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Remarks on Interpolation of Bilinear Operators by Methods Associated to Polygons.

P. FERNÁNDEZ-MARTÍNEZ (*)

Sunto. – *Studiamo l'interpolazione di operatori bilineari secondo il metodo dei poligoni. Dimostriamo un teorema per operatori che agiscono da due K-spazi su un'altro K-spazio, e proviamo l'ottimalità di alcuni risultati precedenti.*

Summary. – *We study interpolation of bilinear operators by the polygons methods. We prove an interpolation theorem of type $K \times K$ into K spaces, and show the optimality of the precedings results.*

Interpolation of multilinear operators has been studied by different authors and it shows to have a variety of interesting applications in analysis. See the book by Bergh and Löfström [3], or the papers by Lions Peetre [16], Peetre [17], Zafran [19], Favini [13] Janson [15] Astashkin [2], and the more recent work by Cobos, Cordeiro and Martínez [4].

These methods, introduced by Cobos and Peetre in [9], deal with N-tuples of normed spaces and present a strong geometrical component, as we can see in [9] or in [6]. There are others methods interpolating several (more than two) spaces. Among those, the methods described by Sparr in 1974, see [18], and the methods introduced by Fernandez in 1979, see [14]. The polygons methods coincide with Sparr spaces (when the associated polygon is the simplex) and with Fernandez Spaces (when the unit square is the associated polygon), so they can be consider as a link between these other methods.

The above mentioned paper by Cobos, Cordeiro and Martinez is devoted to the study of bilinear interpolation in the context of the polygons methods. Three N-tuples are involved, $\overline{A} = \{A_1, \dots, A_N\}$, $\overline{B} = \{B_1, \dots, B_N\}$, $\overline{E} = \{E_1, \dots, E_N\}$, and they work with bounded linear operators $T : \Sigma(\overline{A}) \times \Sigma(\overline{B}) \to \Sigma(\overline{E})$ whose restrictions $T : A_j \times B_j \to E_j$, $1 \leq, j \leq N$ are bounded. It is

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shown that, under the appropiate hypothesis,

$$T : \overline{A}_{(\alpha,\beta),p;J} \times \overline{B}_{(\alpha,\beta),q;K} \to \overline{E}_{(\alpha,\beta),r;K} \tag{1}$$

$$T : \overline{A}_{(\alpha,\beta),p;J} \times \overline{B}_{(\alpha,\beta),q;J} \to \overline{E}_{(\alpha,\beta),r;J}. \tag{2}$$

Remained to know if the range of the first operator could be reduced to a J-space, or if the domain of the latter operator could be enlarged to a $J \times K$-space. In this note we give counterexamples showing none of these options are possible, and in this sense establishing the optimality of the results in [4].

Despite of the fact that under the usual hypothesis we cannot have a general bilinear interpolation theorem of the type

$$T : \overline{A}_{(\alpha,\beta),p;K} \times \overline{B}_{(\alpha,\beta),q;K} \to \overline{E}_{(\alpha,\beta),r;K}, \tag{3}$$

i.e. we cannot enlarge the domain of the interpolated operator in (1), we establish a theorem of this type under new and necessary conditions in § 2.

1. – Interpolation methods associated to polygons.

Subsequently $\overline{A} = \{A_1, \dots, A_N\}$ will stand for a N-tuple of normed vector spaces continuously embedded in a common Hausdorff topological vector space $\mathcal{U}$. Under these conditions we can consider the sum $\Sigma(\overline{A}) = A_1 + \dots + A_N$ and endowed it with the norm

$$\|a\|_{\Sigma(\overline{A})} = \inf\Big\{\sum_{1 \leq j \leq N} \|a_j\|_{A_j}, \text{ where } a = \sum_{1 \leq j \leq N} a_j\Big\}.$$

Similarly we consider the intersection $\Delta(\overline{A}) = A_1 \cap \dots \cap A_n$ with the norm

$$\|a\|_{\Delta(\overline{A})} = \max_{1 \leq j \leq N}\{\|a\|_{A_j}\}.$$

Assume $\Pi = \overline{P_1, \dots, P_N}$ with vertices $P_j = (x_j, y_j)$. Given any two positive parameters $t, s > 0$, and aided by the polygon, we renorm the space A_j with the norm $t^{x_j} s^{y_j} \|\cdot\|_{A_j}$. Now the norm in the sum is given by the K-functional

$$K(t, s, a; \overline{A}) = \inf\Big\{\sum_{1 \leq j \leq N} t^{x_j} s^{y_j} \|a_j\|_{A_j}, \text{ where } a = \sum_{1 \leq j \leq N} a_j\Big\}.$$

Similarly, the norm in the intersection is given by the J-functional

$$J(t, s, a; \overline{A}) = \max_{1 \leq j \leq N}\{t^{x_j} s^{y_j} \|a\|_{A_j}\}.$$

Let $(\alpha, \beta) \in \operatorname{Int} \Pi$ and $1 \leq p \leq \infty$. We define the space $\overline{A}_{(\alpha,\beta),p;K}$ as the set of all those $a \in \Sigma(\overline{A})$ for which the norm

$$\|a\|_{(\alpha, \beta), p; K} = \left(\sum_{m, n \in \mathbb{Z}} (2^{-\alpha m - \beta n} K(2^m, 2^n, a))^p \right)^{1/p}$$

is finite (usual modifications for $p = \infty$).

We can also define a space by means of the norms on the intersection. Precisely, the space $\overline{A}_{(\alpha, \beta), p; J}$ consists of all those $a \in \Sigma(\overline{A})$ for which there exist representations $a = \sum_{m, n \in \mathbb{Z}} u_{m, n}$, convergence in $\Sigma(\overline{A})$ and $(u_{m, n}) \in \Delta(\overline{A})$, verifying that

$$\|a\|_{(\alpha, \beta), p; J} = \inf_{a = \sum_{m, n \in \mathbb{Z}} u_{m, n}} \left\{ \left(\sum_{m, n \in \mathbb{Z}} (2^{-\alpha m - \beta n} J(2^m, 2^n, u_{m, n}))^p \right)^{1/p} \right\}$$

is finite (usual modifications for $p = \infty$).

$\overline{A}_{(\alpha, \beta), p : K}$ and $\overline{A}_{(\alpha, \beta), p : J}$ are interpolation spaces for the N-tuple $\overline{A}$. The following (strict) inclusions hold

$$\Delta(\overline{A}) \hookrightarrow \overline{A}_{(\alpha, \beta), p : J} \hookrightarrow \overline{A}_{(\alpha, \beta), p : K} \hookrightarrow \Sigma(\overline{A}).$$

For examples and more information about these methods see [5], [6], [7], [9] and [12].

2. – Bilinear interpolation.

We start by fixing some notation. The following hypothesis will be refered as $(\mathcal{H})$.

$(\mathcal{H})$ Let $\Pi = \overline{P_1, \ldots, P_N}$ be a convex polygon, $(\alpha, \beta) \in \operatorname{Int} \Pi$ an interior point. Let $\overline{A} = \{A_1, \ldots, A_N\}$, $\overline{B} = \{B_1, \ldots, B_N\}$ and $\overline{E} = \{E_1, \ldots, E_N\}$ be Banach N-tuples. We shall asssume that $1 \leq p, q, r \leq \infty$ verify $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. $T : \Sigma(\overline{A}) \times \Sigma(\overline{B}) \to \Sigma(\overline{E})$ will stand for a bilinear bounded operator whose restrictions $T : A_j \times B_j \to E_j$, $1 \leq j \leq N$, are bounded.

Cobos, Cordeiro and Martínez proved in [4] that under these conditions the following are bounded bilinear operators

$$T : \overline{A}_{(\alpha, \beta), p; J} \times \overline{B}_{(\alpha, \beta), q; K} \to \overline{E}_{(\alpha, \beta), r; K} \tag{4}$$

$$T : \overline{A}_{(\alpha, \beta), p; J} \times \overline{B}_{(\alpha, \beta), q; J} \to \overline{E}_{(\alpha, \beta), r; J}. \tag{5}$$

They also showed, by means of a counterexample, that we cannot expect a general bilinear interpolation theorem of the type

$$T : \overline{A}_{(\alpha, \beta), p; K} \times \overline{B}_{(\alpha, \beta), q; K} \to \overline{E}_{(\alpha, \beta), r; K}. \tag{6}$$

These results are, in some sense, the best possible. Next counterexample shows that we cannot improve result (4) by reducing the range of the interpolated operator to a J-space (despite of the fact that $\Delta(\overline{B})$ is dense in $\Sigma(\overline{B})$). In other words, we cannot improve (5) by enlarging the domain to a $J \times K$-space.

COUNTEREXAMPLE 2.1. – Let $\Pi = \overline{(0,0),(1,0),(0,1),(1,1)}$ be the unit square, and choose the tuples $\overline{A} = \{l_1, l_1, l_1, l_1\}$, $\overline{B} = \{l_1, c_0, c_0, l_1\}$ and $\overline{E} = \{l_1, l_\infty, l_\infty, l_1\}$. Clearly, $\Sigma(\overline{A}) = l_1$, $\Sigma(\overline{B}) = c_0$ and $\Sigma(\overline{E}) = l_\infty$.

The convolution operator, $\phi(a, b)(n) = (a * b)(n) = \sum_{m \in Z} a(n-m)\, b(m)$, $\phi : l_1 \times c_0 \to l_\infty$, is a bilinear bounded operator, and its restrictions to the spaces of the tuples are

$$\phi : l_1 \times l_1 \to l_1 \tag{7}$$

$$\phi : l_1 \times c_0 \to l_\infty, \tag{8}$$

both bounded. Now it makes sense to interpolate, and so we obtain $\overline{A}_{(\frac{1}{2},\frac{1}{2}),\,1;\,J} = l_1$, $\overline{B}_{(\frac{1}{2},\frac{1}{2}),\,\infty;\,K} = c_0$, and

$$\overline{E}_{(\frac{1}{2},\frac{1}{2}),\,\infty;\,J} = (l_1, l_\infty, l_\infty, l_1)_{(\frac{1}{2},\frac{1}{2}),\,\infty;\,J} \hookrightarrow (l_1, l_\infty, l_\infty, l_1)^{S}_{(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}),\,\infty;\,J} = \tag{9}$$

$$(l_1, l_1, l_\infty, l_\infty)^{S}_{(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}),\,\infty;\,J} = (l_1, l_\infty)_{\frac{1}{2},\,\infty;\,J} = l_{2,\,\infty}.$$

In case that under the hypothesis ($\mathcal{H}$) a theorem of type

$$T : \overline{A}_{(\alpha,\beta),\,p;\,J} \times \overline{A}_{(\alpha,\beta),\,q;\,K} \to \overline{E}_{(\alpha,\beta),\,r;\,J} \tag{10}$$

held, we would have that the operator

$$\phi : l_1 \times c_0 \to \overline{E}_{(\frac{1}{2},\frac{1}{2}),\,\infty;\,J} \hookrightarrow l_{2,\,\infty}$$

is bounded. However it is easy to check that $c_0 \subset \phi(l_1 \times c_0)$ which makes impossible for the range of ϕ to be contained in $l_{2,\,\infty}$.

The following theorem shows that we may only expect general results of type (4)

THEOREM 2.2. – *Let $\overline{E}$ be a Banach N-tuple. Assume that, for all $\overline{A}$, $\overline{B}$ and T satisfying ($\mathcal{H}$), $\overline{E}$ verifies that*

$$T : \overline{A}_{(\alpha,\beta);\,p;\,J} \times \overline{B}_{(\alpha,\beta),\,q;\,K} \to \overline{E}_{(\alpha,\beta),\,r;\,J}$$

is a bounded operator. Then J and K-methods coincide on $\overline{E}$.

PROOF. – The choice $\overline{A} = \{\mathbb{R}, \ldots, \mathbb{R}\}$, $\overline{B} = \overline{E}$ and the operator $T : \mathbb{R} \times \Sigma(\overline{E}) \to \Sigma(\overline{E})$ defined by $T(\lambda, x) = \lambda x$ for some $\lambda \in \mathbb{R}$ verifies ($\mathcal{H}$). So, using the hypothesis, the operator $T : \mathbb{R} \times \overline{B}_{(\alpha,\beta),\,q;\,K} \to \overline{E}_{(\alpha,\beta),\,r;\,J}$ is bounded. In particular, for $p = 1$, $q = r$, and $1 \leqslant q \leqslant \infty$ this shows that

$$\overline{E}_{(\alpha,\beta),\,q;\,K} \hookrightarrow \overline{E}_{(\alpha,\beta),\,q;\,J}. \qquad \blacksquare$$

It must be said, and it is easy to prove by means of the Closed Graph Theorem, that if T maps $\overline{A}_{(\alpha,\beta),\,p;\,J} \times \overline{B}_{(\alpha,\beta),\,q;\,K}$ into $\overline{E}_{(\alpha,\beta),\,r;\,J}$, then

$$T : \overline{A}_{(\alpha,\beta),\,p;\,J} \times \overline{B}_{(\alpha,\beta),\,q;\,K} \to \overline{E}_{(\alpha,\beta),\,r;\,J}$$

is a bilinear bounded operator.

Now we turn to the result of type (6). Cobos, Cordeiro and Martínez showed by means of a counterexample that we cannot have a general result of this type. However, if our polygon is the simplex, and one of the initial tuples, say $\overline{B}$, is a functional lattice tuple (see [7]), then we obtain results of type (6) since J and K-methods coincide on $\overline{B}$, see [1]. This suggests that under additional hypothesis one may have a type (6) bilinear interpolation theorem. In order to have an idea of what type of hypothesis, additional to those of $(\mathcal{H})$, are needed we study the following simple case:

EXAMPLE 2.3. – Let (B_0, B_1) a compatible couple of functional Banach lattices (see [7]), such that $B_0 \cap B_1$ is dense in each B_i, $i = 0, 1$. Let Π be the unit square and choose $\overline{B} = \{B_0, B_1, B_1, B_0\}$. If a bilinear interpolation theorem of type (6) holds for $\overline{B}$ and for all tuples $\overline{A}$ and $\overline{E}$, then Theorem 2.4 shows that

$$\overline{B}_{(\frac{1}{2},\frac{1}{2}),\,1;\,K} = \overline{B}_{(\frac{1}{2},\frac{1}{2}),\,1;\,J}.$$

Use now Example 1.25 of [10] to prove that the norms $\|\cdot\|_{B_0}$ and $\|\cdot\|_{B_1}$ are equivalent on $B_0 \cap B_1 = \Delta(\overline{B})$. Since $\Delta(\overline{B})$ is dense in B_0 and B_1 we have to conclude that B_0 and B_1 are the same space, and so $\overline{B}$ is a degenerated 4-tuple. In particular, the crossed restrictions

$$T : A_i \times B_j \to E_i, \qquad \forall i, j$$

are bounded.

Next result shows that, despite of what happens when we deal with the simplex (Sparr spaces), working with functional Banach lattices does not improve the result, in the following sense: whenever we have a type (6) bilinear interpolation theorem, for all tuples $\overline{A}$ and $\overline{E}$, what we really have is a type (4) bilinear interpolation theorem.

THEOREM 2.4. – *Let Π be a convex polygon and $(\alpha, \beta) \in \operatorname{Int} \Pi$. Assume that $\overline{B}$ is a functional Banach lattice N-tuple, such that $\Delta(\overline{B})$ is dense in B_j, $1 \leqslant j \leqslant N$, and that $1 \leqslant p, r \leqslant \infty$, $1 \leqslant q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If for any tuples $\overline{A}$, $\overline{E}$ such that $T : \Sigma(\overline{A}) \times \Delta(\overline{B}) \to \Sigma(\overline{E})$ is a bilinear bounded operator verifying that the restrictions $T : A_j \times (\Delta(\overline{B}), \|\cdot\|_{B_j}) \to E_j$,*

$1 \leq j \leq N$, *are bounded, T can be extended to a bounded bilinear operator*

$$T : \overline{A}_{(\alpha,\beta),p;K} \times \overline{B}_{(\alpha,\beta),q;K} \to \overline{E}_{(\alpha,\beta),q;K},$$

then $\overline{B}_{(\alpha,\beta),q;K} = \overline{B}_{(\alpha,\beta),q;J}$.

PROOF. – Consider the N-tuple $\overline{B'} = \{B_1', \ldots, B_N'\}$ and recall the equalities $\Sigma(\overline{B'}) = \Delta(\overline{B})' = \Delta(\overline{B})^*$, see [11]. Choose $\overline{E} = \{\mathbb{R}, \ldots, \mathbb{R}\}$ a degenerated N-tuple, and consider the operator defined by $T(\varphi, b) = \langle b, \varphi \rangle$. By hypothesis we can extend T to

$$T : \overline{B'}_{(\alpha,\beta),p;K} \times \overline{B}_{(\alpha,\beta),q;K} \to \overline{E}_{(\alpha,\beta),r;K}$$

bilinear and bounded operator. By choosing $\frac{1}{p} + \frac{1}{q} = 1$, $r = \infty$, the fact that T is bounded shows that

$$\overline{B}'_{(\alpha,\beta),p;K} \hookrightarrow (\overline{B}_{(\alpha,\beta),q;K})'.$$

Now use that $\overline{B}$ is a Banach lattice tuple and the chain of inclusions

$$(\overline{B}_{(\alpha,\beta),q;J})' = \overline{B'}_{(\alpha,\beta),p;K} \hookrightarrow (\overline{B}_{(\alpha,\beta),q;K})' \hookrightarrow (\overline{B}_{(\alpha,\beta),q;J})'$$

to obtain the equality $(\overline{B}_{(\alpha,\beta),q;K})' = (\overline{B}_{(\alpha,\beta),q;J})'$. This shows that the norms $\|\cdot\|_{(\alpha,\beta),q;K}$ and $\|\cdot\|_{(\alpha,\beta),q;J}$ are equivalent on $\Delta(\overline{B})$. Now, since $\Delta(\overline{B})$ is dense in both spaces $\overline{B}_{(\alpha,\beta),q;K}$ and $\overline{B}_{(\alpha,\beta),q;J}$, and the latter are functions spaces, we conclude that

$$\overline{B}_{(\alpha,\beta),q;J} = \overline{B}_{(\alpha,\beta),q;K}$$

with equivalence of norms. ■

Now we prove that under weaker hypothesis than those just mentioned (bounded crossed restrictions) a bilinear interpolation theorem of type (6) holds, see also [15]. We will work with mappings of type (Π) which are affine mappings, associated to the polygon Π, defined as

$$R\begin{pmatrix} u \\ v \end{pmatrix} = Q + U\begin{pmatrix} u \\ v \end{pmatrix},$$

for $(u, v) \in \mathbb{R}^2$. Here $Q \in \mathbb{R}^2$, U is an isomorphism of $\mathbb{R}^2$ and R verifies that for each $1 \leq j \leq N$, $RP_j \in \operatorname{Int} \Pi$ or $RP_j = P_k$, for some $1 \leq k \leq N$. These mappings transform the polygon Π into another convex polygon, $R(\Pi) = \overline{Q_1, \ldots, Q_N}$, contained in Π. Moreover $R(\operatorname{Int} \Pi) = \operatorname{Int} R(\Pi)$, in particular if $(\alpha, \beta) \in \operatorname{Int} \Pi$ then $Q = R(\alpha, \beta) \in \operatorname{Int} R(\Pi)$. Let R be such a mapping, then the reiteration theorem in [5] and the fact that J and K-methods coincide on the tuple $\overline{l_\infty} = (l_\infty(2^{-mx_1 - ny_1}), \ldots, l_\infty(2^{-mx_N - ny_N}))$ yield that

$$\{(\overline{l_\infty})_{Q_1,q_1;K}, \ldots, (\overline{l_\infty})_{Q_N,q_N;K}\}_{(\alpha,\beta),q;K} = (\overline{l_\infty})_{Q,q;K}. \tag{11}$$

Recall the K-method associated to polygons can be described as a maximal interpolation functor. Namely, for any Banach N-tuple $\overline{A}$

$$\overline{A}_{(\alpha,\beta),q;K} = H[\overline{l_\infty}, l_q(2^{-\alpha m-\beta n})](\overline{A}), \tag{12}$$

the maximal Banach space, A, such that A and $l_q(2^{-\alpha m-\beta n})$ are interpolation spaces with respect to $\overline{A}$ and $\overline{l_\infty}$, see [8]. Now, from (11) and (12), we conclude that for any Banach N-tuple $\overline{A}$

$$(\overline{A}_{R(P_1),q_1;K}, \dots, \overline{A}_{R(P_N),q_N;K})_{(\alpha,\beta),q;K} \hookrightarrow \overline{A}_{R(\alpha,\beta),q;K} \tag{13}$$

with norm $\leqslant 1$.

THEOREM 2.5. – *Let Π, $\overline{A}$, $\overline{B}$ and T as in $(\mathcal{H})$. Let R be a mapping of type (Π) and let T verify that for $Q_i = R(P_i)$ and $1 \leqslant i \leqslant n$ the restrictions $T: A_i \times B_j \to \overline{E}_{Q_i,p_i}$ are bounded. Then*

$$T: \overline{A}_{(\alpha,\beta),q;K} \times \overline{B}_{(\alpha,\beta),p;K} \to \overline{E}_{(\alpha,\beta),r;K}$$

for $\dfrac{1}{r} \leqslant \max\left\{\dfrac{1}{p}, \dfrac{1}{q}\right\}$, *and* $(\alpha,\beta) \in R(\mathrm{Int}\,\Pi)$.

The proof follows the ideas of Janson in [15].

PROOF. – For some fixed $b \in B_j$, $1 \leqslant j \leqslant N$, consider the operator $T_b: A_i \to \overline{E}_{Q_i,p_i}$ defined by $T_b(a) = T(a,b)$, linear and bounded with norm $\|T\|_{A_i,\overline{E}_{Q_i,p_i}} \leqslant \|T\|_{i,j}\|b\|_j$. Interpolating with parameters $(\alpha,\beta) \in \mathrm{Int}\,\Pi$, $1 \leqslant q \leqslant \infty$, and using the inclusion of equation (13), we obtain

$$T_b: \overline{A}_{(\alpha,\beta),q;K} \to \overline{E}_{R(\alpha,\beta),q;K}$$

is bounded with norm $\|T_b\| \leqslant \max\limits_{1 \leqslant i,j \leqslant N} \|T\|_{i,j}\|b\|_{B_j}$.

Now fix $a \in \overline{A}_{(\alpha,\beta),q;K}$ and consider the operator $T_a: \Sigma(\overline{B}) \to \overline{E}_{R(\alpha,\beta),q;K}$. T_a is a bounded linear operator whose restrictions to each B_j remains bounded. Hence by interpolating with parameters $(\alpha_1,\beta_1) \in \mathrm{Int}\,\Pi$, $1 \leqslant p \leqslant \infty$ we conclude that

$$\|T_a\|_{\overline{B}_{(\alpha_1,\beta_1),p;K},\overline{E}_{R(\alpha,\beta),q;K}} \leqslant C\|T\|\|a\|_{(\alpha,\beta),q;K}.$$

Now it is easy to show that for each pair $(a,b) \in \overline{A}_{(\alpha,\beta),q;K} \times \overline{B}_{(\alpha_1,\beta_1),p;K}$,

$$\|T(a,b)\|_{\overline{C}_{R(\alpha,\beta),q;K}} \leqslant \|T_a b\|_{\overline{C}_{R(\alpha,\beta),q;K}} \leqslant C\|T\|\|a\|_{(\alpha,\beta),q;K}\|b\|_{(\alpha_1,\beta_1),p;K}$$

which shows that

$$T: \overline{A}_{(\alpha,\beta),q;K} \times \overline{B}_{(\alpha_1,\beta_1),p;K} \to \overline{E}_{R(\alpha,\beta),q;K}.$$

By reversing the order of interpolation and using the inclusion relationship between the interpolated spaces we obtain the statement of the theorem. ■

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