m \leqslant n$ *for which there exists an* m*-cover lying in* $\mathcal{C}_n$.

2. – Summary of the known *n*-covers.

In this section, the main existence results for n-covers are summarised. For more information on general results stated here without proof, refer to [24].

The simplest examples of n-covers consist of the union of n spreads from a partial packing of $PG(3, q)$. Since there exist packings of $PG(3, q)$ (ie partial packings which partition the lineset of $PG(3, q)$ into spreads) for all prime powers q (refer to [4], [12], [13], [41] and [42]), it follows trivially that examples of n-covers exist for all admissible values of n, that is for $n = 1, \ldots, q^2 + q + 1$.

The first example of a proper n-cover of $PG(3, q)$ was a 2-cover of $PG(3, 2)$ provided by Bruen and Ott in a private communication to Ebert (refer to [15]). Motivated by their construction, Ebert [15] proceeded to construct an infinite class of proper 2-covers of $PG(3, q)$ for all odd prime powers q. His construction employs Singer's cyclic representation of $PG(3, q)$ (refer to Appendix A.1 for a description of the representation). Let β be a primitive element of $GF(q^4)$ with q odd and t be an arbitrary odd integer satisfying $1 \leqslant t \leqslant q$. Ebert proved that the union of the orbits of the two lines

$$l = \left\langle \beta^0, \beta^{\frac{t(q^2+1)}{2}} \right\rangle$$

$$l' = \left\langle \beta^0, \beta^{\frac{(t+q+1)(q^2+1)}{2}} \right\rangle$$

under the action of $\langle \beta^{(q+1)} \rangle$ is a proper 2-cover of $PG(3, q)$ with q odd.

Using a similar approach, he also constructed sporadic examples of proper 2-covers for $q = 2$, 4 and 8, and a symplectic proper 2-cover for $q = 3$. For $q = 2$, the proper 2-cover is projectively equivalent to the example of Bruen and Ott. It has since been shown (see [36]) that all proper 2-covers of $PG(3, 2)$ are projectively equivalent. The proof may be argued along the following lines. First, it can be shown that any proper 2-cover of $PG(3, 2)$ contains a 5-lateral. The 5 vertices and the 5 non-vertices of the 5-lateral are then respectively the points of two elliptic quadrics $\mathcal{E}_1$ and $\mathcal{E}_2$ of $PG(3, 2)$. Using this fact, it is straightforward to show that the 5-lateral is uniquely extensible to a 2-cover

$\mathcal{C}_2$ of $PG(3, 2)$ and that the 10 lines of $\mathcal{C}_2$ are all tangent to $\mathcal{E}_2$. As such, $\mathcal{C}_2$ is embedded in a general linear complex $\mathcal{L}$ of $PG(3, 2)$ and is complementary to a spread $\mathcal{S}$ in $\mathcal{L}$. As all general linear complexes of $PG(3, 2)$ are projectively equivalent and the collineation group of $\mathcal{L}$ acts transitively on the (regular) spreads lying in $\mathcal{L}$, it is then immediate that all proper 2-covers of $PG(3, 2)$ are projectively equivalent. As an aside, the full collineation group of the proper 2-cover is isomorphic to S_5, the symmetric group on five letters (see [35]). It is not known if proper 2-covers exist for $q = 2^e$ with $e \geq 4$. However, in [36], it is shown that symplectic proper n-covers exist for all $q = 2^e$ for some n between 2 and q inclusive (where n may be depend on q). The construction exploits a result of Bagchi and Sastry [1] which states that for q even, any spread in a general linear complex meets all regular and Lüneburg spreads lying in the general linear complex in at least one line. Thus, the removal of a regular or Lüneburg spread from a general linear complex $\mathcal{L}$ ensures that $\mathcal{L} \setminus \mathcal{S}$ contains no spread of $PG(3, q)$ and so is either itself a proper n-cover with $n = q$ or it contains a proper n-cover with $1 < n < q$.

Whenever an Hermitian surface $H(3, q^2)$ in $PG(3, q^2)$ possesses a hemisystem (that is a set of lines of $H(3, q^2)$ such that each point of $H(3, q^2)$ lies on exactly $\left(\frac{q+1}{2}\right)$ of the lines [24]), then there exists a symplectic $\left(\frac{q+1}{2}\right)$-cover of $PG(3, q)$. It is noted that hemisystems can only exist for q odd and are only known to exist for $q = 3$. The construction originally described for $q = 3$ in [40] goes as follows: Take a hemisystem of $H(3, q^2)$ with q odd. As is well-known (see [38]), the points and lines of $H(3, q^2)$ give rise to a generalised quadrangle (with incidence defined as set inclusion) which is isomorphic to the dual of the generalised quadrangle determined by the points and lines of an elliptic quadric $Q(5, q)$ in $PG(5, q)$ (again with incidence defined as set inclusion). The image of the hemisystem on $Q(5, q)$ is a set S of points meeting each line of $Q(5, q)$ in exactly $\left(\frac{q+1}{2}\right)$ points. The intersection of $Q(5, q)$ with a non-tangent hyperplane is a parabolic quadric $Q(4, q)$ embedded in $Q(5, q)$. By counting the flags in the set

$$\{(P, l) \mid P \in S,\ l \in Q(4, q),\ P \mathrm{I} l\}$$

in two different ways, it is immediate that $Q(4, q)$ contains $\left(\frac{(q+1)(q^2+1)}{2}\right)$ points of S with $\left(\frac{q+1}{2}\right)$ points of S lying on each line of $Q(4, q)$. Since the generalised quadrangles determined by $Q(4, q)$ and a general linear complex $\mathcal{L}$ in $PG(3, q)$ are dual, it follows that the image of $S \cap Q(4, q)$ on $\mathcal{L}$ is a $\left(\frac{q+1}{2}\right)$-cover of $PG(3, q)$. Note that this $\left(\frac{q+1}{2}\right)$-cover is not necessarily guaranteed to be proper. However, when $q = 3$, the resulting symplectic 2-cover of $PG(3, 3)$ is proper and is also projectively equivalent to the symplectic proper

2-cover of $PG(3, 3)$ constructed by Ebert. In fact, it can be shown that all symplectic proper 2-covers of $PG(3, 3)$ are projectively equivalent. In order to prove this, it is necessary to discuss briefly the classification of the 20-caps in $PG(4, 3)$.

A form of classification of the 20-caps in $PG(4, 3)$ was first determined by Pellegrino in [39]. He showed that the 20-caps fall broadly into two classes, namely Δ-caps and Γ-caps. In [21], Hill further refined this classification by showing that all Δ-caps are projectively equivalent and that there are 8 projectively distinct Γ-caps. It can be shown that each Δ-cap lies on a parabolic quadric of $PG(4, 3)$ and that each Γ-cap lies on an elliptic cone of $PG(4, 3)$. Using results from [21], it can also be shown that each cap uniquely determines the quadric in which it is embedded. In particular then, a 20-cap of $PG(4, 3)$ embedded in a parabolic quadric of $PG(4, 3)$ is automatically a Δ-cap.

Now consider two symplectic 2-covers $\mathcal{C}_2$ and $\mathcal{C}_2'$ of $PG(3, 3)$. Each of $\mathcal{C}_2$ and $\mathcal{C}_2'$ consists of 20 lines embedded in a general linear complex which cover the pointset of $PG(3, 3)$ twice. Since all general linear complexes are projectively equivalent, it can be assumed without loss of generality that $\mathcal{C}_2$ and $\mathcal{C}_2'$ are embedded in the same general linear complex $\mathcal{L}$. Let $W(3)$ be the generalised quadrangle determined by $\mathcal{L}$. The dual of $W(3)$ is isomorphic to $Q(4, 3)$ which is the generalised quadrangle determined by a parabolic quadric $P(4, 3)$ of $PG(4, 3)$. Hence, there exists a duality δ which maps $W(3)$ onto $Q(4, 3)$. Under the duality, $\mathcal{C}_2$ and $\mathcal{C}_2'$ map to two 20-caps K_{20} and K_{20}' on $Q(4, 3)$. Because the points and lines of $Q(4, 3)$ are the points and generators of $P(4, 3)$, it follows that K_{20} and K_{20}' are also 20-caps of $PG(4, 3)$ embedded in the parabolic quadric $P(4, 3)$. From the discussion in the previous paragraph, it is immediate that K_{20} and K_{20}' are both Δ-caps and so there is a collineation σ of $PG(4, 3)$ mapping K_{20} to K_{20}'. Since a Δ-cap uniquely determines the parabolic quadric of $PG(4, 3)$ in which it is embedded, it can be concluded that σ fixes $P(4, 3)$. Therefore, it follows that the composite mapping $\delta^{-1}\sigma\delta$ acts as a collineation of $W(3)$ which maps $\mathcal{C}_2$ to $\mathcal{C}_2'$. Finally, since every collineation of $W(3)$ is induced from a collineation of $PG(3, 3)$ (see [50], proposition 4.6.2, p. 154), the mapping $\delta^{-1}\sigma\delta$ lifts to a collineation of $PG(3, 3)$, proving the result.

Returning now to the discussion on n-covers, another strategy which has been employed to construct proper n-covers of $PG(3, q)$ involves starting with a $(q + 1)$-cover of $PG(3, q)$ such as a general linear complex or a long Singer line orbit and then trying to extract proper n-covers from them. This technique has already been used herein in the case of a general linear complex for q even to construct the proper 2-cover of $PG(3, 2)$. For $q = 3$, the removal of a (regular) spread from a general linear complex results in a symplectic proper 3-cover of $PG(3, 3)$. (The proof of this relies on the fact that there are only two projectively distinct spreads in $PG(3, 3)$, namely the regular spread and the subregular spread of index 1. Of these, only the former is symplectic. Furthermore, any two regular spreads in a general linear complex intersect in at least one line. Hence, the complement of a regular spread in a general linear complex is proper (see [36]). Similarly, the long Singer line orbits for $q = 2$ both constitute proper 3-covers of $PG(3, 2)$ (for these line orbits which are listed in the appendix it is easily shown that neither contains a spread (see [36]). Both these 3-covers are projectively equivalent to each other. Moreover, all known proper 3-covers of $PG(3, 2)$ are projectively equivalent to the long Singer line orbits and their collineation groups are isomorphic to the group

$$\langle \beta, \varrho \,|\, \beta^{15} = 1,\ \varrho^6 = 1,\ \beta^5 = \varrho^2 \rangle$$

which has order 30 (see [36]). The homographies of $PG(3, 2)$ corresponding to β and ϱ can be represented by the matrices below:

$$\beta \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \varrho \equiv \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

No examples of proper 4-covers of $PG(3, 2)$ are known, but if one exists, then its complement is a proper 3-cover which is not projectively equivalent to a long Singer line orbit. Proper 5-, 6- and 7-covers do not exist in $PG(3, 2)$ (see [36]).

Finally, for $q = 3$, a sporadic example of a proper 3-cover of $PG(3, 3)$ which is projectively distinct from the symplectic proper 3-cover was constructed in [36] partly by brute force via the determination of a partial packing of $PG(3, 3)$ of degeneracy 3. The complement of this partial packing is a proper 3-cover of $PG(3, 3)$ consisting of the lines numbered:

45,	54,	57,	62,	65,	66,	68,	73,	74,	81,
83,	84,	86,	89,	90,	93,	96,	98,	99,	100,
101,	102,	106,	109,	111,	113,	114,	116,	117,	118

(refer to the list of lines of $PG(3, 3)$ in the appendix). In particular, this 3-cover is not symplectic because it possesses at least one set of three concurrent lines which are not coplanar, for example 81, 102 and 116.

3. – Quasi-n-multiple designs and finite sperner spaces.

Let D be a balanced incomplete block design (BIBD) with v points, b blocks, k points in each block, r blocks through each point and λ blocks through each pair of distinct points. The parameters (v, b, r, k, λ) of D are related via the identities below (see [6]):

$$bk = vr$$

$$\lambda(v-1) = r(k-1).$$

As such, knowledge of v, k and λ is sufficient to uniquely determine the parameters of a BIBD. The BIBD obtained by taking $n \geqslant 1$ copies of each block of D is known as an *n-multiple design* and has parameters $(v, nb, nr, k, n\lambda)$ (note: in [6], n is required to be strictly greater than one, but for the sake of convenience and because it is felt to be more natural, this condition is relaxed throughout this paper). As such, any BIBD with parameters $(v, nb, nr, k, n\lambda)$ for some integer $n \geqslant 1$ is referred to as a *quasi-n-multiple design*. Clearly, an n-multiple design is also a quasi-n-multiple design, but the converse is not true in general (see for example [6], [7], [26], [27], [31] and [51]). A quasi-n-multiple design D is said to be *indecomposable* (or *irreducible*) if its block set cannot be partitioned into two subsets with each subset giving rise to a BIBD defined on the points of D with (by necessity) the same v and k (see [7]). A BIBD D is said to be *resolvable* if the block set of D can be partitioned into subsets known as *resolution classes*, such that the blocks in each resolution class partition the point set of D.

Let $S = (\mathcal{P}, \mathcal{B}, \mathfrak{I})$ be a finite incidence structure endowed with an equivalence relation *parallelism* on its line set. Then S is known as a *finite Sperner space* (alternatively, a *finite weak affine space* [3]) if it satisfies the following three axioms:

AXIOM 1. – Any two points are incident with exactly one line;

AXIOM 2. – For every point P and line l, there exists a unique line l' parallel to l and incident with P;

AXIOM 3. – Each line contains exactly $k \geqslant 2$ points for some integer k.

Resolvable BIBDs with $\lambda = 1$ such as finite translation planes and the classical affine spaces $AG(n, q)$ are also finite Sperner spaces (see also [33], [37], and [44]). Accordingly, such designs are referred to herein as *Sperner designs*. In the sequel, it is demonstrated how classes of quasi-n-multiple Sperner designs can be constructed from n-covers of $PG(3, q)$ by generalising the Bruck-Bose construction of finite translation planes of degree 2 over their kernel from spreads of $PG(3, q)$ (see [8]).

CONSTRUCTION 3.1. – Let Σ_{4t} be a finite $4t$-dimensional projective space and let Σ_{4t-1} be a fixed hyperplane of Σ_{4t}. In [45] p. 23, Segre proved that

$PG(n, q)$ can be partitioned into m-dimensional projective spaces of order q if and only if $m+1$ divides $n+1$ (see also [23] p. 83). It follows that Σ_{4t-1} can be partitioned into $\frac{q^{4t}-1}{q^4-1}$ 3-dimensional projective spaces Σ_3^i $\left(\text{where } i \text{ goes from } 1 \text{ to } \frac{q^{4t}-1}{q^4-1}\right)$ because 4 divides $4t$. Let $\mathcal{C}_n^i$ be an n-cover of the space Σ_3^i for each i. We now define an incidence structure S in the following manner:

The points of S are the points of $\Sigma_{4t} \backslash \Sigma_{4t-1}$.

Each block of S is a plane of Σ_{4t} which meets Σ_{4t-1} in exactly the $q+1$ points of a line lying in one of the n-covers $\mathcal{C}_n^i$.

The incidence relation is the incidence relation of Σ_{4t} restricted to the sets of points and planes defined above.

The incidence structure S constructed above is in fact a BIBD. This is proven in:

THEOREM 3.1. – *S is a BIBD with parameters* $v = q^{4t}$, $b = nq^{4t-2}\frac{(q^{4t}-1)}{(q^2-1)}$, $r = n\frac{(q^{4t}-1)}{(q^2-1)}$, $k = q^2$ *and* $\lambda = n$.

PROOF. – The points of S are the points of $\Sigma_{4t} \backslash \Sigma_{4t-1}$ which is equivalent to a $4t$-dimensional affine space. Hence the number of points in S is $v = q^{4t}$.

The blocks of S are the planes of Σ_{4t} not lying in Σ_{4t-1} which meet Σ_{4t-1} in a line of one of the n-covers $\mathcal{C}_n^i$. Hence $k = q^2$. Also through each line in the union of the $\frac{q^{4t}-1}{q^4-1}$ n-covers there pass $\frac{q^{4t}}{q^2} = q^{4t-2}$ distinct planes of Σ_{4t} which do not lie in Σ_{4t-1}. Furthermore, as each n-cover has $n(q^2+1)$ distinct lines, it follows that the number of blocks in S is

$$n(q^2+1)\, q^{4t-2}\frac{(q^{4t}-1)}{(q^4-1)} = nq^{4t-2}\frac{(q^{4t}-1)}{(q^2-1)}.$$

Similarly, as each point of $\Sigma_{4t} \backslash \Sigma_{4t-1}$ defines a unique plane (not lying in Σ_{4t-1}) with each line in the union of the $\frac{q^4t-1}{q^4-1}$ n-covers. It is immediate then that each point of S lies in

$$n(q^2+1)\frac{(q^{4t}-1)}{(q^4-1)} = n\frac{(q^{4t}-1)}{(q^2-1)}$$

distinct blocks.

Finally, given a pair of arbitrary distinct points in $\Sigma_{4t} \backslash \Sigma_{4t-1}$, the line l passing through them will meet Σ_{4t-1} in a single point. This point will lie on n distinct lines of one of the n-covers and these will each define with l, a plane of

Σ_{4t} not lying in Σ_{4t-1}. Therefore each pair of distinct points of S, lies in n distinct blocks of S.

It is noted that none of the blocks of the BIBD S is repeated and for any two distinct points P_1 and P_2 of S, the blocks containing P_1 and P_2 meet pairwise in a set of q points which is uniquely determined by P_1 and P_2. It is also resolvable because it admits a resolution in which each resolution class corresponds to the set of all planes of $\Sigma_{4t} \backslash \Sigma_{4t-1}$ which meet in a common line of one of the n-covers $\mathcal{C}_n^i$. If each n-cover $\mathcal{C}_n^i$ is a spread of Σ_3^i, then the BIBD S is a finite Sperner design because the resolution acts as a parallelism on the line set of S. Hence, for $n \geq 1$, the construction furnishes examples of quasi-n-multiple Sperner designs (when $t = 1$, the quasi-n-multiple Sperner designs are also quasi-n-multiple affine designs).

LEMMA 3.1. – *Let $D(n)$ denote a decomposable design constructed via the above technique and let D' be one of the designs in the decomposition. If the number of blocks of D' through each pair of distinct points of D' is m (for some integer $m < n$), then each of the individual n-covers used in the construction contains an m-cover.*

PROOF. – Since D' is a component of the decomposition of $D(n)$, it has the same parameters v and k as $D(n)$. From the two identities which relate the parameters of a BIBD, it then follows that D' has parameters $v = q^{4t}$, $b = mq^{4t-2}\dfrac{(q^{4t}-1)}{(q^2-1)}$, $r = m\dfrac{(q^{4t}-1)}{(q^2-1)}$, $k = q^2$ and $\lambda = m$, and so D' is a quasi-m-multiple (Sperner) design.

Consider the blocks of D' through a fixed point P of D'. Each such block corresponds to a plane of Σ_{4t} meeting Σ_{4t-1} in a line in the union of the n-covers. Let the totality of lines defined by the blocks of D' through the point P be $\mathcal{C}$. Since there are $m\dfrac{(q^{4t}-1)}{(q^2-1)}$ blocks in D' containing P, the set $\mathcal{C}$ contains $m\dfrac{(q^{4t}-1)}{(q^2-1)}$ lines.

Now let R be an arbitrary point of Σ_{4t-1}. Each line in Σ_{4t} has at least three points, so a point Q can be chosen on the line through P and R with $Q \neq P, R$. Q is also a point of D'. Therefore there are exactly m blocks of D' containing P and Q. Each block gives rise to a line of $\mathcal{C}$ through R. Thus there are exactly m lines of $\mathcal{C}$ through R. Furthermore, if Σ_3^i is the unique three dimensional space (in the partition of Σ_{4t-1}) which contains R, then these m lines lie in Σ_3^i. As R is arbitrary, it can be concluded that the subset $\mathcal{C}_m^i$ of the lines of $\mathcal{C}$ which lie in a given Σ_3^i (belonging to the partition of Σ_{4t-1}) satisfies the property that through each point of Σ_3^i there pass exactly m lines of $\mathcal{C}_m^i$. This implies that $\mathcal{C}_m^i$ is an m-cover of Σ_3^i. Finally $\mathcal{C}_m^i$ is contained in $\mathcal{C}_n^i$ because $\mathcal{C}$ lies in the union of the n-covers. Thus each n-cover contains an m-cover.

THEOREM 3.2. – *Let $D(n)$ be a design arising via Construction 3.1 with respect to the n-covers $\{\mathcal{C}_n^i\}$. Then $D(n)$ is indecomposable if and only if*

$$\bigcap_i \mathrm{SPEC}\,(\mathcal{C}_n^i) = \{n\}.$$

PROOF. – ($\Rightarrow$) Let the design be indecomposable. Assume that

$$\bigcap_i \mathrm{SPEC}\,(\mathcal{C}_n^i) \neq \{n\}.$$

Then there exists a positive integer m lying in the intersection of the spectra and satisfying $1 \leqslant m < n$. This implies that each n-cover $\mathcal{C}_n^i$ can be partitioned into distinct m and $(n-m)$-covers. Denoting the designs corresponding to these covers by $D(m)$ and $D(n-m)$, it follows that

$$D(m) \cup D(n-m) = D(n).$$

Therefore $D(n)$ is decomposable, a contradiction. Consequently $\bigcap_i \mathrm{SPEC}\,(\mathcal{C}_n^i) = \{n\}$.

($\Leftarrow$) Let $\bigcap_i \mathrm{SPEC}\,(\mathcal{C}_n^i) = \{n\}$.

Assume the design $D(n)$ is decomposable. Then by Lemma 3.1, there exists an integer m satisfying $1 \leqslant m < n$, such that $D(n)$ is the union of two subdesigns $D(m)$ and $D(n-m)$. Again by Lemma 3.1, this implies that each n-cover contains an m-cover and an $(n-m)$-cover. Hence

$$\bigcap_i \mathrm{SPEC}\,(\mathcal{C}_n^i) \supseteq \{m, n-m\}, \quad \text{a contradiction}.$$

Therefore the design is indecomposable.

COROLLARY 3.1. – *Let $D(n)$ be constructed via an n-cover $\mathcal{C}_n$ of $PG(3, q)$. Then $D(n)$ is indecomposable if and only if $\mathcal{C}_n$ is proper. (That is $\mathrm{SPEC}\,(\mathcal{C}_n) = \{n\}$.)*

EXAMPLE 3.1. – Let the points of the projective space $PG(4, 2)$ (with respect to homogeneous coordinates $(x_0, x_1, x_2, x_3, x_4)$ over $GF(2)$) be labelled as shown below:

P_1	(1 0 0 0 0)	P_6	(1 1 0 0 0)	P_{11}	(1 0 1 0 0)
P_2	(0 1 0 0 0)	P_7	(0 1 1 0 0)	P_{12}	(0 1 0 1 0)
P_3	(0 0 1 0 0)	P_8	(0 0 1 1 0)	P_{13}	(1 1 0 1 0)
P_4	(0 0 0 1 0)	P_9	(1 1 1 0 0)	P_{14}	(1 0 0 1 0)
P_5	(1 1 1 1 0)	P_{10}	(0 1 1 1 0)	P_{15}	(1 0 1 1 0)
A_1	(0 0 0 0 1)	A_7	(1 0 1 0 1)	A_{12}	(1 1 1 0 1)
A_2	(1 0 0 0 1)	A_8	(1 0 0 1 1)	A_{13}	(1 1 0 1 1)
A_3	(0 1 0 0 1)	A_9	(0 1 1 0 1)	A_{14}	(1 0 1 1 1)
A_4	(0 0 1 0 1)	A_{10}	(0 1 0 1 1)	A_{15}	(0 1 1 1 1)
A_5	(0 0 0 1 1)	A_{11}	(0 0 1 1 1)	A_{16}	(1 1 1 1 1)
A_6	(1 1 0 0 1)				

The points P_i for $i = 1, \ldots, 15$ are the fifteen points of Σ_3, the 3-dimensional projective space with equation $x_4 = 0$ embedded in $PG(4, 2)$. The sixteen points A_i are the points of the 4-dimensional affine space constructed from $PG(4, 2)$ by removing Σ_3.

Employing Construction 3.1 with the proper 2-cover of $PG(3, 2)$ whose lines are listed in section 4, the resolvable quasi-2-multiple affine design $D(2)$ so obtained is:

$\{1, 2, 3, 6\}$	$\{1, 2, 15, 16\}$	$\{1, 10, 12, 14\}$
$\{4, 7, 9, 12\}$	$\{3, 6, 11, 14\}$	$\{2, 9, 11, 13\}$
$\{5, 8, 10, 13\}$	$\{4, 7, 10, 13\}$	$\{3, 5, 7, 16\}$
$\{11, 14, 15, 16\}$	$\{5, 8, 9, 12\}$	$\{4, 6, 8, 15\}$
$\{1, 3, 4, 9\}$	$\{1, 6, 8, 10\}$	$\{1, 7, 13, 15\}$
$\{2, 6, 7, 12\}$	$\{2, 3, 5, 13\}$	$\{2, 4, 10, 16\}$
$\{5, 10, 11, 15\}$	$\{4, 12, 14, 15\}$	$\{3, 8, 11, 12\}$
$\{8, 13, 14, 16\}$	$\{7, 9, 11, 16\}$	$\{5, 6, 9, 14\}$
$\{1, 4, 5, 11\}$	$\{1, 9, 13, 14\}$	$\{1, 5, 12, 16\}$
$\{2, 7, 8, 14\}$	$\{2, 10, 11, 12\}$	$\{2, 8, 9, 15\}$
$\{3, 9, 10, 15\}$	$\{3, 4, 8, 16\}$	$\{3, 7, 10, 14\}$
$\{6, 12, 13, 16\}$	$\{5, 6, 7, 15\}$	$\{4, 6, 11, 13\}$
$\{1, 7, 8, 11\}$		
$\{2, 4, 5, 14\}$		
$\{3, 12, 13, 15\}$		
$\{6, 9, 10, 16\}$		

where the numbers are the subscripts of the affine points A_i and the lines are grouped into resolution classes. By Corollary 3, it is immediate that the design is indecomposable.

It is noted that not all quasi-n-multiple affine designs arise from this construction. In [27], Jungnickel constructs quasi-2-multiple affine designs from existing affine designs of any order by permuting the points (and therefore also the blocks) of the designs and then adjoining the two block sets. These designs are all decomposable and often contain repeated blocks. Quasi-2-multiple affine designs on 9 points have been studied by Morgan in [31] and Mathon and Rosa in [29]. Of special interest is entry (30) on p. 248 of [31] (alternatively on p.314 of [29]), which has no repeated blocks and so as a consequence, satisfies the property that the two blocks intersect in 0, 1 or 2 points (cf. the property of the quasi-n-multiple affine designs arising from Construction 3 that any two blocks intersect in 0, 1 or q points). The Hall triple system constructed in [37] also furnishes an example of a quasi-multiple affine design possessing this property with $q = 3$. However, the most remarkable quasi-multiple affine designs to date are those with parameters (36,84,14,6,2) and (36,126,21,6,3) (see [19] p.296, [30] p.283 and [51]). These are remarkable because no (36,42,7,6,1)-design exists (a design corresponding to these parameters would be an affine plane of order 6 which is known not to exist. See [19], pp.175-176).

4. – Regular n-Covers.

As mentioned in section 3, the technique for constructing quasi-n-multiple Sperner designs described in Construction 3.1 generalises the Bruck-Bose construction of finite translation planes of degree 2 over their kernel. As such, the notions of derivation and more generally net replacement for translation planes both extend naturally to the quasi-n-multiple Sperner designs. In particular, the reversal of a switching set embedded in one of the constituent n-covers will result in a new quasi-n-multiple Sperner design provided none of the lines in the opposite switching set lies in the n-cover. This amounts to replacing a translation net in the design (refer to [17]). With the aim of producing new quasi-n-multiple Sperner designs from existing ones via the replacement of translation nets of size $q + 1$, the remainder of this section is dedicated to investigating the existence of reguli in n-covers of $PG(3, q)$.

To demonstrate that examples of reguli lying in an n-cover $\mathcal{C}_n$ and having no transversal in $\mathcal{C}_n$ do exist, some explicit examples are given for small values of q. The line numbers correspond to the numbering scheme used in the listings of the Singer line orbits of $PG(3, 2)$ and $PG(3, 3)$ which appear in Appendix A.2.

The Proper 2-Cover of $PG(3, 2)$.

The lines of the 2-cover are:

1	2	3	8	10
12	21	22	26	35

The three lines 8, 26 and 35 form a regulus with transversals 4, 9, 11. (Reversing the regulus produces a 2-cover which is the union of two disjoint spreads.)

The Proper 3-Cover of $PG(3, 2)$.

The lines of the 3-cover are:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

The three lines 1, 3 and 9 form a regulus with transversals 16, 17, 20. (Reversing the regulus produces a 3-cover which is the union of three disjoint spreads.)

A Proper Ebert 2-Cover of $PG(3, 3)$.

The lines of the 2-cover are:

41	45	49	53	57	61	65	69	73	77
81	85	89	93	97	101	105	109	113	117

The two rows correspond to the orbits of lines 41 and 81 under the action of the group $\langle \beta^4 \rangle$. The four lines 81, 85, 101 and 105 form a regulus with transversals 1, 21, 126, 130. (Reversing the regulus produces another proper 2-cover containing the 5-lateral with lines 41, 65, 49, 73 and 57. It is not known if it belongs to one of the existing classes of 2-covers.)

The Proper Symplectic 2-Cover of $PG(3, 3)$.

The lines of the 2-cover are:

1	9	17	25	33	83	87	91	95	99
103	107	111	115	119	121	123	125	127	129

The four lines 17, 33, 91 and 103 form a regulus with transversals 20, 30, 97, 117. (Reversing the regulus produces another proper 2-cover containing the 5-lateral with lines 1, 125, 95, 87 and 119. It is not known if it belongs to one of the existing classes of 2-covers. It is known at least that it is not symplectic. This can be argued as follows: Let the original 2-cover be $\mathcal{C}_2$ and the «switched» 2-cover be $\mathcal{C}_2'$. Assume $\mathcal{C}_2'$ is symplectic. Since two distinct general linear complexes in $PG(3, 3)$ intersect in the 10 lines of a regular spread and $\mathcal{C}_2$ and $\mathcal{C}_2'$ have 16 lines in common, it follows that the two 2-covers lie in the same general linear complex. However, the lines which uniquely complete $\mathcal{C}_2$ to a general linear complex are:

5	13	21	29	37	41	45	49	53	57
61	65	69	73	77	122	124	126	128	130

and so by inspection it is evident that the lines 20, 30, 97 and 117 of the regulus in $\mathcal{C}_2'$ do not lie in the general linear complex determined by $\mathcal{C}_2$, α contradiction.)

The Proper Symplectic 3*-Cover of* $PG(3, 3)$.

The lines of the 3-cover are:

1	5	9	13	17	21	25	29	33	37
41	45	49	53	57	61	65	69	73	77
83	87	91	95	99	103	107	111	115	119

The four lines 17, 33, 91 and 103 form a regulus with transversals 20, 30, 97, 117. (Reversing the regulus produces another 3-cover, but it is not known if it belongs to one of the existing classes of 3-covers. It is known at least that it is not symplectic because it contains three non-coplanar concurrent lines, for example lines 20, 29 and 95 which meet in point number 19.)

A Proper 3*-Cover of* $PG(3, 3)$ *Complementary to a Partial Packing.*

The lines of the 3-cover are:

45	54	57	62	65	66	68	73	74	81
83	84	86	89	90	93	96	98	99	100
101	102	106	109	111	113	114	116	117	118

The four lines 62, 96, 98 and 118 form a regulus with transversals 18, 78, 104, 127. (Reversing the regulus produces another 3-cover, but it is not known if it belongs to one of the existing classes of 3-covers.) Note: The spreads in the partial packing of $PG(3, 3)$ of degeneracy 3 which gives rise to this 3-cover are listed in Appendix B.

The examples above indicate that it is not an uncommon situation (at least for small values of q) to find reguli in an n-cover $\mathcal{C}_n$ of $PG(3, q)$ which have no transversal in $\mathcal{C}_n$. In general, an n-cover of $PG(3, q)$ may contain more than one regulus satisfying this property. Currently however, with the exception of various spreads, especially the regular spreads, there are no results for general q which guarantee the existence of such reguli in n-covers. In the remainder of the section, a related problem on the existence of reguli (not necessarily satisfying the above property) is discussed.

More precisely, n-covers $\mathcal{C}_n$ satisfying the regularity property stated in the following definition are examined.

DEFINITION 4.1. – *An n-cover $\mathcal{C}_n$ of $PG(3, q)$ is said to be a* regular *n-cover if it satisfies the property that for any three pairwise skew lines of $\mathcal{C}_n$, the regulus determined by the lines lies wholly in $\mathcal{C}_n$.*

For $q = 2$, each n-cover is trivially a regular n-cover because the regulus defined by three arbitrary pairwise skew lines simply consists of those three lines. In contrast, the only known examples of regular n-covers for $q > 2$ (up to projective equivalence) are the regular spread, the general linear complex and the line set of $PG(3, q)$. It is conjectured that these examples account for all of the regular n-covers. In the sequel, several non-existence results are established which lend credence to this conjecture.

It is immediate that a regular 1-cover, that is a spread, of $PG(3, q)$ actually does contain at least one regulus since any three distinct lines of the spread are pairwise skew and so the spread contains the regulus determined by them. It is also straightforward to prove that a regular 2-cover $\mathcal{C}_2$ contains reguli, by noting that for each line m of $\mathcal{C}_2$, the $q + 1$ lines of $\mathcal{C}_2$ which meet m are the lines of a regulus (any two of the $q + 1$ lines are pairwise skew because if they intersected, then they would be coplanar with m contradicting the duality of $\mathcal{C}_2$. Thus, any three of the $q + 1$ lines determine a regulus lying in $\mathcal{C}_2$ which has m as a transversal. Finally, since $\mathcal{C}_2$ is a 2-cover, the lines of the regulus can only be the lines of $\mathcal{C}_2$ meeting m). However, for $n \geqslant 3$ it is not so immediate since it first needs to be shown that the n-cover of $PG(3, q)$ contains three pairwise skew lines. Fortunately, the result is valid for all n. This is now shown.

LEMMA 4.1. – *A regular n-cover of $PG(3, q)$ always contains a regulus.*

PROOF. – Let l be a line of the n-cover $\mathcal{C}_n$. The number of lines of $\mathcal{C}_n$ which intersect l (other than l itself) is $(q+1)(n-1)$. Hence, for $q \geqslant 2$, there are:

$$\begin{aligned} N &= n(q^2 + 1) - [(q+1)(n-1) + 1] \\ &= nq(q-1) + q \\ &> n \end{aligned}$$

lines of $\mathcal{C}_n$ which are skew to l. Assume now that each line of $\mathcal{C}_n$ skew to l intersects every other line of $\mathcal{C}_n$ skew to l. Then it follows that these N lines skew to l are all concurrent or they are all coplanar. However, since $\mathcal{C}_n$ is an n-cover and by Theorem 1 also a dual n-cover, the number of lines through each point and the number of lines lying in each plane is $n < N$ which yields a contradiction. It follows then that at least two of the lines l' and l'' of $\mathcal{C}_n$ skew to l are also skew to each other. Finally, because $\mathcal{C}_n$ is regular, it contains the regulus determined by l, l' and l'' which proves the result.

THEOREM 4.1. – *Let $\mathcal{C}_n$ be a regular n-cover of $PG(3, q)$ embedded in a regular m-cover $\mathcal{C}_m$ with $n < m$ and $q \geq 4$. Then the complement of $\mathcal{C}_n$ in $\mathcal{C}_m$ is not regular.*

PROOF. – Let $\mathcal{C}_{m-n}$ denote the complement of $\mathcal{C}_n$ in $\mathcal{C}_m$ and assume that it is regular. The proof proceeds by constructing a regulus which contains at least three lines of $\mathcal{C}_{m-n}$, but which is not wholly contained in $\mathcal{C}_{m-n}$.

Choose two skew lines l_1 and l_2 in $\mathcal{C}_n$ (by Lemma 4.1, two such lines exist because $\mathcal{C}_n$ contains a regulus). The maximum number of lines of $\mathcal{C}_{m-n}$ meeting l_1 or l_2 is $2(q+1)(m-n)$ (which corresponds to the case that no line of $\mathcal{C}_{m-n}$ meets both l_1 and l_2). Hence the number of lines in $\mathcal{C}_{m-n}$ skew to both l_1 and l_2 is at least

$$\begin{aligned}&(m-n)(q^2+1) - 2(q+1)(m-n)\\ &= (m-n)((q-1)^2 - 2)\end{aligned}$$

which is positive for all $q \geq 4$. Therefore, $\mathcal{C}_{m-n}$ possesses at least one line l_3 skew to both l_1 and l_2.

Now consider the regulus $\mathcal{R}$ determined by l_1, l_2 and l_3. By construction, $\mathcal{R}$ intersects both $\mathcal{C}_n$ and $\mathcal{C}_{m-n}$. Furthermore, it intersects $\mathcal{C}_n$ in exactly the two lines l_1 and l_2 because if it contained a third line of $\mathcal{C}_n$, then it would be wholly contained in $\mathcal{C}_n$ by the regularity of $\mathcal{C}_n$, contradicting the fact that l_3 is in $\mathcal{C}_{m-n}$. Hence, since $q \geq 4$ and $\mathcal{C}_m$ is regular, $\mathcal{R}$ contains at least three lines of $\mathcal{C}_{m-n}$, but is not wholly contained in $\mathcal{C}_{m-n}$, and so $\mathcal{C}_{m-n}$ is not regular.

THEOREM 4.2. – ([9], Theorem 4.3, p. 436) *Let $\mathcal{R}$ be a regulus in $PG(3, q)$ and l be a line skew to each line of $\mathcal{R}$. Then there exists a unique regular spread $\mathcal{S}$ which contains both $\mathcal{R}$ and l.*

LEMMA 4.2. – *A regular n-cover $\mathcal{C}_n$ of $PG(3, q)$, $q \geq 3$, which contains a line l skew to every line of some regulus $\mathcal{R}$ in $\mathcal{C}_n$, also contains the regular spread $\mathcal{S}$ determined by $\mathcal{R}$ and l.*

PROOF. – The proof proceeds by demonstrating that every line of $\mathcal{S}$ can be recovered from $\mathcal{R}$ and l using only the regularity property shared by $\mathcal{C}_n$ and $\mathcal{S}$.

Let l_i for $i = 1, \dots, q+1$ be the lines of $\mathcal{R}$ and let $\mathcal{R}_j$ be the regulus determined by l, l_1 and l_j for $j = 2, \dots, q+1$. Clearly, for each $j \neq k$, the reguli $\mathcal{R}_j$ and $\mathcal{R}_k$ intersect in exactly the two lines l and l_1. (If they shared a third line then they would be identical. In particular, they would both contain the three distinct lines l_1, l_j and l_k of $\mathcal{R}$ in which case they would also be identical to $\mathcal{R}$. However, by hypothesis $\mathcal{R}$ does not contain l, contradicting the fact that both $\mathcal{R}_j$ and $\mathcal{R}_k$ do contain l by construction.)

Counting the lines in the union of the q reguli, it follows that $\mathcal{U} = \bigcup_{j=2}^{q+1} \mathcal{R}_j$ accounts for

$$q(q-1)+2 = q^2 - q + 2$$

lines of $\mathcal{S}$. This leaves $q-1$ lines m_k for $k = 1, \ldots, q-1$ of $\mathcal{S}$ unaccounted for. To show that these remaining lines also lie in $\mathcal{C}_n$, the following argument can be used.

Let $\mathcal{R}(l, l_1, m_1)$ be the regulus determined by l, l_1 and m_1. It is noted that $\mathcal{R}(l, l_1, m_1)$ and $\mathcal{U}$ have exactly the two lines l and l_1 in common because if they shared a third line, then $\mathcal{R}(l, l_1, m_1)$ would lie wholly in $\mathcal{U}$, contradicting the fact that m_1 lies in $\mathcal{S} \setminus \mathcal{U}$. Hence $\mathcal{R}(l, l_1, m_1) = \{l, l_1\} \cup (\mathcal{S} \setminus \mathcal{U})$ because it lies in $\mathcal{S}$.

Now consider the regulus $\mathcal{R}(l, l_2, m_1)$. It cannot contain any line of $\mathcal{S} \setminus \mathcal{U}$ other than m_1, otherwise it would coincide with $\mathcal{R}(l, l_1, m_1)$, contradicting the fact that l_2 does not lie in $\mathcal{R}(l, l_1, m_1)$. Hence, because $q \geqslant 3$ there exists at least a third line l^* of $\mathcal{U}$ other than l and l_2 which lies in $\mathcal{R}(l, l_2, m_1)$.

It now follows that every line of $\mathcal{S}$ lies in $\mathcal{C}_n$ because the lines in $\mathcal{R}$ and l determine $\mathcal{U}$ via the regularity property, then likewise the lines l, l_2 and l^* in $\mathcal{U}$ determine m_1 and finally the lines l and l_1 in $\mathcal{U}$ and m_1 determine the remaining lines of $\mathcal{S} \setminus \mathcal{U}$.

THEOREM 4.3. – *Let $\mathcal{C}_n$ be a regular n-cover of $PG(3, q)$ with $q \geqslant 2n \geqslant 4$. Then $\mathcal{C}_n$ is not proper.*

PROOF. – Let $\mathcal{R}$ be a regulus in $\mathcal{C}_n$ (such a regulus exists by Lemma 4.1). The maximum number of lines of $\mathcal{C}_n$ which meet at least one line of $\mathcal{R}$ (which occurs when no line of $\mathcal{C}_n \setminus \mathcal{R}$ meets more than one line of $\mathcal{R}$) is $(n-1)(q+1)^2 + (q+1)$. Hence, the number of lines of $\mathcal{C}_n$ which are skew to every line of $\mathcal{R}$ is at least

$$\begin{aligned} &n(q^2+1) - (n-1)(q+1)^2 - (q+1) \\ &= q(q+1-2n) > 0. \end{aligned}$$

Thus, there is at least one line l of $\mathcal{C}_n$ which is skew to every line of $\mathcal{R}$. By Lemma 4.2, the regular spread determined by $\mathcal{R}$ and l lies in $\mathcal{C}_n$ and so $\mathcal{C}_n$ is not proper.

The first of the main non-existence results for regular n-covers can now be proven.

THEOREM 4.4. – *Let $\mathcal{C}_2$ be a 2-cover of $PG(3, q)$ with $q \geq 4$. Then $\mathcal{C}_2$ is not regular.*

PROOF. – Assume $\mathcal{C}_2$ is regular.

By Theorem 4.3, $\mathcal{C}_2$ contains a regular spread $\mathcal{S}_1$. Let $\mathcal{R}$ be a regulus in $\mathcal{S}_1$ and $\mathcal{S}' = \mathcal{C}_2 \setminus \mathcal{S}_1$.

Now choose a line l_1 of $\mathcal{R}$. Since $\mathcal{C}_2$ is regular, the $q+1$ lines of $\mathcal{C}_2$ meeting l_1 in a single point form a regulus $\mathcal{R}_2$ by the remark in the paragraph preceding Theorem 4.1. By construction, $\mathcal{R}_2$ lies in $\mathcal{S}'$ and so there exists a line l_2 of $\mathcal{S}'$ which is skew to every line of $\mathcal{R}_2$. Thus, by Lemma 4.2, $\mathcal{R}_2$ and l_2 determine a unique regular spread $\mathcal{S}_2$ lying in $\mathcal{C}_2$.

Consider $\mathcal{S}_1 \cap \mathcal{S}_2$. The intersection $\mathcal{S}_1 \cap \mathcal{S}_2$ is non-empty, otherwise $\mathcal{C}_2 = \mathcal{S}_1 \cup \mathcal{S}_2$, ie $\mathcal{C}_2$ is the union of two regular spreads contradicting Theorem 4.1. In addition, $\mathcal{S}_1 \neq \mathcal{S}_2$ because l_1 lies in $\mathcal{S}_1$, but not in $\mathcal{S}_2$. Hence

$$|\mathcal{S}_1 \cap \mathcal{S}_2| = 1, 2 \text{ or } q+1 .$$

If $|\mathcal{S}_1 \cap \mathcal{S}_2| = 1$ or 2, then $|\mathcal{S}_2 \cap \mathcal{S}'| = q^2$ or $q^2 - 1$ respectively. Now $\mathcal{S}_2 \cap \mathcal{S}'$ is a partial spread of degeneracy at most $2 < q+1$ which lies in a spread. Furthermore, a partial spread with less than $q+1$ lines cannot possess a switching set because it has too few lines to cover the points on any line of a putative switching set. Thus $\mathcal{S}_2 \cap \mathcal{S}'$ can be uniquely extended to a spread. It follows then that $\mathcal{S}_2 = \mathcal{S}'$ and so that

$$\mathcal{C}_2 = \mathcal{S}_1 \cup \mathcal{S}' = \mathcal{S}_1 \cup \mathcal{S}_2,$$

which again contradicts the result in Theorem 4.1. Thus, $|\mathcal{S}_1 \cap \mathcal{S}_2| = q+1$ and so $\mathcal{S}_1 \cap \mathcal{S}_2$ is a regulus $\mathcal{R}_3$.

It follows that $\mathcal{C}_2$ can be written as the union of the four pairwise disjoint linesets $\mathcal{R}_3$, $\mathcal{S}_1 \setminus \mathcal{R}_3$, $\mathcal{S}_2 \setminus \mathcal{R}_3$ and $\mathcal{S}' \setminus \mathcal{S}_2$.

Now the set of points $\mathcal{P}$ covered by the lines of $\mathcal{S}_1 \setminus \mathcal{R}_3$ is the same set of points covered by $\mathcal{S}_2 \setminus \mathcal{R}_3$. Thus $\mathcal{P}$ is doubly covered by the lines of $\mathcal{S}_1 \setminus \mathcal{R}_3$ and $\mathcal{S}_2 \setminus \mathcal{R}_3$. Hence, the lines of $\mathcal{R}_3$ and $\mathcal{S}' \setminus \mathcal{S}_2$ doubly cover the set of points of $PG(3, q) \setminus \mathcal{P}$, and so $\mathcal{S}' \setminus \mathcal{S}_2$ comprises the $q+1$ lines of $\mathcal{R}_3^{\text{opp}}$ the opposite regulus of $\mathcal{R}_3$. Let l_3 be a line of $\mathcal{S}_1 \setminus \mathcal{R}_3$. Then l_3 is skew to every line of $\mathcal{R}_3^{\text{opp}}$, and so $\mathcal{R}_3^{\text{opp}}$ and l_3 determine a unique regular spread $\mathcal{S}_3 \subseteq \mathcal{C}_2$ by Lemma 4.2.

Since $\mathcal{R}_3 = \mathcal{S}_1 \cap \mathcal{S}_2$ and $\mathcal{R}_3^{\text{opp}} \subseteq \mathcal{S}_3$, it follows that $\mathcal{S}_3' \subseteq \mathcal{S}_1' \cup \mathcal{S}_2'$ where

$$\mathcal{S}_3' = \mathcal{S}_3 \setminus \mathcal{R}_3^{\text{opp}},$$

$$\mathcal{S}_1' = \mathcal{S}_1 \setminus \mathcal{R}_3,$$

$$\mathcal{S}_2' = \mathcal{S}_2 \setminus \mathcal{R}_3.$$

From this set inclusion and the fact that $\mathcal{S}_1'$ and $\mathcal{S}_2'$ are disjoint, it is immediate that

$$q^2 - q = |\mathcal{S}_3' \cap \mathcal{S}_1'| + |\mathcal{S}_3' \cap \mathcal{S}_2'|.$$

Therefore, at least one of $|\mathcal{S}_3' \cap \mathcal{S}_1'|$ and $|\mathcal{S}_3' \cap \mathcal{S}_2'|$ is $\geqslant (q^2 - q)/2$. Now since $q \geqslant 4$,

$$\begin{aligned} & (q-4)(q+1) \geqslant 0 \\ \Rightarrow & (q^2 - 3q - 4) \geqslant 0 \\ \Rightarrow & (q^2 - q)/2 \quad \geqslant q + 2 . \end{aligned}$$

Thus, at least one of $|\mathcal{S}_3' \cap \mathcal{S}_1'|$ and $|\mathcal{S}_3' \cap \mathcal{S}_2'|$ is $\geqslant q+2$. It follows that one of $|\mathcal{S}_3 \cap \mathcal{S}_1|$ and $|\mathcal{S}_3 \cap \mathcal{S}_2|$ is $\geqslant q+2$. Therefore, since $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are all regular, either $\mathcal{S}_3 = \mathcal{S}_1$ or $\mathcal{S}_3 = \mathcal{S}_2$. However, by construction, $\mathcal{S}_3$ does not contain $\mathcal{R}_3$, while $\mathcal{S}_1$ and $\mathcal{S}_2$ both contain $\mathcal{R}_3$, and so $\mathcal{S}_3 \neq \mathcal{S}_1$ or $\mathcal{S}_2$, a contradiction.

Therefore $\mathcal{C}_2$ is not regular.

While the argument employed in Theorem 4.4 is not valid for $q = 3$, it is noted however, that the above result does still hold when $q = 3$. This is established in:

THEOREM 4.5. – *Let $\mathcal{C}_2$ be a 2-cover of $PG(3, 3)$. Then $\mathcal{C}_2$ is not regular.*

PROOF. – Assume $\mathcal{C}_2$ is a regular 2-cover of $PG(3, 3)$.

By the comment after Definition 4.1, $\mathcal{C}_2$ contains a regulus and a transversal to the regulus. At most 13 lines of $\mathcal{C}_2$ meet the lines of the regulus, and so at least 3 lines of $\mathcal{C}_2$ are skew to every line of the regulus.

Therefore, by Lemma 4.2, $\mathcal{C}_2$ contains a regular spread $\mathcal{S}$. Let m be an arbitrary line of $\mathcal{C}_2 \setminus \mathcal{S}$.

Now, by Theorem 4.4 in [9], all regular spreads of $PG(3, q)$ are projectively equivalent, and by Theorem 4.5 of [9] the collineation group fixing a regular spread of $PG(3, q)$ is transitive on the ordered quadruples

$$\{l_1, l_2, l_3; m\}$$

where l_1, l_2 and l_3 are three pairwise skew lines of $PG(3, q)$ and m is a transversal of the three lines. Hence, $\mathcal{S}$ and m can be mapped by a collineation of $PG(3, 3)$ to the regular spread and line

$$\begin{aligned} \mathcal{S} = & \{0, 10, 20, 30\} \quad \{3, 13, 23, 33\} \quad \{6, 16, 26, 36\} \quad \{9, 19, 29, 39\} \\ & \{1, 11, 21, 31\} \quad \{4, 14, 24, 34\} \quad \{7, 17, 27, 37\} \\ & \{2, 12, 22, 32\} \quad \{5, 15, 25, 35\} \quad \{8, 18, 28, 38\} \end{aligned}$$

and

$$m = \{38, 0, 16, 23\},$$

where the regular spread $\mathcal{S}$ is the short Singer line orbit (iv) in Appendix A for $q = 3$ and m is a line arbitrarily chosen from one of the three long Singer orbits. To complete the proof, it is shown by direct construction that this configuration of lines cannot lie in a regular 2-cover.

Consider the three pairwise skew lines $\{1, 11, 21, 31\}$, $\{7, 17, 27, 37\}$ and $\{0, 16, 23, 38\}$. Since $\mathcal{C}_2$ is assumed to be regular, the fourth line of the regulus determined by these three lines, namely $\{28, 10, 12, 35\}$, also lies in $\mathcal{C}_2$ (to verify that this line completes the regulus, note that two of the transversals of these four lines are $\{0, 1, 28, 37\}$ and $\{11, 7, 10, 38\}$).

Similarly, the fourth line of the regulus determined by the lines $\{2, 12, 22, 32\}$, $\{4, 14, 24, 34\}$ and $\{0, 16, 23, 38\}$, namely $\{28, 33, 39, 7\}$, also lies in $\mathcal{C}_2$ (to verify that this line completes the regulus, note that two of the transversals of these four lines are $\{2, 23, 28, 34\}$ and $\{12, 4, 33, 38\}$).

Hence, $\mathcal{C}_2$ possesses three distinct lines $\{8, 18, 28, 38\}, \{28, 10, 12, 35\}$ and $\{28, 33, 39, 7\}$ all of which pass through point 28. This contradicts the fact that $\mathcal{C}_2$ is a 2-cover. Therefore $PG(3, 3)$ possesses no regular 2-covers.

LEMMA 4.3. – *Let $\mathcal{C}_n$ be an n-cover of $PG(3, q)$ with $q \geq n + 2 \geq 5$. Then there exist distinct points P_1, P_2 and P_3 and distinct lines of $\mathcal{C}_n$ l, l_{11}, l_{12}, l_{21} and l_{3j} for $j = 1, \ldots, n-1$ such that*

(a) $l_{11} \cap l = \{P_1\} = l_{12} \cap l$

(b) $l_{21} \cap l = \{P_2\}$

(c) $l_{3j} \cap l = \{P_3\}$, $j = 1, \ldots, n-1$

(d) $l_{ij} \cap l_{km} = \phi$ for all $i \neq k$.

PROOF. – Let l be an arbitrary line of $\mathcal{C}_n$ and P_3 be an arbitrary point on l. Denote the $n-1$ lines of $\mathcal{C}_n \backslash \{l\}$ through P_3 by l_{3j} for $j = 1, \ldots, n-1$.

Let the $q+1$ planes of $PG(3, q)$ through l be labelled π_k for $k = 1, \ldots, q+1$. Without loss of generality, it can be assumed that the totality of lines l_{3j}, $j = 1, \ldots, n-1$ lies in the planes π_k for $k = 1, \ldots, n-1$.

Since $q \geq n+2$, it follows that there exist at least $(q+1) - (n-1) \geq (n+3) - (n-1) = 4$ planes through l which contain none of the lines l_{3j}, $j = 1, \ldots, n-1$. Each of these $q - n + 2$ planes contains exactly $n-1$ lines of $\mathcal{C}_n \backslash \{l\}$ and these meet l in at most $n-1$ distinct points. Note that none of these points is P_3 because the lines l and l_{3j}, $j = 1, \ldots, n-1$ account for all the lines of $\mathcal{C}_n$ through P_3.

If the points so-determined across all the $q - n + 2$ planes are distinct, then the total number of these points is less than or equal to the number of points on $l \backslash \{P_3\}$, that is

$$(q-n+2)(n-1) \leqslant q \, .$$

However, since $q \geqslant n+2$, it follows that

$$\begin{aligned} & q > n-1 \\ \Rightarrow\ & q(n-2) > (n-1)(n-2) \\ \Rightarrow\ & q(n-1) - (n-1)(n-2) > q \\ \Rightarrow\ & (q-n+2)(n-1) > q \, . \end{aligned}$$

Hence, at least two of these points are identical. Let one such point be P_1 and label two of the lines which determined P_1 as l_{11} and l_{12}.

Now l_{11} and l_{12} possibly lie in distinct planes through l (none of which is π_k, $k = 1, \ldots, n-1$). From the previous observation that there are at least four planes through l which contain none of the lines l_{3j}, $j = 1, \ldots, n-1$, there are at least two other planes through l which contain none of the lines l_{11}, l_{12}, l_{3j}, $j = 1, \ldots, n-1$.

Let one of the remaining two planes be π. Since l_{11}, l_{12} and l account for three of the lines through P_1, the plane π can have at most $n-3$ lines of $\mathcal{C}_n$ through P_1. Hence, at least one line of $\mathcal{C}_n \backslash \{l\}$ in π meets l in a point other than P_1. Choose such a line as l_{21} and label the point $l_{21} \cap l$ as P_2.

By construction $P_2 \neq P_1$ and $P_2 \neq P_3$ since l and l_{3j}, $j = 1, \ldots, n-1$ account for all the lines of $\mathcal{C}_n$ through P_3 (note that it has already been demonstrated that $P_1 \neq P_3$). Furthermore, for any $i, k \in \{1, 2, 3\}$ with $i \neq k$, the lines l_{ij} and l_{km} lie in distinct planes intersecting in the line l. As $l_{ij} \cap l = \{P_i\}$, $l_{km} \cap l = \{P_k\}$ and by construction $P_i \neq P_k$, it follows that these lines are skew.

THEOREM 4.10. – *Let $\mathcal{C}_n$ be an n-cover of $PG(3, q)$ with $q \geqslant n+2 \geqslant 5$. Then $\mathcal{C}_n$ is not regular.*

PROOF. – Assume $\mathcal{C}_n$ is regular.

Consider the labelled configuration of points and lines as in Lemma 4.3 and define the n reguli $\mathcal{R}_i = \mathcal{R}(l_{11}, l_{21}, l_{3i})$ for $i = 1, \ldots, n-1$ and $\mathcal{R}_n = \mathcal{R}(l_{12}, l_{21}, l_{31})$.

Since $\mathcal{R}_i \cap \mathcal{R}_j = \{l_{11}, l_{21}\}$ for each pair $i, j < n$, $i \neq j$, it follows that the set of lines

$$S_1 = \{l\} \bigcup_{i=1}^{n-1} \{\mathcal{R}_i \backslash \{l_{11}, l_{21}\}\}$$

covers the set of points of $l \backslash \{P_1, P_2, P_3\}$ n times. Likewise, the set of lines

$$S_2 = \{l\} \bigcup_{i=1}^{n-1} \{l_{ij} \mid i = 1, \ldots, 3, j = 1, \ldots, n-1\}$$

covers the set of points $\{P_1, P_2, P_3\}$ n times (where l_{ij} is the jth line of $\mathcal{C}_n \backslash \{l\}$ through point P_i).

As a consequence, any other regulus in $\mathcal{C}_n$ with l as a transversal must lie in $S_1 \cup S_2$. In particular, this holds for $\mathcal{R}_n$. Now, besides l_{12}, l_{21} and l_{31}, $\mathcal{R}_n$ has at least $q - 2 \geqslant n$ other lines which by necessity lie in S_1. Choose n of these lines. By the pigeon-hole principle, at least two of these lines must lie in the same set $\mathcal{R}_k \backslash \{l_{11}, l_{12}\}$ for some k, since there are $n-1$ such truncated reguli.

In addition, $\mathcal{R}_k \cap \mathcal{R}_n \supseteq \{l_{21}\}$. Hence, $|\mathcal{R}_k \cap \mathcal{R}_n| \geqslant 3$ and so $\mathcal{R}_k = \mathcal{R}_n$.

However, $l_{12} \in \mathcal{R}_n$ while $l_{12} \notin \mathcal{R}_k$ and so $\mathcal{R}_k \neq \mathcal{R}_n$, giving a contradiction. Therefore, $\mathcal{C}_n$ is not regular.

5. – Conclusion.

The main properties and known examples of n-covers of $PG(3, q)$ have been reviewed. Moreover, it has been shown how n-covers may be employed in the construction of quasi-n-multiple Sperner designs and the indecomposability of these designs has been shown to be related to the spectra of the n-covers used in the construction. In the simplest case, the designs are indecomposable if and only if the n-cover is proper.

Through consideration of the problem of deriving new examples of quasi-n-multiple Sperner designs from existing ones constructed via Construction 3.1, the notion of a regular n-cover has also been introduced and several non-existence results for regular n-covers have been established. For $q = 2$, it is trivial to show that all n-covers are regular. For $q > 2$, it is conjectured that the only regular n-covers of $PG(3, q)$ (up to projective equivalence) are the regular spread, the general linear complex and the line set of $PG(3, q)$. However, this remains an open problem.

A. – Singer's theorem.

The material in section 4 of the paper makes use of Singer's theorem [46] which exploits an algebraic correspondence between the representation of elements in the finite field $GF(q^{n+1})$ and the points of $PG(n, q)$ to establish that $PG(n, q)$ admits a collineation group which cyclically permutes its pointset. The action of this collineation group on the lineset of $PG(n, q)$ depends on the dimension n, but in the case of interest, namely $n = 3$, the lines of $PG(3, q)$

fall into q orbits of size $(q+1)(q^2+1)$ and a single orbit of size q^2+1 (consisting of the lines of a regular spread) [18]. These orbits are referred to as *long* and *short Singer line orbits*. In the remainder of the appendix, Singer's construction is described and the line orbits are listed for $q=2$ and 3.

A.1. *Construction of Singer's Cyclic Collineation of* $PG(3, q)$.

Let f be a primitive monic polynomial of degree $n+1$ over $GF(q)$ such that

$$f(x) = x^{n+1} - a_n x^n - \dots - a_2 x^2 - a_1 x - a_0$$

and let β be a zero of f. Then β can be adjoined to $GF(q)$ to construct the extension field $GF(q^{n+1}) = GF(q)(\beta)$. In this case the set $\{1, \beta, \dots, \beta^n\}$ forms a basis for $GF(q^{n+1})$ over $GF(q)$.

In order to express a non-zero element of $GF(q^{n+1})$ as a linear combination of the basis elements, the fact that $f(\beta)=0$ is used, i.e.

$$\beta^{n+1} = a_n \beta^n + \dots + a_1 \beta + a_0.$$

Two elements β^i, β^j of $GF^*(q^{n+1}) = \langle \beta \rangle$ are then said to be *similar* if and only if

$$\beta^i \cdot \beta^{-j} = \beta^{i-j} \in GF^*(q) = \{1, \beta^v, \beta^{2v}, \dots, \beta^{(q-2)v}\}$$

where $v = (q^{n+1}-1)/(q-1)$. It is evident that the property of similarity is an equivalence relation.

Now since each element of $GF(q^{n+1})$ is uniquely expressible as a linear combination of $1, \beta, \dots, \beta^n$ over $GF(q)$, a coordinate vector $(x_0, x_1, \dots, x_n)$ can be associated with each element

$$\beta^j = \sum_{i=0}^{n} x_i \beta^i.$$

It then follows that two elements β^i, β^j of $GF^*(q^{n+1})$ are similar if and only if their corresponding coordinate vectors are scalar multiples of each other where the scalar belongs to $GF^*(q)$.

Consequently the correspondence extends to one between $GF^*(q^{n+1})$ and $PG(n, q)$ in the following manner:

The points of $PG(n, q)$	$\leftrightarrow$	The equivalence classes determined by the similarity relation on the elements of $GF^*(q^{n+1})$.

The lines of $PG(n, q)$	$\leftrightarrow$	The sets of the form $\{\lambda_1\beta^i+\lambda_2\beta^j \mid \lambda_1, \lambda_2 \in GF(q), \lambda_1, \lambda_2$ not both $0\}$ where β^i and β^j are representatives from distinct equivalence classes.
The incidence in $PG(n,q)$	$\leftrightarrow$	Set inclusion.

Consider the mapping ϕ where

$$\begin{array}{rccc}\phi: & GF^*(q^{n+1}) & \to & GF^*(q^{n+1}) \\ & \beta^i & \mapsto & \beta^{i+1}.\end{array}$$

Then the elements

$$\begin{array}{ll}\beta^i & \\ \phi(\beta^i) & = \beta^{i+1} \\ \phi^2(\beta^i) & = \beta^{i+2} \\ \quad\vdots & \\ \phi^{v-1}(\beta^i) & = \beta^{i+v-1}\end{array}$$

are pairwise dissimilar, while β^i and $\phi^v(\beta^i)=\beta^{i+v}$ are similar. Hence ϕ permutes the equivalence classes cyclically in cycles of length $v=(q^{n+1}-1)/(q-1)$. Clearly ϕ induces a collineation Φ of $PG(n, q)$ because it permutes the point set of $PG(n, q)$ and also maps the line $\langle\beta^i, \beta^j\rangle$ to the line $\langle\beta^{i+1}, \beta^{j+1}\rangle$.

The cyclic collineation group $\langle\Phi\rangle$ (the Singer group) therefore cyclically permutes the point-set of $PG(n, q)$ as required.

Note: the collineation Φ corresponds to the homography with matrix

$$\begin{bmatrix} 0 & & & & & a_0 \\ 1 & & & & & a_1 \\ & 1 & & & & a_2 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ & & & & 1 & a_n \end{bmatrix}.$$

(This can be shown by expanding $\phi(\beta^i)=\beta^{i+1}$ as $\beta.\ \beta^i=\beta(x_0+x_1\beta+x_2\beta^2+\ldots+x_n\beta^n)$ and replacing β^{n+1} by $a_0+a_1\beta+\ldots+a_n\beta^n$). In addition, where it doesn't lead to confusion, $\langle\beta\rangle$ is often simply written in place of $\langle\Phi\rangle$.

A.2. *Singer orbits of* $PG(3, q)$.

In the sequel, the orbits of the points and lines of $PG(3, q)$ under the action of a Singer group are listed for the cases $q = 2$ and $q = 3$.

q=2: Let β be a primitive element of $GF(2^4)$ satisfying $\beta^4 = \beta + 1$. Then the powers of β represent the points of $PG(3, 2)$ as listed below (β^i is represented by its exponent i):

1	$(0, 1, 0, 0)$	6	$(0, 0, 1, 1)$	11	$(0, 1, 1, 1)$
2	$(0, 0, 1, 0)$	7	$(1, 1, 0, 1)$	12	$(1, 1, 1, 1)$
3	$(0, 0, 0, 1)$	8	$(1, 0, 1, 0)$	13	$(1, 0, 1, 1)$
4	$(1, 1, 0, 0)$	9	$(0, 1, 0, 1)$	14	$(1, 0, 0, 1)$
5	$(0, 1, 1, 0)$	10	$(1, 1, 1, 0)$	$15 \equiv 0$	$(1, 0, 0, 0)$.

By observation the point triples $\{0, 5, 10\}$, $\{0, 1, 4\}$ and $\{0, 2, 8\}$ are lines of $PG(3, 2)$. Under the action of $\langle\beta\rangle$, the lines generate the three line orbits:

$$
\begin{array}{ll}
\text{(i)} & \{\{0+i, 1+i, 4+i\} \mid i = 0, \ldots, 14\} \\
\text{(ii)} & \{\{0+i, 2+i, 8+i\} \mid i = 0, \ldots, 14\} \\
\text{(iii)} & \{\{0+i, 5+i, 10+i\} \mid i = 0, \ldots, 4\}\,.
\end{array}
$$

These orbits are listed below in full. (Note: In contrast to the list of points above, the numbers adjacent to each line are there only for the purpose of indexing the lines and do not correspond to a power of β.)

(i)	1	$\{0, 1, 4\}$	6	$\{5, 6, 9\}$	11	$\{10, 11, 14\}$
	2	$\{1, 2, 5\}$	7	$\{6, 7, 10\}$	12	$\{11, 12, 0\}$
	3	$\{2, 3, 6\}$	8	$\{7, 8, 11\}$	13	$\{12, 13, 1\}$
	4	$\{3, 4, 7\}$	9	$\{8, 9, 12\}$	14	$\{13, 14, 2\}$
	5	$\{4, 5, 8\}$	10	$\{9, 10, 13\}$	15	$\{14, 0, 3\}$
(ii)	16	$\{0, 2, 8\}$	21	$\{5, 7, 13\}$	26	$\{10, 12, 3\}$
	17	$\{1, 3, 9\}$	22	$\{6, 8, 14\}$	27	$\{11, 13, 4\}$
	18	$\{2, 4, 10\}$	23	$\{7, 9, 0\}$	28	$\{12, 14, 5\}$
	19	$\{3, 5, 11\}$	24	$\{8, 10, 1\}$	29	$\{13, 0, 6\}$
	20	$\{4, 6, 12\}$	25	$\{9, 11, 2\}$	30	$\{14, 1, 7\}$
(iii)	31	$\{0, 5, 10\}$				
	32	$\{1, 6, 11\}$				
	33	$\{2, 7, 12\}$				
	34	$\{3, 8, 13\}$				
	35	$\{4, 9, 14\}$				

Orbit (iii) constitutes a regular spread $\mathcal{S}$ of $PG(3, 2)$.

q=3: Let β be a primitive element of $GF(3^4)$ satisfying $\beta^4 = -\beta^3 + 1$. Then the powers of β represent the points of $PG(3, 3)$ as listed below (as for the case $q = 2$, β^i is represented by i):

1 $(0,1,0,0)$	11 $(1,1,0,1)$	21 $(1,-1,0,1)$	31 $(1,0,0,1)$
2 $(0,0,1,0)$	12 $(1,1,1,-1)$	22 $(1,1,-1,-1)$	32 $(1,1,0,-1)$
3 $(0,0,0,1)$	13 $(1,-1,-1,1)$	23 $(1,-1,-1,0)$	33 $(1,-1,-1,-1)$
4 $(1,0,0,-1)$	14 $(1,1,-1,1)$	24 $(0,1,-1,-1)$	34 $(1,-1,1,0)$
5 $(1,-1,0,-1)$	15 $(1,1,1,1)$	25 $(1,0,-1,0)$	35 $(0,1,-1,1)$
6 $(1,-1,1,-1)$	16 $(1,1,1,0)$	26 $(0,1,0,-1)$	36 $(1,0,1,1)$
7 $(1,-1,1,1)$	17 $(0,1,1,1)$	27 $(1,0,-1,-1)$	37 $(1,1,0,0)$
8 $(1,1,-1,0)$	18 $(1,0,1,0)$	28 $(1,-1,0,0)$	38 $(0,1,1,0)$
9 $(0,1,1,-1)$	19 $(0,1,0,1)$	29 $(0,1,-1,0)$	39 $(0,0,1,1)$
10 $(1,0,-1,1)$	20 $(1,0,1,-1)$	30 $(0,0,1,-1)$	$40 \equiv 0$ $(1,0,0,0)$

The following four sets of points represent lines of $PG(3, 3)$: $\{0, 10, 20, 30\}$, $\{0, 3, 4, 31\}$, $\{0, 2, 18, 25\}$ and $\{0, 5, 11, 19\}$. Using them, the four line orbits (under the action of $\langle\beta\rangle$) are:

$$\begin{array}{ll}
\text{(i)} & \{\{0+i, 3+i, 4+i, 31+i\} \mid i = 0, \ldots, 39\} \\
\text{(ii)} & \{\{0+i, 2+i, 18+i, 25+i\} \mid i = 0, \ldots, 39\} \\
\text{(iii)} & \{\{0+i, 5+i, 11+i, 19+i\} \mid i = 0, \ldots, 39\} \\
\text{(iv)} & \{\{0+i, 10+i, 20+i, 30+i\} \mid i = 0, \ldots, 9\}.
\end{array}$$

These orbits are listed below in full. (Note: In contrast to the list of points above, the numbers adjacent to each line are there only for the purpose of indexing the lines and do not correspond to a power of β.)

(i)	1 $\{0,3,4,31\}$	11 $\{10,13,14,1\}$	21 $\{20,23,24,11\}$	31 $\{30,33,34,21\}$
	2 $\{1,4,5,32\}$	12 $\{11,14,15,2\}$	22 $\{21,24,25,12\}$	32 $\{31,34,35,22\}$
	3 $\{2,5,6,33\}$	13 $\{12,15,16,3\}$	23 $\{22,25,26,13\}$	33 $\{32,35,36,23\}$
	4 $\{3,6,7,34\}$	14 $\{13,16,17,4\}$	24 $\{23,26,27,14\}$	34 $\{33,36,37,24\}$
	5 $\{4,7,8,35\}$	15 $\{14,17,18,5\}$	25 $\{24,27,28,15\}$	35 $\{34,37,38,25\}$
	6 $\{5,8,9,36\}$	16 $\{15,18,19,6\}$	26 $\{25,28,29,16\}$	36 $\{35,38,39,26\}$
	7 $\{6,9,10,37\}$	17 $\{16,19,20,7\}$	27 $\{26,29,30,17\}$	37 $\{36,39,0,27\}$
	8 $\{7,10,11,38\}$	18 $\{17,20,21,8\}$	28 $\{27,30,31,18\}$	38 $\{37,0,1,28\}$
	9 $\{8,11,12,39\}$	19 $\{18,21,22,9\}$	29 $\{28,31,32,19\}$	39 $\{38,1,2,29\}$
	10 $\{9,12,13,0\}$	20 $\{19,22,23,10\}$	30 $\{29,32,33,20\}$	40 $\{39,2,3,30\}$
(ii)	41 $\{0,2,18,25\}$	51 $\{10,12,28,35\}$	61 $\{20,22,38,5\}$	71 $\{30,32,8,15\}$
	42 $\{1,3,19,26\}$	52 $\{11,13,29,36\}$	62 $\{21,23,39,6\}$	72 $\{31,33,9,16\}$
	43 $\{2,4,20,27\}$	53 $\{12,14,30,37\}$	63 $\{22,24,0,7\}$	73 $\{32,34,10,17\}$

	44	$\{3,5,21,28\}$	54	$\{13,15,31,38\}$	64	$\{23,25,1,8\}$	74	$\{33,35,11,18\}$
	45	$\{4,6,22,29\}$	55	$\{14,16,32,39\}$	65	$\{24,26,2,9\}$	75	$\{34,36,12,19\}$
	46	$\{5,7,23,30\}$	56	$\{15,17,33,0\}$	66	$\{25,27,3,10\}$	76	$\{35,37,13,20\}$
	47	$\{6,8,24,31\}$	57	$\{16,18,34,1\}$	67	$\{26,28,4,11\}$	77	$\{36,38,14,21\}$
	48	$\{7,9,25,32\}$	58	$\{17,19,35,2\}$	68	$\{27,29,5,12\}$	78	$\{37,39,15,22\}$
	49	$\{8,10,26,33\}$	59	$\{18,20,36,3\}$	69	$\{28,30,6,13\}$	79	$\{38,0,16,23\}$
	50	$\{9,11,27,34\}$	60	$\{19,21,37,4\}$	70	$\{29,31,7,14\}$	80	$\{39,1,17,24\}$

(iii)	81	$\{0,5,11,19\}$	91	$\{10,15,21,29\}$	101	$\{20,25,31,39\}$	111	$\{30,35,1,9\}$
	82	$\{1,6,12,20\}$	92	$\{11,16,22,30\}$	102	$\{21,26,32,0\}$	112	$\{31,36,2,10\}$
	83	$\{2,7,13,21\}$	93	$\{12,17,23,31\}$	103	$\{22,27,33,1\}$	113	$\{32,37,3,11\}$
	84	$\{3,8,14,22\}$	94	$\{13,18,24,32\}$	104	$\{23,28,34,2\}$	114	$\{33,38,4,12\}$
	85	$\{4,9,15,23\}$	95	$\{14,19,25,33\}$	105	$\{24,29,35,3\}$	115	$\{34,39,5,13\}$
	86	$\{5,10,16,24\}$	96	$\{15,20,26,34\}$	106	$\{25,30,36,4\}$	116	$\{35,0,6,14\}$
	87	$\{6,11,17,25\}$	97	$\{16,21,27,35\}$	107	$\{26,31,37,5\}$	117	$\{36,1,7,15\}$
	88	$\{7,12,18,26\}$	98	$\{17,22,28,36\}$	108	$\{27,32,38,6\}$	118	$\{37,2,8,16\}$
	89	$\{8,13,19,27\}$	99	$\{18,23,29,37\}$	109	$\{28,33,39,7\}$	119	$\{38,3,9,17\}$
	90	$\{9,14,20,28\}$	100	$\{19,24,30,38\}$	110	$\{29,34,0,8\}$	120	$\{39,4,10,18\}$

(iv)	121	$\{0,10,20,30\}$	124	$\{3,13,23,33\}$	127	$\{6,16,26,36\}$	130	$\{9,19,29,39\}$
	122	$\{1,11,21,31\}$	125	$\{4,14,24,34\}$	128	$\{7,17,27,37\}$		
	123	$\{2,12,22,32\}$	126	$\{5,15,25,35\}$	129	$\{8,18,28,38\}$		

Orbit (iv) consists of the lines of a regular spread of $PG(3, 3)$.

B. – A complete partial packing of $PG(3, 3)$ of Degeneracy 3.

The following linesets and line orbits, based on the action of appropriate subgroups or individual elements of the Singer group on lines of $PG(3, 3)$, are spreads in a complete partial packing of $PG(3, 3)$ of degeneracy 3. The complement of the union of these spreads in the lineset of $PG(3, 3)$ is the proper 3-cover described in section 4.

$$\mathcal{S}_0 = (\{0,\ 10,\ 20,\ 30\})^{\langle\beta\rangle},\ \ (\textit{the short Singer line orbit})$$

$$\mathcal{S}_1 = (\{0,\ 3,\ 4,\ 31\})^{\langle\beta^8\rangle} \cup (\{2,\ 5,\ 6,\ 33\})^{\langle\beta^8\rangle},$$

$$\mathcal{S}_2 = (\{4,\ 7,\ 8,\ 35\})^{\langle\beta^8\rangle} \cup (\{6,\ 9,\ 10,\ 37\})^{\langle\beta^8\rangle},$$

$$\mathcal{S}_3 = \mathcal{S}_1^{\{\beta\}}$$

$$\mathcal{S}_4 = \mathcal{S}_2^{\{\beta\}}$$

$$\mathcal{S}_5 = \begin{array}{lll} \{0, 2, 18, 25\} & \{6, 27, 32, 38\} & \{12, 19, 34, 36\} \\ \{1, 17, 24, 39\} & \{7, 14, 29, 31\} & \{13, 20, 35, 37\} \\ \{3, 5, 21, 28\} & \{8, 10, 26, 33\} & \\ \{4, 9, 15, 23\} & \{11, 16, 22, 30\} & \end{array}$$

$$\mathcal{S}_6 = \mathcal{S}_5^{\{\beta^2\}}$$

$$\mathcal{S}_7 = \mathcal{S}_5^{\{\beta^{20}\}}$$

$$\mathcal{S}_8 = \begin{array}{lll} \{0, 7, 22, 24\} & \{4, 10, 18, 39\} & \{14, 19, 25, 33\} \\ \{1, 6, 12, 20\} & \{5, 26, 31, 37\} & \{16, 21, 27, 35\} \\ \{2, 23, 28, 34\} & \{8, 15, 30, 32\} & \\ \{3, 9, 17, 38\} & \{11, 13, 29, 36\} & \end{array}$$

$$\mathcal{S}_9 = \begin{array}{lll} \{0, 16, 23, 38\} & \{4, 11, 26, 28\} & \{10, 15, 21, 29\} \\ \{1, 22, 27, 33\} & \{5, 13, 34, 39\} & \{12, 14, 30, 37\} \\ \{2, 17, 19, 35\} & \{6, 8, 24, 31\} & \\ \{3, 18, 20, 36\} & \{7, 9, 25, 32\} & \end{array}$$

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Martin Oxenham: Surveillance Systems Division
Defence Science and Technology Organisation, Edinburgh SA 5111
martin.oxenham@dsto.defence.gov.au

Rey Casse: Department of Pure Mathematics
University of Adelaide, Adelaide SA 5005
rcasse@maths.adelaide.edu.au

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