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Existence and Decay in Non Linear Viscoelasticity.

JAIME E. MUÑOZ RIVERA (*) - FÉLIX P. QUISPE GÓMEZ

Sunto. – *In questo lavoro si studia l'esistenza, l'unicitá e il decadimento di soluzioni a una classe di equazioni viscoelastiche in uno spazio di Hilbert H separabile, dato da:*

$$u_{tt} + M([u])\,Au - \int_0^t g(t-\tau)\,N([u])\,Au\,d\tau = 0, \quad \text{in } L^2(0, T; H)$$

$$u(0) = u_0, \qquad u_t(0) = u_1,$$

dove con $[u(t)]$ *si denota*

$$[u(t)] = \left((u(t), u_t(t), (Au(t), u_t(t)), \|A^{\frac{1}{2}} u(t)\|^2, \|A^{\frac{1}{2}} u_t(t)\|^2, \|Au(t)\|^2\right) \in \mathbb{R}^5,$$

$A : D(A) \subset H \to H$ *è un operatore autoaggiunto non-negativo,* $M, N : \mathbb{R}^5 \to \mathbb{R}$ *sono funzioni di classe* C^2 *e* $g : \mathbb{R} \to \mathbb{R}$ *è una funzione di classe* C^3 *verificante condizioni opportune. Mostriamo che esistono soluzioni globali nel tempo per piccoli dati iniziali. Quando* $[u(t)] = \|A^{1/2}u\|^2$, $M : \mathbb{R} \to \mathbb{R}$ *e* $N = 1$, *si mostra l'esistenza globale per grandi dati iniziali* (u_0, u_1) *presi negli spazi* $D(A) \times D(A^{1/2})$ *a condizione che siano abbastanza prossimi a dati analitici. È anche dimostrato un tasso uniforme di decadimento.*

Summary. – *In this work we study the existence, uniqueness and decay of solutions to a class of viscoelastic equations in a separable Hilbert space H given by*

$$u_{tt} + M([u])\,Au - \int_0^t g(t-\tau)\,N([u])\,Au\,d\tau = 0, \quad \text{in } L^2(0, T; H)$$

$$u(0) = u_0, \qquad u_t(0) = u_1,$$

where by $[u(t)]$ *we are denoting*

$$[u(t)] = \left((u(t), u_t(t), (Au(t), u_t(t)), \|A^{\frac{1}{2}} u(t)\|^2, \|A^{\frac{1}{2}} u_t(t)\|^2, \|Au(t)\|^2\right) \in \mathbb{R}^5,$$

$A : D(A) \subset H \to H$ *is a nonnegative, self-adjoint operator,* $M, N : \mathbb{R}^5 \to \mathbb{R}$ *are* C^2*-functions and* $g : \mathbb{R} \to \mathbb{R}$ *is a* C^3*-function with appropriates conditions. We show that there exists global solution in time for small initial data. When* $[u(t)] = \|A^{1/2}u\|^2$, $M : \mathbb{R} \to \mathbb{R}$ *and* $N = 1$, *we show the global existence for large initial data* (u_0, u_1) *taken in the space* $D(A) \times D(A^{1/2})$ *provided they are close enough to Gevrey data. Uniform rate of decay is also proved.*

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1. – Introduction.

The nonlinear wave equation

$$u_{tt} + M(\|A^{1/2}u\|^2)\,Au = 0$$

was studied for several authors see for example [1, 2, 5, 6, 7, 9, 20]; but until now the question about the global existence of solution for initial data taken in the usual Sobolev's Spaces remains open. To obtain global solution to a class relative to the above equation, several authors [14, 15, 18] to name but a few, have considered damping terms as A^2u, Au_t, or $A^{\alpha}u_t$ which gives strong estimates resulting in the convergence of the nonlinear term of the approximated solution. In Nishihara [19] the author consider the wave equation with linear frictional damping and show the existence of global solution for a class of large initial data in $D(A)$ spaces, non analytical but close to an analytical data (analytical in the sense of Gevrey functions). Nishihara's result is an important improvement about the question of existence of solution for the nonlinear Kirchhoff equation with weak dissipation, because it provides a large space where the initial data can be taken to produce large existence result. In this paper we consider the viscoelastic nonlinear wave equation of memory type. The system in question is the following

$$u_{tt} + M([u])\,Au - \int_0^t g(t-\tau)\,N([u])\,Au\,d\tau = 0, \quad \text{in } L^2(0,T;H) \tag{1.1}$$

$$u(0) = u_0, \qquad u_t(0) = u_1, \tag{1.2}$$

where by H we are denoting separable Hilbert space and by $[u(t)]$ the nonlinear argument of N and M of the form

$$[u(t)] = \Big((u(t),\,u_t(t), (Au(t),\,u_t(t)),\ \|A^{\frac{1}{2}}u(t)\|^2,\ \|A^{\frac{1}{2}}u_t(t)\|^2,\ \|Au(t)\|^2\Big) \in \mathbb{R}^5.$$

Here $\|\cdot\|$ and $(\cdot,\,\cdot)$ denote the norm and the inner product defined over H. By A we are denoting an unbounded nonnegative self-adjoint operator satisfying

$$A : D(A) \subset H \to H\,,$$

and

[V1] The embedding $D(A^r) \hookrightarrow D(A^s)$ is compact for any $r > s \geqslant 0$. On M and N we impose the following hypotheses

[V2] The functions $M, N : \mathbb{R}^5 \to \mathbb{R}$ are C^2 and $g : \mathbb{R} \to \mathbb{R}$ is a C^3-function.

The main result of this paper is to show the global existence of solution to (1.1)–(1.2) provided the initial data is small. For large initial data we show the global existence of solution for the Kirchhoff equation with memory, that is when $[u] = \|A^{1/2}u\|^2$, $M : \mathbb{R} \to \mathbb{R}$ and $N = 1$, see section 6 for details. Additionally, we show that the solutions of the different Kirchhoff's models we study in this paper, decay with the same rate as the relaxation function g.

The remaining part of this work is organized as follows. In the next section 2 we prove the existence of local solutions while in section 3 we show that the solution has uniform rate of decay for the linear problem that is to say, when the relaxation function decays exponentially, then the corresponding solution also decays exponentially. When the relaxation decays polynomially the solution also decays polynomially with the same rate as the relaxation. In section 4 we consider a special case of the nonlinear viscoelastic equation, where the nonlinearity works only on the memory stress. We will show in this case that there exists only one global large solution, for initial data taken in the usual Sobolev Space. In section 5 we use the uniform rate of decay, obtained in section 3, to show the global existence of solutions to equation (1.1) when the initial data is small. In section 6 we consider the existence of Gevrey (analytical) solutions to (1.1) for any Gevrey initial data. Finally, in section 7 we show the existence of large solutions in the usual Sobolev's spaces $D(A)$ provided the initial data is close enough to an Gevrey initial data.

2. – Existence of solutions.

The local existence is summarized in the following theorem:

THEOREM 2.1. – *Suppose that* $(u_0, u_1) \in D(A^{3/2}) \times D(A)$ *and that the hypotheses* [V1], [V2] *holds. Then there exists* $T > 0$ *and only one function*

$$u : [0, T] \to H$$

solution the equation (1.1), *satisfying*

$$u \in C^2([0, T[, D(A^{1/2})) \cap C^1([0, T[, D(A)) \cap C([0, T[, D(A^{3/2})).$$

In addition if the initial data satisfies

$$(u_0, u_1) \in D(A^{1+l}) \times D(A^{\frac{1}{2}+l}), \qquad l \geq 1/2,$$

then there exists only one solution of (1.1), *such that*

$$u \in C([0, T], D(A^{1+l})) \cap C^1([0, T], D(A^{\frac{1}{2}+l})) \cap C^2([0, T], D(A^l)).$$

Finally, if $T_{\max}$ *is the maximal time for which the solution exists, then we have*

$$T_{\max} < \infty \;\Rightarrow\; \limsup_{t \to T_{\max}} \left\{ \|Au_t(t)\|^2 + \|A^{3/2} u(t)\|^2 \right\} = \infty . \qquad \blacksquare$$

PROOF. – Let us introduce the space $\mathcal{Y}$ given by

$$\mathcal{Y} = \left\{ v \in C([0, T], D(A)), \; v_t \in C([0, T], D(A^{1/2})) \right\},$$

and let us define the norm

$$\|v\|_{\mathcal{Y}} = \sup_{t \in [0, T]} \left\{ \|Av(t)\|^2 + \|A^{1/2} v_t(t)\|^2 \right\}.$$

Let us denote by

$$\mathcal{W}(\eta, T) = \left\{ \begin{array}{c} w \in \mathcal{Y} : w_t, \; A^{1/2} w \in \mathcal{Y}, \\ w(0) = u_0, \; w_t(0) = u_1 \text{ and } |||w||| < \eta \end{array} \right\}$$

where by $|||\cdot|||$ we are denoting

$$|||w||| = \sup_{t \in [0, T]} \left\{ \|A^{3/2} w\|^2 + \|Aw_t\|^2 + \|A^{1/2} w_{tt}\|^2 \right\},$$

and η is a positive number to be fixed later. It is easy to verify that $\mathcal{W}$ is a closed subspace of $\mathcal{Y}$. Let us define the operator

$$\mathcal{T} : \mathcal{W} \to W$$

$$w \mapsto \mathcal{T}w = u$$

where u is a solution of the equation

$$u_{tt} + M([w])\, Au = \int_0^t g(t - \tau)\, N([w])\, Aw(\tau)\, d\tau , \tag{2.3}$$

$$u(0) = u_0 \in D(A), \qquad u_t(0) = u_1 \in D(A^{\frac{1}{2}}).$$

Our starting point is to show that the operator $\mathcal{T}$ is invariant on $\mathcal{W}(\eta, T))$ for T small enough, that is $\mathcal{T}(\mathcal{W}(\eta, T)) \subset \mathcal{W}(\eta, T)$ then we will show that the restriction of $\mathcal{T}$ to $\mathcal{W}(\eta, T)$ is a contraction in $\mathcal{Y}$ for some $\eta > 0$ and T small enough.

Using the continuity of $t \mapsto w(t) \in \mathcal{W}(\eta, T)$ together with the hypotheses on M and N we conclude that there exist positive constants satisfying

$$0 < m_0 \leqslant M([w(t)]) \leqslant m_1 \tag{2.4}$$

$$0 < n_0 \leqslant N([w(t)]) \leqslant n_1 .$$

Let us introduce the following functionals

$$E(t, v) = \|A^{\frac{1}{2}} v\|^2 + \|v_t\|^2, \qquad L(t, v) = E(t, Av).$$

Note that for any $w \in \mathcal{W}$ we have that $|[w(t)]|_{\mathbb{R}^5} < c_2 \eta$ where c_2 is a positive constant depending on the embedding $D(A) \hookrightarrow D(A^{\frac{1}{2}}) \hookrightarrow H$. Let us consider $w \in \mathcal{W}$, $v_0 \in D(A^{3/2})$ and $v_1 \in D(A)$. Under these conditions is easy to verify that the solution v of

$$v_{tt} + M([w])\, Av = \underbrace{\int_0^t g(t-\tau)\, N([w])\, Aw(\tau)\, d\tau}_{:= F(t)},$$

$$v(0) = v_0, \qquad v_t(0) = v_1,$$

satisfies

$$L(t, v) \leqslant \left\{ c_3 L(0, v) + \frac{1}{c_2 m_0} \|F(t)\|^2 + \frac{1}{2c_2} \int_0^t \|F'(s)\|^2 ds \right\} e^{c_4(\eta) t}, \tag{2.5}$$

where $c_2 = \min\left\{\frac{1}{2}, \frac{m_0}{4}\right\}$, $c_3 = \frac{1}{c_2} \max\left\{\frac{1}{2}, \frac{m_1}{2}\right\}$ and $c_4(\eta) = \frac{(c_1 \mu \eta + 1)}{2c_2}$.

Let us denote by

$$\mu = \sup_{|\sigma| < c_2 \eta} \{ |\partial^\alpha M(\sigma)|, |\partial^\alpha N(\sigma)|;\ |\alpha| \leqslant 2 \}.$$

Then for any $w \in \mathcal{W}(\eta, T)$ it follows that

$$\left| \frac{d}{dt} M([w(t)]) \right| \leqslant \mu c_1 \eta, \qquad \left| \frac{d}{dt} N([w(t)]) \right| \leqslant \mu c_1 \eta$$

$$\|(g * Aw)(t)\|^2 \leqslant \eta \left(\int_0^t g(\tau)\, d\tau \right)^2,$$

where $c_1 > 0$ and c_2 are positive constant depending only on embedding $D(A) \hookrightarrow D(A^{\frac{1}{2}}) \hookrightarrow H$. On the other hand, we get

$$\|F(t)\|^2 = \|g * N([w])\, Aw\|^2 \leqslant n_1^2 \eta \left(\int_0^t g(\tau)\, d\tau \right)^2.$$

$$\begin{aligned} \|F'(t)\|^2 &= \|g(0) N([w])\, Aw + g' * N([w])\, Aw\|^2 \\ &\leqslant 2\{ (g(0)\, n_1)^2 \|Aw\|^2 + (cn_1)^2 \|g * Aw\|^2 \} \\ &\leqslant 2\eta\varrho \left\{ 1 + \left(\int_0^t g(\tau)\, d\tau \right)^2 \right\}, \end{aligned}$$

where c_5 is a positive constant and $\varrho = (g(0)\, n_1)^2 + (c_5\, n_1)^2$. Hence, from (2.5) we get

$$L(t,\, u) \leqslant \left\{ c_3 L(0,\, u) + \frac{n_1^2\, \eta \left(\int_0^t g(\tau)\, d\tau \right)^2}{c_2\, m_0} + \frac{\varrho\eta}{c_2} \left[1 + \left(\int_0^t g(\tau)\, d\tau \right)^2 \right] t \right\} e^{c_4(\eta) t}.$$

Since $\int_0^t g(\tau)\, d\tau \to 0$ when $t \to 0$, then there exists $T_0 > 0$ such that

$$\left(\int_0^{T_0} g(\tau)\, d\tau \right)^2 < \frac{m_0\, c_2}{n_1^2\, \eta}, \qquad 1 + \left(\int_0^{T_0} g(\tau)\, d\tau \right)^2 < 2,$$

$$T_0 < \min \left\{ \frac{1}{c_4(\eta)},\, \frac{1}{\varrho\eta} \right\}.$$

Therefore, for any $t \in [0,\, T_0]$ we have

$$L(t,\, u) \leqslant \left\{ c_3 L(0,\, u) + \left(1 + \frac{2}{c_2} \right) \right\} e =: K_0.$$

From equation (2.3) it follows

$$\|A^{1/2} u_{tt}\|^2 \leqslant 4 \left\{ m_1^2 \|A^{3/2} u\|^2 + n_1^2 \|g * A^{3/2} w\|^2 \right\},$$

which implies

$$\|A^{1/2} u_{tt}\|^2 \leqslant m_1^2 L(t,\, u) + 4 n_1^2\, \eta \left(\int_0^{T_0} g(\tau)\, d\tau \right)^2$$

$$\leqslant 4 m_1^2 K_0 + 4 n_1^2\, \eta \left(\int_0^{T_0} g(\tau)\, d\tau \right)^2.$$

Taking $\eta = (2 + 5 m_1^2) K_0$ and choosing $T_0 > 0$ such that

$$4 n_1^2\, \eta \left(\int_0^{T_0} g(\tau)\, d\tau \right)^2 \leqslant m_1^2 K_0,$$

from where it follows that

$$|||u||| = \sup_{t \in [0,\, T_0]} \left\{ \|A^{3/2} u\|^2 + \|A u_t\|^2 + \|A^{1/2} u_{tt}\|^2 \right\} \leqslant \eta\, .$$

which proves that $\mathcal{C}$ is invariant.

Now we will show that there exists T_1, $0 < T_1 < T_0$ such that the restriction of $\mathcal{T}$ over $\mathcal{W}(\eta, T_1)$ is a contraction in $\mathcal{Y}$. Let us take w^1 and $w^2 \in \mathcal{W}(\eta, T_1)$. Denoting by $u^i = \mathcal{T}w^i$, $i = 1, 2$; $U = u^1 - u^2$ and $W = w^1 - w^2$. We have

$$U_{tt} + M([w^1])AU = \{M([w^1]) - M([w^2])\}\, Au^2 +$$

$$g * \{N([w^1]) - N([w^2])\}\, Aw^1 + g * N([w^2])\, AW.$$

Using multiplicative techniques and the mean value theorem we get

$$\|A^{1/2}U_t\|^2 + \|AU\|^2 \leqslant$$

$$\frac{1}{c_6}\left(K_1 T_3 + \frac{2n_1^2\left(\int_0^t g(\tau)\, d\tau\right)^2}{m_0}\right)\|W\|_{\mathcal{Y}}^2 + \frac{(\mu c_1 \eta + 1)}{c_6}\int_0^t \{\|A^{1/2}U_t\|^2 + \|AU\|^2\}\, d\tau\,,$$

where $c_6 = \min\left\{\frac{m_0}{2}, 1\right\}$. Using Gronwall's inequality we arrive at

$$\|A^{1/2}U_t\|^2 + \|AU\|^2 \leqslant \frac{1}{c_6}\left(K_1 T_1 + \frac{2n_1^2\left(\int_0^t g(\tau)\, d\tau\right)^2}{m_0}\right)\|W\|_{\mathcal{Y}}^2 e^{\gamma T_1},$$

where $\gamma = \frac{(\mu c_1 \eta + 1)}{c_6}$ and K_1 are positive constants. Taking $T_1 > 0$ such that

$$\frac{1}{c_6}\left(K_1 T_1 + \frac{2n_1^2\left(\int_0^{T_1} g(\tau)\, d\tau\right)^2}{m_0}\right)\cdot e^{\gamma T_1} < 1\,,$$

we have that $\mathcal{T}|_{\mathcal{W}(\eta, T)}$ is a contraction on $\mathcal{Y}$. From where the existence result follows. To show the uniqueness let us take two solutions u^1 and u^2 of (2.3). Denoting by $U = u^1 - u^2$ we have that

$$U_{tt} + M([u^1])\, AU =$$

$$\left\{M([u^1]) - M([u^2])\right\} Au^2 - g * \left\{N([u^1]) - N([u^2])\right\} Au^1 + g * N([u^2])\, AU$$

$$U(0) = U_t(0) = 0\,.$$

Using multiplicative techniques we are able to show that

$$\|U_t(t)\|^2 + \frac{m_0}{2}\|A^{1/2}U(t)\|^2 \leqslant C_1(1 + t)\int_0^t \|U_t(\tau)\|^2 + \|A^{1/2}U(\tau)\|^2\, d\tau.$$

From Gronwall's inequality we get that $U = 0$, which completes the proof. ■

3. – Asymptotic behaviour: Linear Case.

In this section, we study the asymptotic behaviour of the equation

$$u_{tt} + Au - \int_0^t g(t-\tau)\, Au(\tau)\, d\tau = f .$$

To prove the exponential decay of the solutions we use the following hypotheses on g:

$$0 < g(t) \in C^3, \qquad -\kappa g(t) \leqslant g'(t) \leqslant -cg(t) \tag{3.2}$$

$$|g''(t)| \leqslant Cg(t), \tag{3.3}$$

$$\alpha =: 1 - N([0]) \int_0^\infty g(\tau)\, d\tau > 0 \tag{3.4}$$

to facilitate our computation we introduce the notations

$$(g\Box f)(t) = \int_0^t g(t-\tau) \|f(\tau) - f(t)\|^2 d\tau \quad \text{and}$$

$$(\eta * v)(t) = \int_0^t \eta(t-\tau)\, v(\tau)\, d\tau .$$

The following Lemma will play an important role in the sequel.

LEMMA 3.1. – *Let us denote by X a Hilbert space. Consider η a $C^1(\mathbb{R})$-function and $\phi \in C^1([0, T]; X)$. Under this conditions the following identity holds*

$$2(\eta * \phi)\, \phi' = -\eta(t)\, |\phi|^2 - \frac{d}{dt}\left\{ \eta\Box\phi - \left(\int_0^t \eta d\tau \right) |\phi|^2 \right\} + \eta'\,\Box\phi . \tag{3.5}$$

PROOF. – It is sufficient to differentiate the expression

$$\eta\Box\phi - \left(\int_0^t \eta d\tau \right) |\phi|^2. \qquad \blacksquare$$

Let us introduce the energy functional,

$$E(t, v) = \frac{1}{2}\left\{ \|v_t\|^2 + \left(1 - \int_0^t g d\tau \right) \|A^{1/2} v\|^2 + g\Box A^{1/2} v \right\}. \tag{3.6}$$

Multiplying equation (3.1) by u_t and applying Lemma 3.1 we have that

$$\frac{d}{dt}E(t,u) = -\frac{1}{2}g(t)\|A^{1/2}u\|^2 + \frac{1}{2}g'\Box A^{1/2}u + (f, u_t). \tag{3.7}$$

Let us introduce the function $w = u - g * u$. A simply computation yields

$$w_t = u_t - g(0)\,u - g'^* u$$

$$w_{tt} = u_{tt} - g(0)\,u_t - g'(0)\,u - g''^* u$$

So, w satisfies:

$$w_{tt} + Aw + g(0)\,w_t + \tag{3.8}$$

$$g(0)^2 u + g(0)\,g' * u + g'(0)\,u + g'' * u = f \text{ in } L^2(0,T;H)$$

$$w(0) = u_0(x); \qquad w_t(0) = u_1(x). \tag{3.9}$$

Note that the function w transform equation (1.1) into a wave equation with frictional damping except for the remaining terms on u. The idea now is to estimate the terms on u and to use the simple dissipation on w to prove the uniform rate of decay. To do this, let us introduce the functional

$$\mathcal{E}(t) = \frac{1}{2}\left\{\|w_t\|^2 + \|A^{1/2}w\|^2 + g(0)(w, w_t) + \frac{g(0)^2}{2}\|w\|^2\right\}$$

Our method consist in introducing functions whose derivatives have the terms $-\|w_t\|^2$, $-\|A^{1/2}w\|^2$. The starting point of this process is to establish an inequality which we summarized in following Lemma:

LEMMA 3.2. – *Under the above conditions, the solution of equation* (3.12) *satisfies the following inequality*

$$\frac{d}{dt}\mathcal{E}(t) \leqslant -\left(\frac{g(0)}{2} - \delta\right)\left\{\|w_t\|^2 + \|A^{1/2}w\|^2\right\} +$$

$$C_\delta\left\{g(t)\|u\|^2 + g\Box u\right\} + \left(f, w_t + \frac{g(0)}{2}w\right)$$

where δ *is a small constant to be fixed, later.*

PROOF. – Multiplying equation (3.8) by w_t we have,

$$\frac{1}{2}\frac{d}{dt}\left\{\|w_t\|^2 + \|A^{1/2}w\|^2\right\} + g(0)\|w_t\|^2 = \tag{3.10}$$

$$(f, w_t) - g(0)(\{g(0)\,u + g' * u, w_t - (\{g'(0)\,u + g'' * u\}, w_t.$$

Note that

$$g(0)\,u + g' * u = g(t)\,u + \int_0^t g'(t-\tau)\{u(\tau) - u(t)\}\,d\tau$$

$$g'(0)\,u + g'' * u = g'(t)\,u + \int_0^t g''(t-\tau)\{u(\tau) - u(t)\}\,d\tau\,.$$

Inserting the above identities into relation (3.10) we get

$$(\{g(0)\,u + g' * u\},\, w_t) \leq c_\delta\{g(t)\|u(t)\|^2 + g\Box u\} + \frac{\delta}{2}\|w_t\|^2$$

$$(\{g'(0)\,u + g'' * u\},\, w_t) \leq c_\delta\{g(t)\|u(t)\|^2 + g\Box u\} + \frac{\delta}{2}\|w_t\|^2$$

From where it follows that

$$\text{(3.11)} \qquad \frac{1}{2}\frac{d}{dt}\{\|w_t\|^2 + \|A^{1/2}w\|^2\} + $$
$$(g(0) - \delta)\|w_t\|^2 \leq c_\delta\{g(t)\|u(t)\|^2 + g\Box u\} + (f,\, w_t)\,.$$

Multiplying equation (3.8) by w, we have

$$\frac{d}{dt}(w,\, w_t) = \|w_t\|^2 - \|A^{1/2}w\|^2 - g(0)(w,\, w_t) + (f,\, w) -$$
$$g(0)(\{g(0)\,u + g' * u\},\, w) - (\{g'(0)\,u + g''* u\},\, w)\,.$$

Using similar arguments as above, we have

$$\frac{d}{dt}\left\{(w,\, w_t) + \frac{g(0)}{2}\|w(t)\|^2\right\} \leq$$
$$\|w_t\|^2 - (1-\delta)\|A^{1/2}w\|^2 + (f,\, w) + c_\delta\{g(t)\|u(t)\|^2 + g\Box u\}\,.$$

Multiplying the above expression by $g(0)/2$ and adding the product result to (3.11) our conclusion follows. ■

In the following remark we show how the frictional dissipation on w can be used to estimate the expression on u.

REMARK 3.1. – *It is easy to see that there exists a positive constant C for which we have:*

$$\mathcal{E}(t) \leq CE(t),$$

and also

$$\|w_t\|^2 + \|A^{1/2} w\|^2 \geq (1-\delta)\|u_t\|^2 +$$

$$\left(1 - \int_0^t g \, d\tau\right) \|A^{1/2} u\|^2 - c_\delta \{ g(t) \|A^{1/2} u\|^2 + g \Box A^{1/2} u\}$$

We will show only that

$$\|w_t\|^2 \leq cE(t).$$

The others inequalities are similar. Note that

$$w_t = u_t - g(t)\, u - \int_0^t g'(t-\tau)\{u(\cdot,\tau) - u(\cdot,t)\}\, d\tau .$$

From where it follows that

$$\|w_t\|^2 \leq c\{\|u_t\|^2 + g(t)\|u\|^2 + g\Box u\}.$$

Using that $D(A^s) \subset D(A^r)$ has continuous immersion for $s > r$, our conclusion follows. ■

From (3.7) and Lemma 3.2 we have that the functional $\mathcal{L}(t)$ given by

$$\mathcal{L}(t) = \nu E(t) + \mathcal{E}(t)$$

satisfies:

$$c_0 E(t) \leq \mathcal{L}(t) \leq c_1 E(t) \tag{3.12}$$

and also that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\kappa \mathcal{L}(t) + (\nu f, u_t) + \left(f, w_t + \frac{g(0)}{2} w\right). \tag{3.13}$$

The exponential decay of the solution of equation (1.1) is summarized in the following Theorem

THEOREM 3.1. – *Under the same hypothesis as Lemma* 3.2, *with the kernel* g *satisfying conditions* (3.2)-(3.4), *and* $\|f\|^2 \leq ce^{-\gamma t}$, *there is positive constants* κ_0, c_0 *and* κ_1 *such that*

$$E(t, u) \leq (\kappa_0 E(0, u) + c_0)\, e^{-\kappa_1 t}.$$

PROOF. – From (3.13) we have

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\frac{\kappa}{2}\mathcal{L}(t) + C\|f\|^2 \leqslant -\frac{\kappa}{2}\mathcal{L}(t) + C_0 e^{-\gamma t}.$$

From where our result follows. ■

Now we consider kernel which decay polynomially that is we suppose that the kernel g satisfies

$$0 < g(t) \in C^3, \tag{3.14}$$

$$-c_0 g^{1+\frac{1}{p}}(t) \leqslant g'(t) \leqslant -c_1 g^{1+\frac{1}{p}}(t); \qquad |g''(t)| \leqslant c_2 g^{1+\frac{1}{p}}(t) \tag{3.15}$$

$$\beta := \int_0^\infty g^{1-\frac{1}{p}}(\tau)\, d\tau < \infty, \qquad p > 2. \tag{3.16}$$

The relations (3.14)-(3.16) mean that $g \approx (1+t)^{-p}$ as $t \to \infty$, for $p > 2$. We will show that under the above conditions the solution of (1.1) decays polynomially, with the same rate of decay of g. To do this we will use the following Lemma.

LEMMA 3.3. – *Suppose that «g» and «h» are continuous functions satisfying* $g \in L^{1+\frac{1}{q}}(0, \infty) \cap L^1(0, \infty)$ *and* $g^r \in L^1(0, \infty)$ *for some* $0 \leqslant r < 1$, *then, we have that:*

$$\int_0^t |g(t-\tau)\, h(\tau)|\, d\tau \leqslant$$

$$\left\{ \int_0^t |g(t-\tau)|^{1+\frac{1-r}{q}}\, |h(\tau)|\, d\tau \right\}^{\frac{q}{q+1}} \left\{ \int_0^t |g(t-\tau)|^r\, |h(\tau)|\, d\tau \right\}^{\frac{1}{q+1}}.$$

PROOF. – For any t fixed we have:

$$\int_0^t |g(t-\tau)\, h(\tau)|\, d\tau = \int_0^t \underbrace{|g(t-\tau)|^{\frac{r}{q+1}}\, |h(\tau)|^{\frac{1}{q+1}}}_{:=z}\ \underbrace{|g(t-\tau)|^{1-\frac{r}{q+1}}\, |h(\tau)|^{\frac{q}{q+1}}}_{:=v}\, d\tau\,.$$

Note that $z \in L^p(0, \infty)$ and $v \in L^{p'}(0, \infty)$, where $p = q+1$ and $p' = \dfrac{q+1}{q}$.

Using the Hölder's inequality, we have

$$\int_0^t |g(t-\tau)\, h(\tau)\,|d\tau \leqslant$$

$$\left\{\int_0^t |g(t-\tau)\,|^r\,|h(\tau)\,|d\tau\right\}^{\frac{1}{q+1}}\left\{\int_0^t |g(t-\tau)\,|^{1+\frac{1-r}{q}}\,|h(\tau)\,|d\tau\right\}^{\frac{q}{q+1}}.$$

Which completes the proof. ■

LEMMA 3.4. – *Suppose that* $v \in C(0, T; D(A^{1/2}))$ *and* g *are continuous functions satisfying the hypotheses* (3.15)–(3.16), *then for* $0 < r < 1$, *we have*

$$g\Box A^{1/2}v \leqslant 2\left\{\int_0^t g^r\, d\tau \|A^{1/2}v\|^2_{C(0,\,T)}\right\}^{\frac{1}{1+(1-r)p}}\left\{g^{1+\frac{1}{p}}\Box A^{1/2}v\right\}^{\frac{(1-r)p}{1+(1-r)p}},$$

and for $r = 0$,

$$g\Box A^{1/2}v \leqslant 2\left\{\int_0^t \|A^{1/2}v(\tau)\|^2\, d\tau + t\|A^{1/2}v(t)\|^2_H\right\}^{\frac{1}{p+1}}\left\{g^{1+\frac{1}{p}}\Box A^{1/2}v\right\}^{\frac{p}{1+p}}.$$

PROOF. – From hypotheses on v and the Lemma 3.3, we have:

$$g\Box A^{1/2}v = \int_0^t g(t-\tau)\,\underbrace{(A^{1/2}v(t) - A^{1/2}v(\tau))(A^{1/2}v(t) - A^{1/2}v(\tau))}_{=h(\tau)}\, d\tau$$

$$\leqslant \left\{\int_0^t g^r(t-\tau)\, h(\tau)\, d\tau\right\}^{\frac{1}{(1-r)p+1}}\left\{\int_0^t g^{1+\frac{1}{p}}(t-\tau)\, h(\tau)\, d\tau\right\}^{\frac{(1-r)p}{(1-r)p+1}}$$

$$\leqslant \{g^r\Box A^{1/2}v\}^{\frac{1}{(1-r)p+1}}\left\{g^{1+\frac{1}{p}}\Box A^{1/2}v\right\}^{\frac{(1-r)p}{(1-r)p+1}}.$$

For $0 < r < 1$ we have

$$g^r\Box A^{1/2}v = \int_0^t g^r(t-\tau)(A^{1/2}v(t) - A^{1/2}v(\tau))(A^{1/2}v(t) - A^{1/2}v(\tau))\, d\tau$$

$$\leqslant 4\int_0^t g^r(\tau)\, d\tau \|A^{1/2}v\|^2_{C(0,\,T)}.$$

From where it follows the first inequality of Lemma 3.4. To prove the second part, let us take $r=0$. From Lemma 3.3 we have

$$1\,\Box A^{1/2}v=\int_0^t (A^{1/2}v(t)-A^{1/2}v(\tau))(A^{1/2}v(t)-A^{1/2}v(\tau))\,d\tau$$

$$\leq 2t\|A^{1/2}v(t)\|^2+2\int_0^t \|A^{1/2}v(\tau)\|^2\,d\tau\,.$$

Substitution of the above inequality into (3.17) our conclusion follows. The proof is complete. ■

From above Lemmas and taking in mind that the first order energy is bounded we have

$$g\Box A^{1/2}u\leq c_0\left(g^{1+\frac{1}{p}}\Box A^{1/2}u\right)^{\frac{(1-r)p}{1+(1-r)p}}, \tag{3.18}$$

for $0<r<1$.

LEMMA 3.5. – *Under the above conditions and* $f\in C^1([0,T[;H)$, *the solution of equation* (3.8) *satisfies the inequality*

$$\frac{d}{dt}\,\mathcal{E}(t)\leq -\frac{(g(0)-\delta)}{2}\left\{\|w_t\|^2+\|A^{1/2}w\|^2\right\}+$$

$$C_\delta\left\{g(t)\|u\|^2+g^{1+\frac{1}{p}}\Box u\right\}+\left(f,\,w_t+\frac{g(0)}{2}w\right)$$

where δ *is a positive number that will be fixed later.*

PROOF. – The only difference with the proof of the Lemma 3.2, is the estimate of the following term,

$$-\left(w_t,\int_0^t g'(t-\tau)\{u(\tau)-u(t)\}\,d\tau\right)\leq C\left\{g^{1+\frac{1}{p}}\Box u\right\}^{1/2}\|w_t\|$$

The others estimates follow using similar arguments. ■

Now we are in conditions to prove the polynomial decay.

THEOREM 3.2. – *Suppose that the initial data* (u_0,u_1) *is such that*

$$u_0\in D(A),\qquad u_1\in D(A^{1/2}),\qquad \|f\|^2\leq\frac{c}{(1+t)^{p+1}}$$

verifying (3.24), (3.25), *then the solution of equation* (6.5) *satisfies:*

$$E(t, u) \leqslant CE(0, u)(1+t)^{-p},$$

for $p > 2$.

PROOF. – As in Theorem 3.1, we arrive at the following inequality

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\kappa_0 \left\{ \underbrace{\|u_t\|^2 + \|A^{1/2}u\|^2}_{:=\mathcal{N}(t;u)} + g^{1+\frac{1}{p}} \Box A^{1/2}u \right\} + (\nu f, u_t) + \left(f, w_t + \frac{g(0)}{2} w\right).$$

Since the energy is bounded, Lemma 3.4 implies that,

$$\mathcal{N}(t) \geqslant c\mathcal{N}(t)^{\frac{1+(1-r)p}{(1-r)p}}, \qquad g^{1+\frac{1}{p}} \Box A^{1/2}u \geqslant c\{g \Box A^{1/2}u\}^{\frac{1+(1-r)p}{(1-r)p}}.$$

It is not difficult to verify that $\mathcal{L}$ satisfies:

$$c\{E(t, u)\} \leqslant \mathcal{L}(t, u) \leqslant c_1\{\mathcal{N}(t) + g\Box A^{1/2}u\}^{\frac{(1-r)p}{1+(1-r)p}}. \tag{3.19}$$

for ν large enough. From where it follows

$$\frac{d}{dt}\mathcal{L}(t, u) \leqslant -c_2 \mathcal{L}(t, u)^{\frac{1+(1-r)p}{(1-r)p}} + c\|f(t)\|^2.$$

using the hypothesis on f we get

$$\mathcal{L}(t, u) \leqslant C\mathcal{L}(0, u)\frac{1}{(1+t)^{(1-r)p}}.$$

From where it follows that energy decay uniformly to zero. From Lemma 3.4 for $r = 0$ we have

$$\mathcal{N}(t) \geqslant c\mathcal{N}(t)^{\frac{1+p}{p}}, \qquad g^{1+\frac{1}{p}} \Box A^{1/2}u \geqslant c\{g\Box A^{1/2}u\}^{\frac{1+p}{p}}.$$

Using similar reasoning as above we arrive at

$$\mathcal{L}(t, u) \leqslant C\mathcal{L}(0, u)\frac{1}{(1+t)^p}.$$

From where our conclusion follows. ∎

REMARK 3.2. – *The above result says that there exists one functional* $\mathcal{L}$ *associated to the viscoelastic system* (3.1) *satisfying*

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\kappa\mathcal{N}(t) - \frac{\nu}{2}\kappa g\Box A^{1/2}u + (\nu f, u_t) + \left(f, w_t + \frac{g(0)}{2}w\right).$$

while when the kernel decays polynomially we have

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\kappa\mathcal{N}(t) - \frac{\nu}{2}g^{1+\frac{1}{p}}\Box A^{1/2}u + (\nu f,\, u_t) + \left(f,\, w_t + \frac{g(0)}{2}w\right).$$

Note that $\mathcal{L}$ *also depends on* ν. *In fact, from equation* (3.7) *and Lemma* 3.2 *we have that*

$$\frac{d}{dt}\mathcal{L}_\nu(t) \leqslant -\left(\frac{g(0)}{2} - \delta\right)\left\{\|w_t\|^2 + \|A^{1/2}w\|^2\right\}$$

$$-(\nu - C_\delta)\left\{g(t)\|A^{1/2}u\|^2 + g\Box A^{1/2}u\right\} + (\nu f,\, u_t) + \left(f,\, w_t + \frac{g(0)}{2}w\right).$$

From remark 3.1 *it follows that*

$$\frac{d}{dt}\mathcal{L}_\nu(t) \leqslant -\left(\frac{g(0)}{2} - \delta\right)(1-\delta)\left\{\|u_t\|^2 + \left(1 - \int_0^t g\,d\tau\right)\|A^{1/2}u\|^2\right\}$$

$$-(\nu - C_\delta - c_\delta)\left\{g(t)\|A^{1/2}u\|^2 + g\Box A^{1/2}u\right\} + (\nu f, u_t) + \left(f, w_t + \frac{g(0)}{2}w\right).$$

From where we arrive at

$$\frac{d}{dt}\mathcal{L}_\nu(t) \leqslant -\left(\frac{g(0)}{2} - \delta\right)(1-\delta)\left\{\|u_t\|^2 + \left(1 - \int_0^t g d\tau\right)\|A^{1/2}u\|^2 + g\Box A^{1/2}\right\}$$

$$-\left(N - C_\delta - c_\delta + \frac{g(0)}{2} - \delta\right)\left\{g(t)\|A^{1/2}u\|^2 + g\Box A^{1/2}u\right\}$$

$$+(\nu f,\, u_t) + \left(f,\, w_t + \frac{g(0)}{2}w\right)$$

Since $\mathcal{L}$ *satisfies* (3.21), *taking* $\nu/2 > C_\delta + c_\delta - \frac{g(0)}{2} + \delta$ *we have that*

$$\frac{d}{dt}\mathcal{L}_\nu(t) \leqslant -\kappa\mathcal{L}_\nu(t) - \frac{\nu}{2}\left\{g(t)\|A^{1/2}u\|^2 + g\Box A^{1/2}u\right\} + (\nu f, u_t + \left(f, w_t + \frac{g(0)}{2}w\right).$$

Analogously when g *decays polynomially we get*

$$\frac{d}{dt}\mathcal{L}_\nu(t) \leqslant -\kappa\mathcal{N}(t) - \frac{\nu}{2}\left\{g(t)\|A^{1/2}u\|^2 + g^{1+\frac{1}{p}}\Box A^{1/2}u\right\} + (\nu f, u_t) + \left(f, w_t + \frac{g(0)}{2}w\right)$$

From where our conclusion follows. ■

4. – Global large solution and uniform rate of decay.

In this section we show the global existence of solutions for large data of the nonlinear viscoelastic system:

$$u_{tt} + Au - \int_0^t g(t-\tau)\, N([u(\tau)])\, Au\, d\tau = 0\,, \tag{4.1}$$

with initial conditions

$$u(0) = u_0; \qquad u_t(0) = u_1$$

here for $[u]$ we are denoting the functional

$$[u(t)] = \big((u(t),\, u_t(t)), (Au(t),\, u_t(t)),\, \|A^{\frac{1}{2}} u(t)\|^2,\, \|A^{\frac{1}{2}} u_t(t)\|^2,\, \|Au(t)\|^2\big) \in \mathbb{R}^5.$$

We assume that,

$$|N(s_1,\, s_2,\, s_3,\, s_4,\, s_5)| \leq C \tag{4.2}$$

REMARK 4.1. – *Let us denote by σ the number*

$$\sigma = \sup_{|\sigma| < c_2\eta} \{ |\partial^\alpha M(\sigma)|,\, |\partial^\alpha N(\sigma)|;\, |\alpha| \leq 2 \}.$$

and let us introduce the set

$$\mathcal{W}(\eta,\, T) = \{ v \in C^2(0,\, T;\, D(A^{1/2}));$$

$$v \in C(0,\, T;\, D(A^{3/2})),\ v_t \in C^1(0,\, T;\, D(A)) \ \textit{and}\ |||v||| \leq \eta \}$$

where by $|||\cdot|||$ we are denoting the norm

$$|||v|||^2 = \sup_{t \in [0,\, T]} \big\{ \|A^{3/2} v\|^2 + \|A_t^v\|^2 + \|A^{1/2} v_{tt}\|^2 \big\}.$$

Then for any $w \in \mathcal{W}(\eta,\, T)$ we have that for $F = M$ or N the following inequalities:

$$\Big| \frac{d}{dt} F([w(t)]) \Big| \leq \sigma c_3 \eta$$

$$\|(g * Aw)(t)\|^2 \leq \eta \left(\int_0^t g(\tau)\, d\tau \right)^2,$$

holds, where $c_3 > 0$ is a positive constant which depends on the embedding $D(A) \hookrightarrow D(A^{\frac{2}{1}}) \hookrightarrow H$.

In this conditions we are able to show the existence of strong solutions. We suppose that g satisfies

$$g, \qquad g' \in L^1(\mathbb{R}) \tag{4.3}$$

THEOREM 4.1. – *Suppose that N satisfies the hypotheses* (4.2) *and that the initial data satisfies:*

$$u_0 \in D(A^{3/2}), \qquad u_1 \in D(A).$$

Then there exists, only one solution of (4.1) *satisfying:*

$$u \in C^i([0, T[; D(A^{3/2 - i/2})), \qquad i = 0, 1, 2.$$

PROOF. – By the local existence result it is sufficient to show that the second order energy remains bounded for any $t > 0$. Differentiation equation (4.1) with respect to time, we get

$$u_{ttt} + Au_t - g(0)\, N([u])\, Au - \int_0^t g'(t-\tau)\, N([u])\, Au(\tau)\, d\tau = 0.$$

Multiplying the above equation by u_{tt} we have that the functional

$$\mathcal{N}(t; u_t) = \frac{1}{2}\left\{\|u_{tt}\|^2 + \|A^{1/2} u_t\|^2\right\}$$

satisfies

$$\frac{d}{dt}\mathcal{N}(t; u_t) = g(0)\, N([u])(Au, u_{tt}) + \int_0^t g'(t-\tau)\, N([u])(Au(\tau)\, d\tau, u_{tt}). \tag{4.4}$$

From equation (4.1), we have

$$\|Au(t)\|^2 \leq \|u_{tt}(t)\|^2 + C\int_0^t g(t-\tau)\|Au(\tau)\|^2 d\tau$$

and using Gronwall's Lemma we have

$$\|Au(t)\|^2 \leq \|u_{tt}\|^2 + c\int_0^t \|u_{tt}\|^2 d\tau .$$

Substitution of the above inequality into identity (4.4) we conclude that there exists a positive constant C such that:

$$\frac{d}{dt}\mathcal{N}(t;\, u_t) \leqslant C\|u_{tt}(t)\|^2 + C\int_0^t \|u_{tt}(\tau)\|^2\, d\tau\,.$$

Integrating with respect to the time and using Gronwall's Lemma we finally arrive at

$$\mathcal{N}(t;\, u_t) \leqslant \mathcal{N}(0;\, u_t) + C\int_0^t \mathcal{N}(\tau,\, u_t)\, d\tau$$

from where it follows that

$$\mathcal{N}(t;\, u_t) \leqslant \mathcal{N}(0;\, u_t)\, e^{Ct}.$$

Note that $v = A^{1/2}u$ satisfies:

$$v_{ttt} + Av_t - g(0)\, N([u])\, Av - \int_0^t g'(t-\tau)\, N([u])\, Av(\tau)\, d\tau = 0\,.$$

Repeating the above process we conclude that there exists one positive constant C such that:

$$\|A^{1/2}u_{tt}\|^2 + \|Au_t\|^2 + \|A^{3/2}u(t)\|^2 \leqslant C\{\|A^{1/2}u_2\|^2 + \|Au_1\|^2 + \|A^{3/2}u_0\|^2\}, \quad \forall t>0$$

this complete the existence result. ■

To study the uniform rate of decay of equation (4.1) we rewrite it in the following form.

$$u_{tt} + Au - \int_0^t g(t-\tau)\, Au(\tau)\, d\tau = f$$

where f is given by

$$f(t) = \int_0^t g(t-\tau)\{N([u(\tau)]) - N(0)\}\, Au(\tau)\, d\tau \tag{4.5}$$

We suppose, with out loss of generality, that $N(0) = 1$, (otherwise, put $\tilde{g} := N([0])\, g$) then hypothesis (3.4) can be written as

$$1 - \int_0^\infty g(\tau)\, d\tau = \alpha > 0\,. \tag{4.6}$$

In the next theorem we show the exponential decay of the solution.

THEOREM 4.2. – *Let us take* $g \in C^3$ *satisfying the hypotheses* (3.2)-(3.4), $\|\nabla N\| < \delta$ *and with initial data* (u_0, u_1) *in* $D(A^{3/2}) \times D(A)$. *Then the energy associated to system* (4.1), *has exponential decay.*

PROOF. – From remark 3.2, we have the functional $\mathcal{L}$ satisfies:

$$\frac{d}{dt}\mathcal{L}_\nu(t) \leqslant -\kappa \mathcal{L}_\nu(t) - \frac{\nu}{2}\left\{g(t)\|A^{1/2}u\|^2 + g\Box A^{1/2}u\right\} + (\nu f, u_t) + \left(f, w_t + \frac{g(0)}{2}w\right).$$

Note that

$$\begin{aligned}(f, u_t) &= \frac{d}{dt}(f, u) - (f_t, u)\\ &= \frac{d}{dt}(f, u) - g(0)\{N([u(t)]) - N([0])\}(Au, u)\\ &\quad - \int_0^t g'(t-\tau)\{N([u(\tau)]) - N([0])\}(A^{1/2}u, A^{1/2}u)\, d\tau\\ &\leqslant \frac{d}{dt}(f, u) + \delta\|A^{1/2}u\|^2 + c\varepsilon\left\{g\Box A^{1/2}u + \|A^{1/2}u\|^2\right\}.\end{aligned}$$

Similarly we have

$$(f, w_t) \leqslant \frac{d}{dt}(f, w) + g(t)\, c_\delta E(0) + \delta\|A^{1/2}u\|^2 + c\left\{g\Box A^{1/2}u + g(t)\|A^{1/2}u\|^2\right\}$$

where E is defined in (3.6). From the mean value inequality we have that there exists one positive constant C such that

$$|N([u]) - N([0])| \leqslant C\delta.$$

Taking $\nu/2 > c$, the functional $\mathcal{H}$ given by

$$\mathcal{H}(t) = \mathcal{L}(t) + \nu(f, u) + (f, w)$$

satisfies

$$\frac{d}{dt}\mathcal{H}(t) \leqslant -\kappa\mathcal{L}(t) + g(t)\, c_\delta E(0),$$

$$\frac{1}{2}\mathcal{L}(t) \leqslant \mathcal{H}(t) \leqslant 2\,\mathcal{L}(t).$$

From the two above inequalities, it follows that $\mathcal{L}(t) \leqslant \mathcal{L}(0)\, e^{-\gamma t}$ which that the solution decays exponentially. ■

Now we show the polynomial decay of the solution

THEOREM 4.3. – *Under the same hypotheses as Theorem* 4.2, $A \geqslant \alpha_0 > 0$, and g satisfying (3.14)-(3.16) *we have for any initial data*

$$(u_0, u_1) \in D(A^{3/2}) \times D(A)$$

that there is only one solution u *of the equation* (4.1), *that is to say*

$$E(t) \leqslant CE(0)(1+t)^{-p} \qquad \forall t \geqslant 0.$$

PROOF. – From remark 3.2, we have that the functional $\mathcal{L}$ satisfies

$$\frac{d}{dt}\mathcal{L}_\nu(t) \leqslant -\kappa\mathcal{N}(t) -$$

$$\frac{\nu}{2}\left\{g(t)\|A^{1/2}u\|^2 + g^{1+\frac{1}{p}}\Box A^{1/2}u\right\} + (\nu f, u_t) + \left(f, w_t + \frac{g(0)}{2}w\right).$$

As in the proof of Theorem 4.2 we have

$$(f, u_t) \leqslant \frac{d}{dt}(f, u) + \delta\|A^{1/2}u\|^2 + c\varepsilon\left\{g^{1+\frac{1}{p}}\Box A^{1/2}u + \|A^{1/2}u\|^2\right\}$$

$$(f, w_t) \leqslant \frac{d}{dt}(f, w) + \delta\|A^{1/2}u\|^2 + c\left\{g^{1+\frac{1}{p}}\Box A^{1/2}u + g(t)\|A^{1/2}u\|^2\right\}$$

$$\mathcal{N}(t) \geqslant c\mathcal{N}(t)^{\frac{1+(1-r)p}{(1-r)p}}, \qquad g^{1+\frac{1}{p}}\Box A^{1/2}u \geqslant c\left\{g\Box A^{1/2}u\right\}^{\frac{1+(1-r)p}{(1-r)p}}.$$

It is not difficult to verify, that taking N large enough, we have that $\mathcal{L}$ satisfies:

$$c\{E(t, u)\} \leqslant \mathcal{L}(t, u) \leqslant c_1\left\{\mathcal{N}(t) + g^{1+\frac{1}{p}}\Box A^{1/2}u\right\}^{\frac{(1-r)p}{1+(1-r)p}}. \tag{4.7}$$

Since

$$|N([u]) - N([0])| \leqslant \delta,$$

and taking $\sigma/2 > c$ the functional $\mathcal{H}$ given by

$$\mathcal{H}(t) = \mathcal{L}(t) + \nu(f, u) + (f, w)$$

verifies

$$\frac{d}{dt}\mathcal{H}(t) \leqslant -\kappa\mathcal{L}(t)^{\frac{1+(1-r)p}{(1-r)p}} + cg(t)^2.$$

Since

$$\frac{1}{2}\,\mathcal{L}(t) \leqslant \mathcal{H}(t) \leqslant 2\,\mathcal{L}(t).$$

We conclude that

$$\mathcal{L}(t, u) \leqslant C\mathcal{L}(0, u)\,\frac{1}{(1+t)^{(1-r)p}}\,.$$

From where it follows that the energy decays uniform by to zero. Finally, using similar arguments as in the proof of Theorem 3.2 our conclusion follows. ■

5. – Global solutions for small data.

In this section we study the existence of global solutions for small data and also the asymptotic behaviour of the solution to the full nonlinear problem,

$$u_{tt} + M([u])\,Au - \int_0^t g(t-\tau)\,N([u])\,Au(\tau)\,d\tau = 0\,, \tag{5.1}$$

$$u(0) = u_0, \qquad u_t(0) = u_1$$

where, M and N satisfies conditions $[V2]$, $M([0]) > 0$ and $[u]$ is given by

$$[u(t)] = \Big((u(t), u_t(t)), (Au(t), u_t(t)), \|A^{\frac{1}{2}}u(t)\|^2, \|A^{\frac{1}{2}}u_t(t)\|^2, \|Au(t)\|^2\Big) \in \mathbb{R}^5$$

The particular case $A = -\triangle$ and $[u] = \|A^{1/2}u\|^2$, was studied by Torrejon and Young [22]. The authors showed the existence of global solution, for analytical data and the asymptotic stability when $t \to \infty$. To explore the dissipative properties of equation (5.1) let us rewrite the equation in the following form,

$$u_{tt} + M([0])\,Au - N([0])\int_0^t g(t-\tau)\,Au(\tau)\,d\tau = P := R + Q, \tag{5.2}$$

where R and Q are the nonlinear term of equation

$$R(t) = \int_0^t g(t-\tau)\{N([u(\tau)]) - N(0)\}\,Au(\tau)\,d\tau$$

$$Q(t) = \{M([u(t)]) - M([0])\}\,Au(t)$$

We assume hypotheses (3.2)-(3.3) on the kernel g and instead of the hypothesis (3.4) we use hypothesis:

$$M([0]) - N([0]) \int_0^\infty g(t)\, dt > 0 \tag{5.3}$$

For simplicity and without loss of generality, we suppose that $M(0) = 1$, $N(0) = 1$, (otherwise we make the change of variables $t \mapsto \sqrt{M([0])}\, t$, and put $\hat{g} := \frac{M([0])}{N([0])} g$) then hypothesis (5.3) may be written as,

$$1 - \int_0^\infty g(\tau)\, d\tau = \alpha > 0\,. \tag{5.4}$$

Rewriting equation (5.2) we have

$$u_{tt} + Au - \int_0^t g(t-\tau)\, Au(\tau)\, d\tau = P := R + Q\,, \tag{5.5}$$

THEOREM 5.1. – *Let us suppose that hypotheses* [V1] *and* [V2] *holds and let us take g satisfying* (3.2)-(3.4). *Consider $\varepsilon > 0$ such that the initial data*

$$(u_0, u_1) \in D(A^{3/2}) \times D(A)$$

satisfies

$$\|A^{\frac{3}{2}} u_0\|^2 + \|Au_1\|^2 < \varepsilon\,.$$

Then, there exist, only one solution u of equation (5.1), *such that*

$$u \in C^2([0, \infty[, D(A^{1/2})) \cap C^1([0, \infty[, D(A)) \cap C([0, \infty[, D(A^{3/2})).$$

In addition, we have that the energy $E(t)$ defined in (3.6) *satisfies*

$$E(t) \leqslant E(0)\, e^{-\gamma t} \qquad \forall t \geqslant 0 \quad e \quad \gamma > 0.$$

PROOF. – Applying the operator A to equation (5.5) and using remark 3.2 for $v = Au$ we have

$$\frac{d}{dt} \mathcal{L}_\nu(t, Au) \leqslant -\kappa \mathcal{L}_\nu(t, Au) - \frac{\nu}{2} \left\{ g(t)\|A^{3/2}u\|^2 + g \Box A^{3/2} u \right\} + \tag{5.6}$$

$$+ \nu(AP, Au_t) + \left(AP, Aw_t + \frac{g(0)}{2} Aw \right)$$

Since M and N are continuous functions, for all $\delta > 0$, there exists $\varepsilon > 0$, such that

$$|\sigma|_{\mathbb{R}^5} < c_2\,\varepsilon \Rightarrow |M(\sigma) - M([0])| < \delta \quad \text{and} \quad |N(\sigma) - N([0])| < \delta\,.$$

From Theorem 2.1, there exists $0 < T_0 \leqslant T_{\max}$, such that

$$\mathfrak{M}(t, Au) := \|A^{\frac{3}{2}} u(t)\|^2 + \|Au_t\|^2 + g\Box A^{3/2} u \leqslant d\varepsilon \text{ em } [0, T_0[\,.$$

where $d \geqslant 1$ and will be fixed later. Let us consider

$$T^* = \sup\left\{T_1^* > 0 : E(t) \leqslant d\varepsilon \quad \text{in} \quad [0, T_1^*[\right\}$$

We have two cases: (i) $T^* = T_{\max}$, (ii) $T^* < T_{\max}$. The first one implies that the solution u is bounded so, we have $T_{\max} = \infty$. Hence, we only consider case (ii). Suppose that $T^* < T_{\max}$ and $T_{\max} < \infty$ then we have

$$|[u(t)]|_{\mathbb{R}^5} < c_2\,\varepsilon \Rightarrow |M([u]) - M([0])| < \delta \qquad (5.7)$$

$$\text{and} \quad |N([u]) - N([0])| < \delta \text{ in } [0, T^*[\,.$$

Denoting by α_1 the expression,

$$\alpha_1 = \max_{|s| \leqslant c_2\varepsilon} \left\{ \frac{\partial M}{\partial x_i}(s) : i = 1, 2, 3, 4, 5 \right\}.$$

From remark 4.1 we have that

$$\left| \frac{d}{dt}\{M([0]) - M([u])\} \right| \leqslant \qquad (5.8)$$

$$2\alpha_1 \left\{\|A^{1/2} u_{tt}(t)\| + \|A^{3/2} u(t)\| + \|Au_t(t)\|\right\} \leqslant c_3\,\varepsilon\,,$$

note that

$$(AQ, Au_t) = \{M([u]) - M([0])\}(A^2 u, Au_t)$$

$$= -\frac{1}{2}\left(\frac{d}{dt}\{M([u]) - M([0])\}\right)\|A^{3/2} u\|^2$$

$$+ \frac{1}{2}\frac{d}{dt}\left(\{M([u]) - M([0])\}\|A^{3/2} u\|^2\right)$$

$$\leqslant c_3\,\delta\|A^{3/2} u\|^2 + \frac{1}{2}\frac{d}{dt}\left(\{M([u]) - M([0])\}\|A^{3/2} u\|^2\right)$$

$$
\begin{aligned}
(AR, Au_t) &= \frac{d}{dt}(AR, Au) - (AR_t, Au) \\
&= \frac{d}{dt}(AR, Au) - g(0)\{N([u(t)]) - N([0])\}(A^2 u, Au) \\
&\quad + \int_0^t g'(t-\tau)\{N([u(\tau)]) - N([0])\}\, A^2 u(\tau)\, d\tau\, Au \\
&\leqslant \frac{d}{dt}(AR, Au) + \delta g(0)\|A^{3/2}u\|^2 + \delta C\left\{g\Box A^{3/2}u + \|A^{3/2}u\|^2\right\}.
\end{aligned}
$$

From where it follows that

$$
(AP, Au_t) \leqslant c_3\,\varepsilon\left\{\|A^{3/2}u\|^2 + g\Box A^{3/2}u\right\} + \frac{d}{dt}\left\{\frac{1}{2}\left\{M([u]) - M([0])\right\}\|A^{3/2}u\|^2 + (AR, Au)\right\}.
$$

Similarly, we have

$$
\left(AP, Aw_t + \frac{1}{2}Aw\right) \leqslant c_3\,\delta\left\{\|A^{3/2}u\|^2 + g\Box A^{3/2}u\right\} + \frac{1}{2}\frac{d}{dt}\left(\{M([u]) - M([0])\}\|A^{3/2}u\|^2\right) + \frac{d}{dt}(AR, Au).
$$

Denoting by

$$
\mathcal{S}(t) = 2\{M([u]) - M([0])\}\|A^{3/2}u\|^2 + 2(AR, Au).
$$

From (5.6) and taking ε and δ small enough it follow that

$$
\frac{d}{dt}\{\mathcal{L}(t) - \mathcal{S}(t)\} \leqslant -\frac{\kappa}{2}\mathcal{L}(t);
$$

$$
|\,\mathcal{S}(t)\,| < \frac{c_0}{2}\,\delta\,\mathcal{M}(t).
$$

Recalling the definition of $\mathcal{S}$ we have

$$
\frac{1}{2}\,\mathcal{M}(t) \leqslant \mathcal{L}(t) - \mathcal{S}(t) \leqslant 2\,\mathcal{M}(t), \tag{5.9}
$$

and

$$
\frac{d}{dt}\{\mathcal{L}(t) - \mathcal{S}(t)\} \leqslant -\frac{\kappa}{2}\{\mathcal{L}(t) - \mathcal{S}(t)\},
$$

which implies that

$$\mathcal{L}(t) - \mathcal{S}(t)) \leqslant \{\mathcal{L}(0) - \mathcal{S}(0)\}\, e^{-\gamma t},$$

where $\gamma = \frac{\kappa}{2}$. From above inequality together with (5.9) we have

$$\mathcal{M}(t) \leqslant \frac{2}{c_0}\{\mathcal{L}(0) - \mathcal{S}(0)\}\, e^{-\gamma t} \leqslant \frac{4c_1}{c_0}\mathcal{M}(0)\, e^{-\gamma t} \qquad \forall t \in [0, T^*[.$$

The next step is to show that $T_{\max} = \infty$. To do it we reason by contradiction. Let us suppose that $T^* < T_{\max} < \infty$ and that $T^* = T_1^*$. Thus we have

$$\mathcal{M}(t) \leqslant d\,\mathcal{M}(0)e^{-\gamma t} < d\varepsilon e^{-\gamma t}. \tag{5.10}$$

Letting $t \to T^* = T_1^*$, it follows that

$$\mathcal{M}(T_1^*) \leqslant de^{-\gamma T_1^*}\varepsilon < d\varepsilon,$$

which is a contradiction to the maximality of T_1^*. Hence, $T_{\max} = \infty$ so, and the solution is global in time. From where our conclusion follows. ■

6. – Analytical solutions.

In this section we deal with the existence of analytical solutions for systems (1.1) with M and N satisfying the hypotheses of section 1. We prove the existence of solutions for large A-Gevrey vector data. The idea is to use the local existence result and show that the expression

$$\|A^{1/2}u_t(t)\|^2 + \|Au(t)\|^2 + \|u_{tt}(t)\|^2$$

remains bounded for any $t > 0$. To do this we introduce the concept of Gevrey function (also known as analytical function). The existence of solutions for large non Gevrey data is an open problem.

A field of Hilbert spaces is an applications $\lambda \mapsto \mathcal{H}(\lambda)$ defined on $\mathbb{R}$, where $\mathcal{H}(\lambda)$ is a Hilbert space. A vectorial field over $\mathbb{R}$ is an application $\lambda \mapsto u(\lambda)$ such that $u(\lambda) \in \mathcal{H}(\lambda)$, for any $\lambda \in \mathbb{R}$. Let us denote by $\mathcal{F}$ the vectorial space given by all the vectors over $\mathbb{R}$ and let us denote by μ a measure over $\mathbb{R}$.

DEFINITION 6.1. – *A field of Hilbert spaces $\lambda \mapsto \mathcal{H}(\lambda)$ is called μ-measurable when there exists a subspace $\mathcal{M} \neq \emptyset$ of $\mathcal{F}$, satisfying the following conditions*

i) The application $\lambda \mapsto \|u(\lambda)\|_{\mathcal{H}(\lambda)}$ is μ-measurable for any $u \in \mathcal{M}$;

ii) For $u \in \mathcal{F}$ the function $\lambda \mapsto (u(\lambda), v(\lambda))_{\mathcal{H}(\lambda)}$ is μ-measurable for any field $v \in \mathcal{M}$, then $u \in \mathcal{M}$;

the elements of $\mathcal{M}$ are called μ-measurable vector field

DEFINITION 6.2. – *A field u of vectors over $\mathbb{R}$ is called square integrable with respect to a measure μ when*

$$\int_{\mathbb{R}} \|u(\lambda)\|^2_{\mathcal{H}(\lambda)}\, d\mu(\lambda) < \infty .$$

Denoting by $\mathcal{H}_0$ the vectorial space given by the vector field square integrable with respect to μ. Let us define in $\mathcal{H}_0$ the following inner product

$$(u, v)_0 = \int_{\mathbb{R}} (u(\lambda), v(\lambda))_{\mathcal{H}(\lambda)}\, d\mu(\lambda), \qquad u, v \in \mathcal{H}_0 .$$

Then the Hilbert space $\mathcal{H}_0$ is called Hilbertian Integral of the spaces field $\lambda \mapsto \mathcal{H}(\lambda)$, and it is denoted by $\mathcal{H}_0 = \int^{\oplus} \mathcal{H}(\lambda)\, d\mu(\lambda)$. Under this conditions we have the following result

THEOREM 6.1. – *Let H be a Hilbert space and let us denote by A a self adjoint operator, positive definite in H, then there exists a bounded positive measure ν over $\mathbb{R}$, a Hilbert space $\mathcal{H}_0 = \int^{\oplus} \mathcal{H}(\lambda)\, d\nu(\lambda)$ and a unitary operator $\mathcal{U}$ from H over $\mathcal{H}_0$, satisfying the following properties*

i) $\mathcal{U}(A^{\alpha} u) = \lambda^{\alpha}\, \mathcal{U}(u)$, $\forall u \in D(A^{\alpha})$, $\alpha \geq 0$,

ii) $\mathcal{U}$ *is an isomorphism from* $D(A^{\alpha})$ *over* $\mathcal{H}_{\alpha}$, $\alpha \geq 0$.

PROOF. – See [10] e [13].

DEFINITION 6.3. – *A function $v \in H$ is called $A(\kappa)$-Gevrey vectors of order $\kappa > 0$ if exists one positive constant γ satisfy the following property:*

$$\int_0^{\infty} e^{\gamma \lambda^{\kappa}}\, |\, \mathcal{U} v(\lambda)\, |^2\, d\mu(\lambda) < \infty , \tag{6.1}$$

where $\mathcal{U}$ is the given unit operator in $\mathcal{H}$. We say that the function v is A-Gevrey vector if it is $A(1)$-Gevrey vector of order one.

The idea we use, is a combination of Arosio and Spagnolo's method [1] and the continuation of local solutions, exploring the dissipative properties of the memory effect. For the case $A = -\Delta$ and periodic boundary conditions in bounded domain of $\mathbb{R}^n$ then A-Gevrey vector is a classic analytical functions, see [1]. Before to proof the existence of solutions, we show the following Lemma.

LEMMA 6.1. – *Let us take* $n \in C(\mathbb{R})$, $g \in C^1(\mathbb{R})$ *and* $v \in C^1([0, T[; \mathcal{H})$, *then we have that:*

$$\left(\int_0^t g(t-\tau)\, n(\tau)\, v(\tau)\, d\tau,\ v_t \right)$$

$$= -\frac{1}{2}\frac{d}{dt}\left\{ \int_0^t g(t-\tau)\, n(\tau)\, |v(t)-v(\tau)|^2 d\tau \right\} + \frac{1}{2}\int_0^t g'(t-\tau)\, n(\tau)\, |v(t)-v(\tau)|^2 d\tau$$

$$+ \frac{1}{2}\frac{d}{dt}\left\{ \int_0^t g(t-\tau)\, n(\tau)\, d\tau\, |v|^2 \right\} - \frac{1}{2}\left\{ g(0)\, n(t) + \int_0^t g'(t-\tau)\, n(\tau)\, d\tau \right\} |v|^2$$

PROOF. – Differentiation the expression

$$\frac{d}{dt}\left\{ \int_0^t g(t-\tau)\, n(\tau)\, |v(t)-v(\tau)|^2 d\tau \right\}$$

we arrived to the required identity. ■

THEOREM 6.2. – *Suppose that the initial data* u_0, u_1 *are A-Gevrey vectors, and that* $M(t) \geqslant C > 0$ *and* $N(t) \geqslant 0$ *are bounded functions, then there exists only one global solution of system* (5.1) *satisfying,*

$$u \in C^1([0, T[; \mathcal{A})$$

where $\mathcal{A}$ *is a set of all Gevrey vector functions*

PROOF. – For $\delta > 0$ denote j_δ a Friedrich's regularization with support on $[-\delta, \delta]$. Let us denote by u the local solution and

$$M_\delta(r) := (M * j_\delta)(r) = \int_{\mathbb{R}} M(s)\, j_\delta(r-s)\, ds\,,$$

where $M(t) := M([u(t)])$ is the extension of M for negative values of t. Similarly for N_δ, denoting by $v := \mathcal{U}u$, then it follows that v satisfies:

$$v_{tt} + M_\delta \lambda v + \int_0^t g(t-\tau)\, N_\delta \lambda v(\tau)\, d\tau = \tag{6.2}$$

$$(M_\delta - M)\, \lambda v + \int_0^t g(t-\tau)(\nu_\delta - \nu)\, \lambda v(\tau)\, d\tau\,.$$

Multiplying (6.2) by v_t and using the Lemma 6.1 we get

$$\frac{d}{dt}\left\{ |v_t|^2+\lambda M_\delta |v|^2+\lambda\int_0^t g(t-\tau)\, N_\delta |v(t)-v(\tau)|^2 d\tau-\lambda\int_0^t g(t-\tau)\, N_\delta(\tau)\, d\tau\, |v|^2\right\}=$$

$$\lambda M_\delta' \,|v|^2+(M_\delta-M)\,\lambda v v_t+\int_0^t g(t-\tau)(N_\delta-N)\,\lambda v\, d\tau v_t$$

$$+\int_0^t g'(t-\tau)\, N_\delta \lambda |v(t)-v(\tau)|^2 d\tau-\lambda\left(g(0)\, N_\delta\, |v|^2-\int_0^t g'(t-\tau)\, N_\delta(\tau)\, d\tau\right)|v|^2.$$

Introducing the following functionals

$$E_\delta(t,\lambda)=|v_t|^2+\lambda M_\delta\,|v|^2+\lambda\int_0^t g(t-\tau)\, N_\delta\,|v(t)-v(\tau)|^2 d\tau$$

$$E(t,\lambda)=|v_t|^2+\lambda|v|^2+\lambda\int_0^t g(t-\tau)\, N_\delta\,|v(t)-v(\tau)|^2 d\tau,$$

we have that there exist positive constants such that

$$c_0 E(t,\lambda)\leqslant E_\delta(t,\lambda)\leqslant c_1 E(t,\lambda)$$

for δ small. From the above inequalities we have,

$$\frac{d}{dt}\left\{E_\delta(t,\lambda)-\lambda\int_0^t g(t-\tau)\, N_\delta(\tau)\, d\tau |v|^2\right\}\leqslant$$

$$\varepsilon\lambda|v|\,|v_t|+\varepsilon\lambda\int_0^t g(t-\tau)\,|v| d\tau|v_t|+c_\delta E_\delta(t,\lambda).$$

Integrating in time we arrive at

$$E(t,\lambda)\leqslant CE(0,\lambda)+\int_0^t\left(\frac{c}{\delta^2}+\varepsilon\lambda\right)E(s,\lambda)\, ds\,.$$

From Gronwall's inequality we have

$$E(t,\lambda)\leqslant cE(0,\lambda)\, e^{cT_*/\delta^2} e^{\varepsilon\lambda T_*}\qquad t\in[0,T_*) \tag{6.3}$$

Using that

$$\lambda^m\leqslant\frac{m!}{\varepsilon^m}e^{\varepsilon\lambda},\qquad m\in\mathbb{N}$$

we get

$$(6.4) \qquad \int_0^\infty \lambda^m E(t,\lambda)\, d\mu(\lambda) \leq c(\gamma, T_*) \int_0^\infty e^{\gamma\lambda} E(0,\lambda)\, d\mu(\lambda),$$

From where the right hand side of the above equation is finite by hypothesis. Therefore from Theorem 2.1 it follows the existence of global solution.

REMARK 6.1. – *From inequality* (6.3) *we deduce that the solution is also A-Gevray. In fact, multiplying the inequality* (6.3) *by* $e^{\varepsilon\lambda}$ *we have:*

$$e^{\varepsilon\lambda} E(t,\lambda) \leq cE(0,\lambda) e^{cT_*/\delta^2} e^{\varepsilon\lambda(T_*+1)}.$$

Taking $\varepsilon = \gamma/(T_*+1)$ *our conclusion follows.* ∎

COROLLARY 6.1. – *Suppose that* M *and* N *be functions satisfying:*

$$M, N : \mathbb{R}^+ \to \mathbb{R}^+$$

$$sN(s) \leq C \int_0^s M(\sigma)\, d\sigma .$$

Let us consider $[u] = \|A^{1/2}u\|^2$. *In this conditions we have for any initial data* u_0, u_1 *A-Gevray, that there exists only one global A-Gevray solution of system* (5.1).

PROOF. – By Theorem 6.2 it is enough to show that $M([u])$ and $N([u])$ are bounded. To do this let us, multiply equation (5.1) by u_t so we have:

$$\frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2 + \widehat{M}(\|A^{1/2}u(t)\|^2)\right\} =$$

$$\left(\int_0^t g(t-\tau)\, N(\|A^{1/2}u(\tau)\|^2)\, A^{1/2}u(\tau)\, d\tau,\, A^{1/2}u_t(t)\right)$$

where $\widehat{M}(s) = \int_0^s M(\sigma)\, d\sigma$. Applying Lemma 6.1 to the above expression we have

$$\frac{d}{dt}\left\{E(t) - 1/2\int_0^t g(t-\tau)\, N(\|A^{1/2}u(\tau)\|^2)\, d\tau \|A^{1/2}u(t)\|^2\right\} =$$

$$\frac{1}{2}\int_0^t g'(t-\tau)\, N(\|A^{1/2}u(\tau)\|^2) \|A^{1/2}u(t) - A^{1/2}u(\tau)\|^2\, d\tau$$

$$-1/2\left\{g(0)\, N(\|A^{1/2}u(t)\|^2) + \int_0^t g'(t-\tau)\, N(\|A^{1/2}u(\tau)\|^2)\, d\tau\right\} \|A^{1/2}u(t)\|^2$$

where

$$E(t) = \frac{1}{2}\left\{\|u_t(t)\|^2 + \widehat{M}(\|A^{1/2}u(t)\|^2) + \int_0^t g(t-\tau)\, N(\|A^{1/2}u(\tau)\|^2)\|A^{1/2}u(t) - A^{1/2}u(\tau)\|^2\, d\tau\right\}.$$

Note that

$$-\int_0^t g'(t-\tau)\, N(\|A^{1/2}u(\tau)\|^2)\, d\tau \|A^{1/2}u(t)\|^2 \leqslant$$

$$-2\int_0^t g'(t-\tau)\, N(\|A^{1/2}u(\tau)\|^2)\|A^{1/2}u(t) - A^{1/2}u(\tau)\|^2\, d\tau$$

$$-\int_0^t g'(t-\tau)\, N(\|A^{1/2}u(\tau)\|^2)\|A^{1/2}u(\tau)\|^2\, d\tau\, .$$

Using the hypotheses on M and N we have that there exists one constant C, such that

$$-\int_0^t g'(t-\tau)\, N(\|A^{1/2}u(\tau)\|^2)\, d\tau \|A^{1/2}u(t)\|^2 \leqslant$$

$$2\int_0^t |g'(t-\tau)|N(\|A^{1/2}u(\tau)\|^2)\|A^{1/2}u(t) - A^{1/2}u(\tau)\|^2\, d\tau$$

$$+2C\int_0^t |g'(t-\tau)|\, \widehat{M}(|A^{1/2}u(t)|^2)\, d\tau$$

from where it follows that there exists a positive constant such that:

$$E(t) \leqslant E(0) + C\int_0^t E(\tau)\, d\tau\, .$$

So, we have that $E(t)$ is bounded. Which implies that $M([u(t)])$ and $N([u(t)])$ are also bounded. Therefore applying the Theorem 6.2, our conclusion follows. ■

Using similar methods as in section 2 and 4 we are able to show the uniform rate of decay of the solutions of

$$u_{tt} + M(\|A^{1/2}u\|^2)\,Au - \int_0^t g(t-\tau)\,Au(\tau)\,d\tau = 0, \tag{6.5}$$

$$u(0) = u_0, \qquad u_t(0) = u_1,$$

for large initial data and arbitrary function M satisfying:

$$m_0 \leqslant M(s), \quad \forall s \geqslant 0, \qquad m_0 - \int_0^t g(\tau)\,d\tau > 0. \tag{6.6}$$

Therefore we have:

THEOREM 6.3. – *Suppose that g satisfies hypotheses* (3.2)-(3.4), *then the solution of equation* (6.5) *decays exponentially, that is*

$$\|u_t(t)\|^2 + \|A^{1/2}u(t)\|^2 + g\Box A^{1/2}u(t) \leqslant c\left\{\|u_1\|^2 + \|A^{1/2}u_0\|^2\right\} e^{-\gamma t}$$

while when g satisfies the hypotheses (3.15)-(3.16), *then solution decays polynomially, that is*

$$E(t) \leqslant CE(0)(1+t)^{-p} \qquad \forall t \geqslant 0. \qquad \blacksquare$$

COROLLARY 6.2. – *With the same hypothesis as in Theorem* 6.3, *we have that*

$$\|A^l u_t(t)\|^2 + \|A^{l+1/2}u(t)\|^2 + g\Box A^l u \leqslant C\left\{\|A^l u_1\|^2 + \|A^{l+1/2}u_0\|^2\right\} e^{-\gamma' t}$$

PROOF. – Note that $v = A^l u$, satisfies the equation

$$v_{tt} + Av - \int_0^t g(t-\tau)\,Av(\tau)\,d\tau = \left(1 - M(\|A^{1/2}u\|^2)\right) Av.$$

Repeating the same arguments used in the proof of Lemma 3.2, we have:

$$\frac{d}{dt}\mathcal{L}(t) \leqslant (Au, u_t)\|Av\|^2 - c_0\mathcal{L}(t).$$

Since Av is bounded and u_t decays exponentially, our conclusion follows. $\blacksquare$

7. – Large solutions in $\mathcal{D}(A)$ spaces.

In this section we show that for a class of initial data in $\mathcal{D}(A)$, there exists global solution for large initial data. Suppose that a function M has the following form:

$$M(s) = 1 + s$$

therefore equation (6.5) may be written as:

$$u_{tt} + (1 + \|A^{1/2} u\|^2)\, Au - \int_0^t g(t-\tau)\, Au(\tau)\, d\tau = 0, \tag{7.1}$$

$$u(0) = u_0, \qquad u_t(0) = u_1$$

Denoting by v_0 and v_1 initial data in $D(A) \times D(A^{1/2})$ we have that there exists only one local solution of equation (7.1).

THEOREM 7.1. – *Suppose that* $(u_0, u_1) \in D(A) \times D(A^{1/2})$ *and* g *satisfy* (3.2)-(3.4). *Then exists* $T > 0$ *and only one function*

$$u : [0, T] \to H$$

solution of (7.1), *satisfying:*

$$u \in C^2([0, T], H) \cap C^1([0, T], D(A^{1/2})) \cap C([0, T], D(A)).$$

Also by Corollary 6.1, for arbitrary A-Gevrey data u_0 and u_1 we have that there exists only one A-Gevrey solution u of (7.1). We will be prove that when v_0 and v_1 are close enough to u_0 and u_1 respectively, then v is bounded uniformly in the norm of $D(A)$, this implies the existence of global solution. We summarized this result in the following theorem.

THEOREM 7.2. – *Let us denote by* $(u_0, u_1) \in D(A) \times D(A^{1/2})$ *and* g *satisfy* (3.2)-(3.4), *such that*

$$\|Au_0 - Av_0\|^2 + \|A^{1/2} u_1 - A^{1/2} v_1\|^2 \leqslant \varepsilon$$

with v_0 *and* v_1 *A-analytical data and* ε *a small positive number. Then the local solution* u *of* (7.1) *is globally defined.*

PROOF. – Denoting by $U = u - v$. Since

$$\|A^{1/2} u\|^2 = \|A^{1/2} U\|^2 + \|A^{1/2} v\|^2 + 2(A^{1/2} U, A^{1/2} v).$$

and recalling that u and v are solutions of equation (7.1), we have that U satisfies:

$$U_{tt} + (1 + \|A^{1/2} U\|^2)\, AU - \int_0^t g(t-\tau)\, AU(\tau)\, d\tau = R, \tag{7.2}$$

$$U(0) = U_0 = u_0 - v_0, \qquad U_t(0) = U_1 = u_1 - v_1$$

where

$$R = -\underbrace{\left\{\|A^{1/2}v\|^2 + 2(A^{1/2}U, A^{1/2}v)\right\}}_{R_1} AU - \underbrace{\left\{\|A^{1/2}U\|^2 + 2(A^{1/2}U, A^{1/2}v)\right\}}_{R_2} Av.$$

Rewriting the equation (7.2) we get

$$U_{tt} + AU - \int_0^t g(t-\tau)\, AU(\tau)\, d\tau = -\|A^{1/2}U\|^2 AU + R := P. \tag{7.3}$$

From the hypotheses and the continuity of solutions it follows that there exists $T_1 < T_{\max}$ for which we have

$$\mathcal{N}(t, A^{1/2}U) := \|AU(t)\|^2 + \|A^{1/2}U_t(t)\|^2 + g\Box AU \leqslant d\varepsilon \quad t \in [0, T_1] \tag{7.4}$$

where $d > 1$ is a number to be fixed later. To show that $T_{\max} = \infty$, we reason by contradiction, suppose that $T_{\max} < \infty$ and denoting by $T^* < T_{\max}$

$$T^* = \sup\left\{T_1;\ \mathcal{N}(t, A^{1/2}U) \leqslant d\varepsilon, \quad t \in [0, T_1]\right\}.$$

From remark 3.2, we have that

$$\frac{d}{dt}\mathcal{L}_\nu(t, A^{1/2}U) \leqslant -\kappa\mathcal{L}_\nu(t, A^{1/2}U) - \frac{\nu}{2}\left\{g(t)\|AU\|^2 + g\Box AU\right\}$$

$$+ (NA^{1/2}P, A^{1/2}U_t) + \left(A^{1/2}P, A^{1/2}W_t + \frac{g(0)}{2}A^{1/2}W\right)$$

where $W = U - g * U$. Thus we have

$$(\|A^{1/2}U\|^2 A^{3/2}U, A^{1/2}U_t) = \frac{1}{2}\|A^{1/2}U\|^2 \frac{d}{dt}\|AU\|^2$$

$$= \frac{1}{2}\frac{d}{dt}\left\{\|A^{1/2}U\|^2\|AU\|^2\right\} - \frac{1}{2}\left\{\frac{d}{dt}\|A^{1/2}U\|^2\right\}\|AU\|^2$$

$$\leqslant \frac{1}{2}\frac{d}{dt}\left\{\|A^{1/2}U\|^2\|AU\|^2\right\} + d\varepsilon\|AU\|^2.$$

On the other hand

$$(A^{1/2}R, A^{1/2}U_t) = -\frac{1}{2}R_1\frac{d}{dt}\|AU\|^2 - R_2(A^{3/2}v, A^{1/2}U_t)$$

$$= \frac{1}{2}\left(\frac{d}{dt}R_1\right)\|AU\|^2 - \frac{1}{2}\frac{d}{dt}(R_1\|AU\|^2)$$

$$+ C\varepsilon\|A^{3/2}v\|\left\{\|A^{1/2}U_t\| + \|A^{1/2}U\|\right\}.$$

Since

$$\left| \frac{d}{dt} R_1 \right| \leq 2\|A^{1/2} v_t\|^2 + 2\|A^{1/2} v\|^2 + \underbrace{\|A^{1/2} U_t\|^2 + \|A^{1/2} U\|^2}_{\leq d\varepsilon},$$

it is easy to see that

$$(A^{1/2} R, A^{1/2} U_t) \leq -\frac{1}{2}\frac{d}{dt}(R_1 \|AU\|^2) + C\varepsilon\mathcal{N}(t, A^{1/2} v) + C\varepsilon\mathcal{N}(t, A^{1/2} U).$$

Similar we have that

$$\left(A^{1/2} P, A^{1/2} W_t + \frac{g(0)}{2} A^{1/2} W\right) \leq$$

$$\frac{1}{2}\frac{d}{dt}\left\{\|A^{1/2} U\|^2 \|AU\|^2 - R_1 \|AU\|^2\right\} + C\varepsilon\mathcal{N}(t, A^{1/2} v)$$

$$+ C\varepsilon\mathcal{N}(t, A^{1/2} U) + C\left\{g\Box A^{1/2} U + g(t)\|A^{1/2} U\|^2\right\}.$$

From where it follows

$$\frac{d}{dt}\left\{\mathcal{L}_\nu(t, A^{1/2} U) - \|A^{1/2} U\|^2 \|AU\|^2 + R_1 \|AU\|^2\right\} \leq$$

$$-\kappa\mathcal{L}_\nu(t, A^{1/2} U) + C\varepsilon\mathcal{N}(t; A^{1/2} v).$$

Using relation (7.4) and recalling that $R_1 \geq 0$ we have that

$$\frac{1}{2}\mathcal{L}_\nu(t, A^{1/2} U) \leq \mathcal{L}_\nu(t, A^{1/2} U) - \|A^{1/2} U\|^2 \|AU\|^2 + R_1 \|AU\|^2 \leq 2\,\mathcal{L}_\nu(t, A^{1/2} U)$$

for ε small enough. From Theorem 6.3, v decay exponential, therefore there exist positive constants C and γ_2 such that

$$\mathcal{L}_\nu(t, A^{1/2} U) \leq \mathcal{L}_\nu(0, A^{1/2} U)\, e^{-\frac{\kappa}{2} t} + C\varepsilon\mathcal{N}(0; A^{1/2} v)\, e^{-\gamma_2 t}$$

$$\leq \varepsilon\big(C + \mathcal{N}(0; A^{1/2} v)\big)\, e^{-\gamma t}$$

Since

$$c_0\,\mathcal{N}(t, A^{1/2} U) \leq \mathcal{L}_\nu(t, A^{1/2} U) \leq c_1\,\mathcal{N}(t, A^{1/2} U)$$

we have that

$$\mathcal{N}(t, A^{1/2} U) \leq \varepsilon c_0\big(C + \mathcal{N}(0; A^{1/2} v)\big)\, e^{-\gamma t}.$$

Taking $d = c_0 \left\{ C + \mathcal{N}(0; A^{1/2} v) \right\}$ and letting $t \to T^*$ we have

$$\mathcal{N}(T^*, A^{1/2} U) \leqslant d\varepsilon e^{-rT^*} < d\varepsilon ,$$

but this is contradictory to the maximality of $T_{\max}$. So we have that $T_{\max} = \infty$, hence u is globally defined and since v decays exponentially we have

$$\|Au\|^2 + \|A^{1/2} u_t\|^2 \leqslant Ce^{-\gamma t},$$

from where our conclusion follows, the proof is now complete. ■

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