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On the Derived Length of Parasoluble Groups.

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Sunto. – *In questa nota si studiano i gruppi che inducono gruppi di automorfismi potenza su ciascun fattore della loro serie derivata. In particolare si prova che i gruppi risolubili con questa proprietà hanno lunghezza derivata al più* 3, *e che questa limitazione è la migliore possibile.*

Summary. – *In this paper groups are considered inducing groups of power automorphisms on each factor of their derived series. In particular, it is proved that soluble groups with such property have derived length at most* 3, *and that this bound is best possible.*

1. – Introduction.

A group G is said to be *parasoluble* if it has a normal series of finite length

$$\{1\} = G_0 < G_1 < \ldots < G_t = G$$

with abelian factors such that each subgroup of G_i/G_{i-1} is normal in G/G_{i-1} for all $i = 1, \ldots, t$. Since every group of power automorphisms is abelian, it follows that parasoluble groups have nilpotent commutator subgroup (recall that an automorphism of a group G is called a *power automorphism* if it maps every subgroup of G onto itself). Clearly, every parasoluble group is *hypercyclic* (i.e. it has an ascending normal series with cyclic factors), and so also locally supersoluble. Moreover, all nilpotent groups are parasoluble, and hence moving from the finite to the infinite case, parasoluble groups seem to be the most natural interpretation of the concept of a supersoluble group. Properties of parasoluble groups can be found in [5] and [2]. The most relevant difficulties in dealing with parasoluble groups arise from the fact that such groups do not in general have a characteristic «parasoluble» series; in fact, there exists a polycyclic group G (with derived length 3) having a supersoluble normal subgroup S which does not contain an abelian non-trivial G-invariant subgroup A such that all subgroups of A are normal in S (see [2]).

We shall say that a group G is an $\mathfrak{X}$*-group* if it acts as a group of power automorphisms on each factor $G^{(\alpha)}/G^{(\alpha+1)}$ of its derived series. Clearly, all groups with perfect commutator subgroup (in particular, all simple groups and all symmetric groups S_n for $n \geqslant 5$) belong to the class $\mathfrak{X}$, while soluble $\mathfrak{X}$-groups are parasoluble; these latter groups will be called *strongly parasoluble*. Note also that $\mathfrak{X}$ contains the class $\mathfrak{W}$ consisting of all groups G such that, if N is a normal subgroup of $G^{(\alpha)}$ for some ordinal α, then N is normal in G; finite soluble groups with this property were introduced and investigated by Weidig [6].

The aim of this article is to study the derived length of strongly parasoluble groups. It is easy to show that such groups have derived length at most 3. On the other hand, $\mathfrak{W}$-groups are clearly related to groups in which normality is a transitive relation (*T-groups*), and it is well-known that soluble T-groups are metabelian (see [3]). We will prove here that finitely generated soluble $\mathfrak{W}$-groups with non-periodic commutator subgroup are metabelian, and we will provide examples to show that this result cannot be extended to arbitrary strongly parasoluble groups and that the assumption that the group is finitely generated cannot be omitted.

Most of our notation is standard and can be found in [4].

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2. – Statements and proofs.

The class $\mathfrak{X}$ is obviously closed with respect to homomorphic images; on the other hand, $\mathfrak{X}$ is not subgroup-closed, since each finite group of order n can be embedded in the symmetric group S_n.

Our first result shows that if G is an $\mathfrak{X}$-group, then $G^{(3)} = G^{(4)}$. In particular, strongly parasoluble groups have derived length at most 3. Recall that a group G is said to be *hypo-abelian* if it has a descending series with abelian factors, or equivalently if G does not contain perfect non-trivial subgroups.

PROPOSITION 1. – *Let G be an hypoabelian $\mathfrak{X}$-group. Then the commutator subgroup G' of G is nilpotent with class at most 2.*

PROOF. – Let α be any ordinal number. Then G acts as a group of power automorphisms on $G^{(\alpha)}/G^{(\alpha+1)}$, and so G' is contained in the centralizer $C_G(G^{(\alpha)}/G^{(\alpha+1)})$. Since the group G is hypoabelian, it follows that G' is hypocentral and $\gamma_i(G') = G^{(i)}$ for each non-negative integer i. In particular, we have

$$\gamma_3(G') = G^{(3)} = (\gamma_2(G'))' \leqslant \gamma_4(G'),$$

so that $\gamma_3(G') = \{1\}$ and G' is nilpotent with class at most 2. ■

In [6] an example is given of a finite soluble $\mathfrak{W}$-group with derived length 3. Observe that the sharpness of the bound of Proposition 1 can also be proved by the following easier construction. Let p be an odd prime, and let

$$P = \langle a, b, c \mid a^p = b^p = c^p = [a, c] = 1, [b, c] = a \rangle$$

be an extraspecial group of order p^3 and exponent p. If x is the automorphism of P defined by the positions

$$a^x = a, \ b^x = b^{-1}, \ c^x = c^{-1},$$

the semidirect product $G = \langle x \rangle \ltimes P$ is a soluble $\mathfrak{W}$-group with derived length 3. On the other hand, all subgroups of the group G have the property $\mathfrak{W}$, while the example produced by Weidig contains a subgroup that is not an $\mathfrak{X}$-group, proving in particular that the class of strongly parasoluble groups is not closed with respect to subgroups.

It has been proved by Robinson [3] that any soluble T-group is metabelian, and that finitely generated soluble T-groups either are finite or abelian; this latter result could suggest that something more can be said on the derived length of strongly parasoluble groups containing elements of infinite order. This is not case, as the following example proves. Let p be an odd prime, and consider the group

$$K = \langle a, b, c \mid a^p = c^p = [a, c] = 1, [b, c] = a \ ;$$

if x is the automorphism of K defined by the positions

$$a^x = a, \ b^x = b^{-1}, \ c^x = c^{-1},$$

the semidirect product $G = \langle x \rangle \ltimes K$ has derived length 3 and non-periodic commutator subgroup, since $G' = \langle a, b^2, c \rangle$ and $G'' = \langle a \rangle$. It is also easy to show that G is strongly parasoluble. On the other hand, G is not a $\mathfrak{W}$-group, as $H = \langle ab^{2p} \rangle$ is a normal subgroup of G' but $H^x \neq H$. It is much more complicated to prove that there exist soluble $\mathfrak{W}$-groups with derived length 3 and non-periodic commutator subgroup. The following lemma is needed.

LEMMA 2. – *Let G be a group such that G/G' is torsion-free and $G' = Z(G)$ is a group of type p^∞ for some prime p. Then every normal subgroup of G either contains or is contained in G'.*

PROOF. – Let N be any normal subgroup of G which does not contain G'. Then $N \cap G'$ is a proper subgroup of G', and so it is finite. Since G/G' is torsion-free, it follows that $N \cap G'$ is the set of all elements of finite order of N. Put $|N \cap G'| = n$, and let T be the subgroup of all elements of finite order of $Z(N)$. Thus T is contained in $N \cap G'$, and there exists a torsion-free subgroup X of $Z(N)$ such that $Z(N) = T \times X$. Therefore $M = Z(N)^n = X^n$ is a torsion-free normal subgroup of G, and so $[M, G] = \{1\}$. It follows that $M = \{1\}$, so that $Z(N)$ is finite, and N is periodic (see [4] Part 1, Theorem 2.23). The lemma is proved. ■

THEOREM 3. – *There exists a soluble $\mathfrak{W}$-group G with non-periodic commutator subgroup such that $G'' = Z(G')$ is a group of type 2^∞.*

PROOF. – Let $\mathbb{Q}_2$ be the ring of all rational numbers whose denominators are powers of 2, and consider in $GL(3, \mathbb{Q}_2)$ the subgroup U consisting of all upper unitriangular matrices. Put also

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $\Gamma = \langle g, U \rangle$. Clearly g has order 2, and

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^g = \begin{pmatrix} 1 & -a & b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

for all elements a, b, c of $\mathbb{Q}_2$, so that $U^g = U$ and $\Gamma = \langle g \rangle \ltimes U$. Let

$$u = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$v = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

be elements of U. Then

$$[u, v] = \begin{pmatrix} 1 & 0 & az - xc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and hence

$$Z(U) = U' = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, b \in \mathbb{Q}_2 \right\},$$

so that we have also $Z(\Gamma) = Z(U)$. Moreover, the identity

$$\left[\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, g\right] = \begin{pmatrix} 1 & -2a & 2ac \\ 0 & 1 & -2c \\ 0 & 0 & 1 \end{pmatrix} \qquad (a, b, c \in \mathbb{Q}_2)$$

yields that $\Gamma' = U'[U, g] = U$. Clearly,

$$V = \left\{ \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; d \in \mathbb{Z} \right\}$$

is a subgroup of U', and the factor group U'/V is of type 2^∞. Put $G = \Gamma/V$, so that $G' = U/V$ and $G'' = U'/V \simeq Z(2^\infty)$; in particular, G' is a nilpotent group and $G'/G'' \simeq U/U'$ is torsion-free. Let

$$w = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

be an element of U such that $[U, \langle w \rangle] \leq V$; then $az - xc$ must be an integer for all choices x, z of elements of $\mathbb{Q}_2$, so that $a = c = 0$ and $w \in U'$. It follows that $Z(G') = G''$. Let N be any normal subgroup of G'. If $N \leq G''$, then N is obviously normal in G. Suppose that N is not contained in G'', so that $G'' \leq N$ by Lemma 2; since g induces by conjugation the inversion on U/U', we obtain also in this case that N is a normal subgroup of G. Therefore G is a $\mathfrak{W}$-group. ■

In order to consider finitely generated $\mathfrak{W}$-groups, we need the following two lemmas, that also give information on the structure of strongly parasoluble groups.

LEMMA 4. – *Let G be a strongly parasoluble group with non-periodic commutator subgroup. Then G'' is contained in $Z(G)$.*

PROOF. – The commutator subgroup G' of G is nilpotent with class at most 2 by Proposition 1, so that G'/G'' is not periodic. Let g be any element of G, and let x, y be elements of G'. Since g acts on G'/G'' as a power automorphism, we have $x^g = x^\varepsilon u$ and $y^g = y^\varepsilon v$, where u, v belong to G'' and $\varepsilon = \pm 1$ (see [1], Corollary 4.2.3). Therefore

$$[x, y]^g = [x^\varepsilon u, y^\varepsilon v] = [x^\varepsilon, y^\varepsilon] = [x, y],$$

and hence G'' is contained in $Z(G)$. ■

LEMMA 5. – *Let G be a strongly parasoluble group with non-periodic commutator subgroup. If G is locally nilpotent, then it is nilpotent with class at most* 2.

PROOF. – The subgroup G'' is contained in $Z(G)$ by Lemma 4. Moreover, G acts as a group of power automorphisms on the non-periodic group G'/G''. Since G is locally nilpotent, it follows that G'/G'' is contained in $Z(G'/G'')$, so that G is nilpotent with class at most 3. Therefore G is metabelian, and hence it has nilpotency class at most 2. ■

Our next lemma is elementary and certainly well-known.

LEMMA 6. – *Let G be a group, and let A be an abelian normal subgroup of G such that every element of G induces by conjugation either the identity or the inversion on A. There either A is contained in $Z(G)$ or $A \cap Z(G)$ has exponent at most* 2.

PROOF. – Suppose that A is not contained in $Z(G)$, and let g be an element of $G \setminus C_G(A)$. If a is any element of $A \cap Z(G)$, we have that $a = a^g = a^{-1}$, so that $a^2 = 1$ and $A \cap Z(G)$ has exponent at most 2. ■

THEOREM 7. – *Let G be a finitely generated soluble $\mathfrak{W}$-group with non-periodic commutator subgroup. Then G is metabelian.*

PROOF. – Obviously, the group G satisfies the maximal condition on subgroups, and G'' is contained in $Z(G)$ by Lemma 4. Then both groups G'/G'' and $Z(G')$ are non-periodic. Assume by contradiction that $G'' \neq \{1\}$, and let N be a subgroup of G'' such that G''/N has prime order p. Replacing G by G/N it can be assumed without loss of generality that $|G''| = p$. Since the group G is not nilpotent by Lemma 5, and it acts on G'/G'' as a group of power automorphisms, it follows that the centralizer $C = C_G(G'/G'')$ has index 2 in G, and hence $G = C\langle x \rangle$ for some $x \in G$. Moreover, Lemma 6 yields that $(G'/G'') \cap Z(G'/G'')$ has exponent at most 2, so that in particular $G' \cap Z(G)$ is finite, and hence $Z(G')$ is not contained in $Z(G)$. As G induces on $Z(G')$ a group of power automorphisms, it follows again from Lemma 6 that $Z(G') \cap Z(G)$ has exponent at most 2, so that $|G''| = 2$. Clearly

$$[G', C, C] \leqslant [G'', C] = \{1\},$$

so that $C' \leqslant Z(G')$ by the Three Subgroup Lemma. Moreover, since x induces by conjugation the inversion on G'/G'', we obtain that

$$[G', \langle x \rangle, C] \leqslant [(G')^2 G'', C] = [(G')^2, C] = [G', C]^2 = \{1\}.$$

On the other hand, $[C, G', \langle x \rangle] = \{1\}$, and it follows again from the Three

Subgroup Lemma that $[\langle x\rangle, C, G'] = \{1\}$. Therefore $[\langle x\rangle, C]$ is contained in $Z(G')$, and $G' = C'[C, \langle x\rangle]$ is abelian. This contradiction proves the theorem. ■

LEMMA 8. – *Let G be a group such that every subgroup of G' is normal in G. Then G is metabelian.*

PROOF. – Obviously G acts on G' as a group of power automorphisms, so that $G/C_G(G')$ is abelian, and hence G' itself is an abelian group. ■

THEOREM 9. – *Let G be a hypercentral $\mathfrak{W}$-group. Then G is metabelian.*

PROOF. – Assume by contradiction that the group G is not metabelian. Since G' is hypoabelian, by Lemma 5 its commutator subgroup G' is periodic, and we may of course suppose that G' is a p-group for some prime number p. Moreover, G'' is contained in $Z(G')$ by Proposition 1. Let N be a proper subgroup of G'' such that G''/N is locally cyclic. Then N is normal in G, and replacing G by G/N it can be assumed without loss of generality that G'' is locally cyclic. Since all normal subgroups of G' are normal in G, it follows from Lemma 8 that G' contains a cyclic non-normal subgroup $\langle x\rangle$ such that $\langle x^p\rangle$ is normal in G. Again replacing G by $G/\langle x^p\rangle$, we may suppose that $x^p = 1$. Let $\langle y\rangle$ be the unique subgroup of order p of G''. Clearly $A = \langle x, G''\rangle$ is an abelian normal subgroup of G, so that also the subgroup $A[p] = \langle x, y\rangle$ is normal in G. Moreover, the hypercentral group G centralizes both $\langle y\rangle$ and $\langle x, y\rangle/\langle y\rangle$, so that $G/C_G(\langle x, y\rangle)$ is abelian and $G' \leqslant C_G(\langle x, y\rangle)$. This contradiction proves the theorem. ■

COROLLARY 10. – *Let G be a hypoabelian $\mathfrak{W}$-group. If G is a p-group for some prime number p, than G is metabelian.*

PROOF. – The factor group $G/G^{(3)}$ is parasoluble, and so also nilpotent (see [2], Lemma 2.3). It follows from Theorem 9 that $G/G^{(3)}$ is metabelian, so that $G'' = G^{(3)} = \{1\}$ and G itself is metabelian. ■

Theorem 9 cannot be extended to the case of hypercentral $\mathfrak{X}$-group, as the following example shows.

Let p be an odd prime, and let

$$C = \langle c_n \mid n \in \mathbb{N}_0,\ c_0 = 1,\ c_{n+1}^p = c_n \text{ for all } n\rangle$$

be a group of type p^∞. Put

$$Q_1 = \operatorname*{Dr}_{n \in \mathbb{N}} \langle x_n\rangle \quad \text{and} \quad Q_2 = \operatorname*{Dr}_{n \in \mathbb{N}} \langle y_n\rangle,$$

where for each positive integer n the elements x_n and y_n have order p^n. Consider the abelian group $Q = Q_1 \times Q_2$, and let

$$C \rightarrowtail N \twoheadrightarrow Q$$

be a central extension such that $[x_n, y_n] = c_n$ for all n. Let α be a non-trivial p-adic integer with $\alpha \equiv 1 (\mathrm{mod}\, p)$. Then N admits an automorphism g acting as α on Q and as α^2 on C. The semidirect product $G = \langle g \rangle \ltimes N$ is a hypercentral $\mathfrak{X}$-group with derived length 3, since $G' = CN^p$ and $G'' = CN^p$.

Observe finally that hypercentral $\mathfrak{W}$-groups need not be nilpotent, as the consideration of the locally dihedral 2-group shows.

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