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$L^{2,\lambda}$-Regularity for Minima of Variational Integrals.

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Sunto. – *In questo lavoro si studia la $L^{2,\lambda}$ - regolarità del gradiente dei minimi locali per funzionali non-lineari.*

Summary. – *The $L^{2,\lambda}$-regularity of the gradient of local minima for nonlinear functionals is shown.*

1. – Introduction.

In this paper we shall consider the problem of the regularity of the derivatives of functions minimizing the variational integral

$$F(u;\,\Omega)=\int_\Omega f(x,\,u,\,Du)\,dx\,, \tag{1}$$

where $\Omega\subset\mathbb{R}^n$, $n>2$ is an bounded open set, $x=(x_1,\,\dots,\,x_n)\in\Omega$, $u:\Omega\to\mathbb{R}^N$, $N>1$, $u(x)=(u^1(x),\,\dots,\,u^N(x))$, $Du=\{D_\alpha u^i\}$, $D_\alpha=\partial/\partial x_\alpha$, $\alpha=1,\,\dots,\,n$, $i=1,\,\dots,\,N$ and $f:\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}$ will be stated below. A local minimum for the functional F is a function $u\in W^{1,2}_{\mathrm{loc}}(\Omega,\,\mathbb{R}^N)$ such that for every $\varphi\in W^{1,2}(\Omega,\,\mathbb{R}^N)$ with $supp\,\varphi\subset\subset\Omega$ we have

$$F(u;\,supp\,\varphi)\leqslant F(u+\varphi;\,supp\,\varphi)\,.$$

For more information see [2], [4], [7].

The aim of this paper is to investigate the $L^{2,\lambda}$-regularity (for a definition see below) of the gradient of minima, directly working with the functional F instead of working with its Euler equation. In fact we shall not suppose any differentiability. In the following we shall suppose

$$f(x,\,u,\,Du)=A^{\alpha\beta}_{ij}(x)\,D_\alpha u^i D_\beta u^j+g(x,\,u,\,Du)\,, \tag{2}$$

where $\alpha,\beta=1,\,\dots,\,n$, $i,j=1,\,\dots,\,N$, $A=(A^{\alpha\beta}_{ij})$ is a matrix of functions with $A^{\alpha\beta}_{ij}\in L^\infty(\Omega)\cap\mathcal{L}_\psi(\Omega)$ (for a definition of $\mathcal{L}_\psi(\Omega)$ see the next paragraph) and

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the following condition of strong ellipticity

$$A_{ij}^{\alpha\beta}(x)\,\xi_\alpha^i\,\xi_\beta^j \geq \nu|\xi|^2,\ a.e.\quad x\in\Omega,\ \forall\,\xi\in\mathbb{R}^{nN};\ \nu>0 \tag{3}$$

holds. Here and in the following summation over repeated indices is understood.

About the function $g=g(x,u,Du)$ we suppose that g is Carathéodory function; i.e. measurable in x for each $(u,z)\in\mathbb{R}^N\times\mathbb{R}^{nN}$ and continuous in (u,z) for almost every $x\in\Omega$. Moreover for almost $x\in\Omega$ and all $(u,z)\in\mathbb{R}^N\times\mathbb{R}^{nN}$ the following condition hold

$$|g(x,u,z)|\leq f(x)+L|z|^\gamma, \tag{4}$$

where $f\in L^p(\Omega)$, $2<p\leq\infty$, $f\geq 0$ a.e. on Ω, $L\geq 0$ and $0\leq\gamma<2$.

From these assumptions it follows, that our functionals (1), (2) are, in general, non differentiable and therefore that $u\notin W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$.

The $L^{2,\lambda}$-regularity of the gradient of minima of the functional (1), (2) we have stated in [3] too. In [3] it was supposed that the coefficients $A_{ij}^{\alpha\beta}$ are continuous. In this paper the coefficients $A_{ij}^{\alpha\beta}$ are discontinuous in general. From such point of view the result of this paper may be seen as a generalization of that from [3] and [5]. The below stated result may be seen as one from first steps to proving BMO-regularity of the gradient of minima for class of functionals defined by (1), (2).

If we want discuss our method of proof, we have to say that its crucial points are assumptions on $A_{ij}^{\alpha\beta}$ and higher integrability of gradient Du (see [10], [6]). Using these two facts we obtain $L^{2,\lambda}$ - regularity of the gradient.

2. – Preliminary results and definitions.

In this part we shall formulate some definitions and results needed for proving of main result.

For the sake of simplicity we denote by $|\cdot|$ the norm in $\mathbb{R}^n$ as well as in $\mathbb{R}^N$ and in $\mathbb{R}^{nN}$. If $x\in\mathbb{R}^n$ and r is a positive real number, we set $B(x,r)=\{y\in\mathbb{R}^n\,|y-x|<r\}$, i.e. the open ball in $\mathbb{R}^n$ and $\Omega(x,r)=\Omega\cap B(x,r)$. In our next considerations we shall denote $B(x,r)$ by $B_r(x)$ too. Denote by

$$u_{x,r}=\frac{1}{|\Omega(x,r)|_n}\int_{\Omega(x,r)}u(y)\,dy =: \fint_{\Omega(x,r)}u(y)\,dy$$

the mean value over the set $\Omega(x,r)$ of the function $u\in L^1(\Omega,\mathbb{R}^N)$, where $|\Omega(x,r)|_n$ is the n-dimensional Lebesgue measure of $\Omega(x,r)$. Beside the usually used spaces $C_0^\infty(\Omega,\mathbb{R}^N)$, Hölder spaces $C^{0,\alpha}(\Omega,\mathbb{R}^N)$, spaces $L^p(\Omega,\mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega,\mathbb{R}^N)$, $W^{k,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, $W^{k,p}_0(\Omega,\mathbb{R}^N)$ (for details see e.g. [7], [9], [11]) we shall use the spaces, stated in the following definitions.

DEFINITION 1. – Let $\lambda \in [0, n]$, $q \in [1, \infty)$. A function $u \in L^q(\Omega, \mathbb{R}^N)$ is said to belong to $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ if

$$\|u\|^q_{L^{q,\lambda}(\Omega, \mathbb{R}^N)} = \sup\left\{\frac{1}{r^\lambda}\int_{\Omega(x,r)} |u(y)|^q dy \;\; x \in \Omega,\, r > 0\right\} < \infty .$$

For more details see [1], [4], [9], [11].

DEFINITION 2. – A function $u \in L^2(\Omega, \mathbb{R}^N)$ is said to belong to $\mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ if

$$[u]_{\psi,\Omega} := \sup\left\{\frac{1}{\psi(r)}\left(\fint_{\Omega(x,r)} |u(y) - u_{x,r}|^2 dy\right)^{1/2} : x \in \Omega,\, r \in (0, \mathit{diam}\,\Omega]\right\} < \infty$$

and by $l_\psi(\Omega, \mathbb{R}^N)$ we denote subspace of all $u \in \mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ such that

$$[u]_{\psi,\Omega,r_0} := \sup\left\{\frac{1}{\psi(r)}\left(\fint_{\Omega(x,r)} |u(y) - u_{x,r}|^2 dy\right)^{1/2} : x \in \Omega,\, r \in (0, r_0]\right\} = o(1)$$

as $r_0 \searrow 0$ and $\psi(r) = 1/(1 + |\ln r|)$.

REMARK 1. – In full generality the spaces $\mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ may be defined for a function $\psi : [0, d] \to [0, \infty)$, which is continuous, non-decreasing and such that $t \to \psi(t)/t$ is almost decreasing, i.e. there exists a constant $K_\psi \geq 1$ such that $K_\psi\, \psi(t)/t \geq \psi(s)/s$, $0 \leq t < s < d$. The $\mathcal{L}_\psi$ classes, introduced in [12] generalize Campanato's $\mathcal{L}^{p,\lambda}$ spaces [1].

In the next proposition the properties of above mentioned spaces are formulated. For the proofs see [1], [7], [9], [12].

PROPOSITION 1. – *For domain $\Omega \subset \mathbb{R}^n$ of the class $C^{0,1}$* (*i.e. Lipschitz class*) *we have*

(i) $L^{q,n}(\Omega, \mathbb{R}^N)$ *is isomorphic to the* $L^\infty(\Omega, \mathbb{R}^N)$.

(ii) *If* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *and* $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$, $n - 2 < \lambda < n$, *then* $u \in C^{0,(\lambda - n + 2)/2}(\Omega, \mathbb{R}^N)$.

(iii) $\mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ *is a Banach space with norm* $\|u\|_{\mathcal{L}_\psi(\Omega, \mathbb{R}^N)} = \|u\|_{L^2(\Omega, \mathbb{R}^N)} + [u]_{\psi,\Omega}$.

(iv) $C^0(\overline{\Omega}, \mathbb{R}^N) \setminus \mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ *and* $\{L^\infty(\Omega, \mathbb{R}^N) \cap l_\psi(\Omega, \mathbb{R}^N)\} \setminus C^0(\overline{\Omega}, \mathbb{R}^N)$ are not empty.

(v) *For* $p \in [1, \infty)$, $\Omega' \subset\subset \Omega$, $r_0 \in (0, dist(\Omega', \partial\Omega))$ *and* $u \in \mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ *set*

$$N_p(u; \psi, \Omega', r_0) = \sup\left\{ \frac{1}{\psi(r)} \left(\fint_{\Omega(x, r)} |u(y) - u_{x, r}|^p \, dy \right)^{1/p} x \in \Omega', \, r \in (0, r_0] \right\}.$$

Then we have for each $u \in \mathcal{L}_\psi(\Omega, \mathbb{R}^N)$

$$N_1(u; \psi, \Omega', r_0) \leq N_p(u; \psi, \Omega', r_0) \leq c(p, n)[u]_{\psi, \Omega, r_0}.$$

The considerations which will used in the next paragraph we may formulated by

LEMMA 1 (*see* [1]). – *Let* $u \in W^{1,2}(B(x_0, R), \mathbb{R}^N)$ *be a weak solution to the system*

$$D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = 0$$

with constant coefficients $A_{ij}^{\alpha\beta}$ *and* (3) *be satisfied. Then for each* $t \in [0, 1]$

$$\int_{B(x_0, tR)} |Du|^2 dx \leq ct^n \int_{B(x_0, R)} |Du|^2 dx$$

holds.

LEMMA 2 (*see* [10], [6]). – *Let* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *be a minimum of the functional* (1), (2), *where* $A_{ij}^{\alpha\beta} \in L^\infty(\Omega) \cap \mathcal{L}_\psi(\Omega)$ *and* (3), (4) *be satisfied. Then* $Du \in L^r_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ *for some* $r > 2$ *and there exist constants* $c = c(\nu, L, \|A\|_\infty)$ *and* $\overline{R} > 0$ *such that for all balls* $B(x, R) \subset \Omega$, $R < \overline{R}$

$$\left(\fint_{B(x, R/2)} |Du|^r dy \right)^{1/r} \leq c \left\{ \left(\fint_{B(x, R)} |Du|^2 dy \right)^{1/2} + \left(\fint_{B(x, R)} |f(x)|^{r/2} dy \right)^{2/r} \right\}$$

holds.

REMARK 2. – Here stated Lemma 2 has a form useful for our situation. It is a special case to that from [10].

LEMMA 3 (*see* [7]). – *Let* $u \in W^{1,2}(B(x_0, R), \mathbb{R}^N)$ *be a solution of Dirichlet problem*

$$D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = D_\alpha h_i^\alpha, \quad u = 0 \;\; on \;\; \partial B(x_0, R)$$

with constant coefficients $A_{ij}^{\alpha\beta}$, (3) *be satisfied and* $h_i^\alpha \in L^r(B(x_0, R))$, $r > 2$.

Then $Du \in L^r(B(x_0, R), \mathbb{R}^{nN})$ *and*

$$\|Du\|_r \leq c\|h\|_r$$

hold, where the constant c *has no dependence on* R.

LEMMA 4 (*see* [8]). – *Let* $\Phi = \Phi(R)$, $R \in (0, d]$, $d > 0$ *be a nonnegative function and let* A, B, C, a, b *be nonnegative constants. Suppose that for all* $t \in (0, 1]$ *and all* $R \in (0, d]$

$$\Phi(tR) \leq (At^a + B)\,\Phi(R) + CR^b$$

holds. Further let $K \in (0, 1)$ *be such that* $\varepsilon = AK^{a-b} + BK^{-b} < 1$. *Then*

$$\Phi(R) \leq cR^b, \qquad \forall R \in (0, d],$$

where $c = \max\{C/K(1-\varepsilon), \sup_{R \in [Kd, d]} \Phi(R)/R^b\}$.

3. – Main result.

Now we may state the following

THEOREM. – *Let* $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ *be a minimum of the functional* (1), (2). *Let* $A_{ij}^{\alpha\beta} \in L^\infty(\Omega) \cap \mathcal{L}_\psi(\Omega)$ *and* (3), (4) *be satisfied. Then*

$$Du \in \begin{cases} L^{2,\, n(1-2/p)}_{\mathrm{loc}}(\Omega, R^{nN}) & \text{if } 2 < p < \infty \\ L^{2,\lambda}_{\mathrm{loc}}(\Omega, R^{nN}) \text{ with arbitrary } \lambda < n & \text{if } p = \infty\,. \end{cases}$$

Therefore,

$$u \in \begin{cases} C^{0,\,1-n/p}(\Omega, R^N) & \text{if } n < p \\ C^{0,\vartheta}(\Omega, R^N) \text{ with arbitrary } \vartheta < 1 & \text{if } p = \infty\,. \end{cases}$$

REMARK 3. – The Hölder-continuity of u stated in previous Theorem immediately follows from Proposition 1(ii).

4. – Proof of Theorem.

We shall prove the Theorem only in situation when $2 < p < \infty$. For $p = \infty$ the proof is analogous with some small modification.

Let $B_R(x_0) \subset\subset \Omega$ and v be a minimum of the functional

$$F^0(v; B_{R/2}(x_0)) = \int_{B_{R/2}(x_0)} (A_{ij}^{\alpha\beta})_{x_0, R/2} D_\alpha v^i D_\beta v^j\, dx \tag{5}$$

among all the functions in $W^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$ taking the values u on $\partial B_{R/2}(x_0)$. From Euler equation for v and from Lemma 1 we have

$$\int_{B_{tR/2}(x_0)} |Dv|^2\, dx \leqslant c_1 t^n \int_{B_{R/2}(x_0)} |Dv|^2\, dx, \qquad t \in (0, 1]. \tag{6}$$

Put $w = u - v$. We have $w \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$.

By standard arguments we obtain, using (6),

$$\int_{B_{tR/2}(x_0)} |Du|^2\, dx \leqslant c_2 \left\{ t^n \int_{B_{R/2}(x_0)} |Du|^2\, dx + \int_{B_{R/2}(x_0)} |Dw|^2\, dx \right\}. \tag{7}$$

In the following we shall estimate the last integral on the right hand side of (7).

From [5] (see Lemma 2.1) we have

$$\begin{aligned} \nu \int_{B_{R/2}(x_0)} |Dw|^2\, dx &\leqslant c_3 (F^0(u; B_{R/2}(x_0)) - F^0(v; B_{R/2}(x_0))) = \\ & c_3 \Bigg\{ \int_{B_{R/2}(x_0)} ((A_{ij}^{\alpha\beta})_{x_0, R/2} - A_{ij}^{\alpha\beta}(x))\, D_\alpha u^i D_\beta u^j\, dx + \\ & \int_{B_{R/2}(x_0)} (A_{ij}^{\alpha\beta}(x) - (A_{ij}^{\alpha\beta})_{x_0, R/2})\, D_\alpha v^i D_\beta v^j\, dx + \\ & \int_{B_{R/2}(x_0)} (-g(x, u, Du))\, dx + \int_{B_{R/2}(x_0)} g(x, v, Dv)\, dx + \\ & F(u; B_{R/2}(x_0)) - F(v; B_{R/2}(x_0)) \Bigg\} = \\ & c_3 \{ I + II + III + IV + F(u; B_{R/2}(x_0)) - F(v; B_{R/2}(x_0)) \} \leqslant \\ & c_3 \{ I + II + III + IV \}. \end{aligned} \tag{8}$$

Notice that $F(u; B_{R/2}(x_0)) - F(v; B_{R/2}(x_0)) \leqslant 0$, since u is a minimizer.

Now we shall estimate the terms I, II, III and IV from (8). In the following we shall denote $(A_{ij}^{\alpha\beta}) =: A$ - matrix of coefficients.

$$\begin{aligned} I &\leqslant \int_{B_{R/2}(x_0)} |(A_{ij}^{\alpha\beta})_{x_0, R/2} - A_{ij}^{\alpha\beta}(x)|\, |D_\alpha u^i|\, |D_\beta u^j|\, dx \\ &\leqslant c_4 \int_{B_{R/2}(x_0)} |A_{x_0, R/2} - A(x)|\, |Du|^2\, dx\,. \end{aligned}$$

Taking into account Lemma 2, Hölder inequality ($r' = r/(r-2)$, $r > 2$) and

Proposition 1(v), we obtain

$$I \leqslant c_4 \Big(\int_{B_{R/2}(x_0)} |A(x) - A_{x_0, R/2}|^{r'} dx \Big)^{1/r'} \Big(\int_{B_{R/2}(x_0)} |Du|^r dx \Big)^{2/r} \tag{9}$$

$$\leqslant c_5 \frac{R^{n/r'}}{1 + |\ln R|} \Big\{ (1 + |\ln R|) \Big(\fint_{B_{R/2}(x_0)} |A(x) - A_{x_0, R/2}|^{r'} dx \Big)^{1/r'} \Big\}$$

$$\times \Big(\int_{B_{R/2}(x_0)} |Du|^r dx \Big)^{2/r}$$

$$\leqslant c_5 N_{r'}(A, \psi, B_{R/2}(x_0), R/2) \frac{R^{n/r'}}{1 + |\ln R|} \Big(\int_{B_{R/2}(x_0)} |Du|^r dx \Big)^{2/r}$$

$$\leqslant c_6(n, r, \|A\|_{\mathfrak{L}_\psi(\Omega, \mathbb{R}^{n^2N^2})}) \frac{R^{n/r'}}{1 + |\ln R|} \Big(\int_{B_{R/2}(x_0)} |Du|^r dx \Big)^{2/r}.$$

Using the inequality from Lemma 2 we may estimate the last integral in (9) by the following manner

$$\Big(\int_{B_{R/2}(x_0)} |Du|^r dx \Big)^{2/r} \leqslant c_7 R^{2n/r} \Big\{ \fint_{B_R(x_0)} |Du|^2 dx + \Big(\fint_{B_R(x_0)} |f(x)|^{r/2} dx \Big)^{4/r} \Big\}. \tag{10}$$

Taking into account the fact that $f \in L^p(\Omega)$, $p > 2$ and $r > 2$ may be choose by such a way that $r/2 < p$ (i.e. $2p/r > 1$), we obtain from Hölder inequality

$$\Big(\int_{B_R(x_0)} |f(x)|^{r/2} dx \Big)^{4/r} \leqslant c_8(n, \|f\|_{L^p(\Omega)}) R^{2n(2p - r)/pr}. \tag{11}$$

Now from (9), (10), (11) we have

$$I \leqslant c_9 \frac{1}{1 + |\ln R|} \int_{B_R(x_0)} |Du|^2 dx + c_{10} R^{n(1 - 2/p)}, \tag{12}$$

where the constants c_9 and c_{10} depend only on parameters of the functional (1), (2).

By the same way as in estimating of I we have

$$II \leqslant \int_{B_{R/2}(x_0)} |A(x) - A_{x_0, R/2}| |Dv|^2 dx. \tag{13}$$

From Euler equation for v and from the fact that $v = u$ on $\partial B_{R/2}(x_0)$ we have for w:

$$D_\alpha((A_{ij}^{\alpha\beta})_{x_0,\,R/2} D_\beta w^j) = D_\alpha((A_{ij}^{\alpha\beta})_{x_0,\,R/2} D_\beta u^j) \quad\text{and}\quad w=0 \quad\text{on}\quad \partial B_{R/2}(x_0).$$

Lemma 3 implies that $Dw \in L^r(B_{R/2}(x_0))$, $r>2$ and because $v = u - w$, we have $v \in L^r(B_{R/2}(x_0))$ and from the inequality in Lemma 3 we obtain

$$\left(\int_{B_{R/2}(x_0)} |Dv|^r\,dx\right)^{1/r} \leqslant c_{11}\left(\int_{B_{R/2}(x_0)} |Du|^r\,dx\right)^{1/r}, \tag{14}$$

where c_{11} not depends on R.

Now (13), (14) using Hölder inequality ($r>2$, $r' = r/(r-2)$) give

$$II \leqslant c_{12}\left(\int_{B_{R/2}(x_0)} |A(x) - A_{x_0,\,R/2}|^{r'}\,dx\right)^{1/r'} \left(\int_{B_{R/2}(x_0)} |Du|^r\,dx\right)^{2/r}.$$

From the last estimate by way analogous to that in estimating of I we obtain

$$II \leqslant c_{13}\frac{1}{1+|\ln R|}\int_{B_R(x_0)} |Du|^2\,dx + c_{14}R^{n(1-2/p)}. \tag{15}$$

From (4) and Young inequality (we may suppose $R<1$) we have

$$\begin{aligned} III &\leqslant \int_{B_{R/2}(x_0)} |g(x,u,Du)|\,dx \leqslant c_{15}(\|f\|_{L^p(\Omega)})\,R^{n(1-1/p)} + L\int_{B_{R/2}(x_0)} |Du|^\gamma\,dx \\ &\leqslant c_{15}(\|f\|_{L^p(\Omega)})R^{n(1-1/p)} + \frac{L\gamma}{2}\varepsilon\int_{B_{R/2}(x_0)} |Du|^2\,dx + c_{16}(n,\varepsilon,\gamma,L)\,R^n \\ &\leqslant \frac{\gamma L}{2}\varepsilon\int_{B_{R/2}(x_0)} |Du|^2\,dx + c_{17}R^{n(1-1/p)}, \qquad \forall\,\varepsilon>0. \end{aligned} \tag{16}$$

By the analogous way as in obtaining of (16) we have:

$$\begin{aligned} IV &\leqslant c_{15}R^{n(1-1/p)} + L\int_{B_{R/2}(x_0)} |Dv|^\gamma\,dx \\ &\leqslant c_{15}R^{n(1-1/p)} + L\gamma\varepsilon\left(\int_{B_{R/2}(x_0)} |Du|^2\,dx + \int_{B_{R/2}(x_0)} |Dw|^2\,dx\right) \\ &\quad + c_{16}(n,\varepsilon,\gamma,L)R^n \\ &\leqslant L\gamma\varepsilon\left(\int_{B_{R/2}(x_0)} |Du|^2\,dx + \int_{B_{R/2}(x_0)} |Dw|^2\,dx\right) + c_{17}R^{n(1-1/p)} \end{aligned} \tag{17}$$

for each $\varepsilon > 0$. The estimates (8), (12), (15), (16), (17) imply

$$(18)\qquad \nu \int_{B_{R/2}(x_0)} |Dw|^2\,dx \leq c_{18}\left(\frac{1}{1+|\ln R|}\int_{B_R(x_0)} |Du|^2\,dx + \varepsilon \int_{B_{R/2}(x_0)} |Du|^2\,dx\right)$$

$$+ c_{19}(R^{n(1-2/p)} + R^{n(1-1/p)}) + c_3\gamma L\varepsilon \int_{B_{R/2}(x_0)} |Dw|^2\,dx.$$

Now we can choose $\varepsilon_0 > 0$, $R_0 = \min\{1, \overline{R}\}$, ($\overline{R}$ is from Lemma 2) such that $\nu - c_3\gamma L\varepsilon > 0$ for all $\varepsilon < \varepsilon_0$. And thus we have for all $R < R_0$, $\varepsilon < \varepsilon_0$ the following estimate

$$(19)\qquad \int_{B_{R/2}(x_0)} |Dw|^2\,dx \leq c_{20}\left(\frac{1}{1+|\ln R|} + \varepsilon\right)\int_{B_R(x_0)} |Du|^2\,dx + c_{21}R^{\lambda},$$

where $\lambda = n(1-2/p)$.

Now from (7) and (19) we obtain for $t \in [0, 1]$

$$(20)\qquad \int_{B_{tR/2}(x_0)} |Du|^2\,dx \leq c_{22}\left(t^n + \varepsilon + \frac{1}{1+|\ln R|}\right)\int_{B_R(x_0)} |Du|^2\,dx + c_{23}R^{\lambda}.$$

For $t \in [1, 2]$ the above inequality is trivial and we have for each $t \in [0, 1]$

$$(21)\qquad \int_{B_{tR}(x_0)} |Du|^2\,dx \leq c_{24}\left(t^n + \varepsilon + \frac{1}{1+|\ln R|}\right)\int_{B_R(x_0)} |Du|^2\,dx + c_{25}R^{\lambda},$$

where the constants c_{24} and c_{25} depend only on above mentioned parameters.

Now from Lemma 4 we get the result by the following manner. If we put $\Phi(R) = \int_{B_R(x_0)} |Du|^2\,dx$, $a = n$, $b = \lambda$, $A = c_{24}$, $B = c_{24}(\varepsilon + 1/(1+|\ln R|))$ and $C = c_{25}$ we can choose $0 < K < 1$ such that $AK^{n-\lambda} < 1/2$.

It is obvious that the constants $\varepsilon_0 > 0$ and $R_0 > 0$ exist such that $BK^{-\lambda} < 1/2$, for all $\varepsilon < \varepsilon_0$, $R < R_0$ and then for all $t \in (0, 1)$, $R < R_0$ the assumptions of Lemma 4 are satisfied and therefore

$$\int_{B_R(x_0)} |Du|^2\,dx \leq c_{26}R^{\lambda}, \qquad \lambda = n(1-2/p).$$

This estimate implies that $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$. The proof is finished. ∎

REMARK 4. – Taking into account Definition 2 and Remark 1 following this definition, we see that for $\psi(t) \equiv 1$, $\mathcal{L}_{\psi}(\Omega) \equiv BMO(\Omega)$ (bouded mean oscillation) and $l_{\psi}(\Omega) \equiv VMO(\Omega)$ (vanishing mean oscillation).

It is a trivial fact that $\mathcal{L}_\psi(\Omega) \subseteq VMO(\Omega)$ if $\psi(t)$ vanishes as t approaches zero. And thus one may prove the above mentioned Theorem (using the same considerations as in above proof) in situation when $A_{ij}^{\alpha\beta} \in L^\infty(\Omega) \cap VMO(\Omega)$ too.

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