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On a Mathematical Model for the Crystallization of Polymers.

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Sunto. – *Consideriamo il modello matematico per la cristallizzazione dei polimeri proposto in* [1], *che descrive l'evoluzione della temperatura, la frazione di volume cristallina, il numero e la posizione dei cristalli. Fondamentalmente, il modello è un sistema di equazioni alle derivate parziali di tipo evolutivo sia di primo che di secondo ordine, con termini non lineari, che sono o Lipschitziani, come in* [1], *o Hölderiani, come in* [3]. *La novità principale in questo articolo è che la funzione* W_{eq} *dipende dalla temperatura (come è nella realtà). Analizzeremo il modello in due diverse condizioni: nel caso di equazioni di tipo non Lipschitziano sfruttiamo la proprietà di monotonicità e la tecnica* L^1 *per provare esistenza e dipendenza continua dai dati della soluzione debole. Per equazioni con più regolarità, usando la tecnica del punto fisso, otterremo risultati di esistenza globale ed unicità per una soluzione classica del problema.*

Summary. – *We consider a mathematical model proposed in* [1] *for the cristallization of polymers, describing the evolution of temperature, crystalline volume fraction, number and average size of crystals. The model includes a constraint* W_{eq} *on the crystal volume fraction. Essentially, the model is a system of both second order and first order evolutionary partial differential equations with nonlinear terms which are Lipschitz continuous, as in* [1], *or Hölder continuous, as in* [3]. *The main novelty here is the fact that* W_{eq} *is a function depending on the temperature* T *(which is actually the case). We analyse the model in two different conditions: for constitutive equations of non-Lipschitz type, we use monotonicity and* L^1*-technique to prove existence and continuous dependence on the data of a weak solution. For more regular constitutive equations, using a fixed point tecnique, we prove a global existence and uniqueness result for a classical solution.*

1. – Introduction.

It is known that phase change in polymers is a complex process exhibiting several peculiarities:

(i) polymers cristallize not as pure crystals, but in structures having a roughly spherical lamellar organization with amorphous inclusions (spherulites),

(ii) spherulites can nucleate and grow in a temperature interval (T_g, T_m) depending on pressure, with a rate also depending on temperature and pressure (T_m is the melting temperature and T_g is the glassy transition temperature),

(iii) a maximum attainable crystal volume fraction (W_{eq}) is defined at any given temperature $T \in (T_g, T_m)$.

The crystallization model we are going to consider is a system coupling the thermal balance equation, with the crystallization kinetics, written as a system of four differential equations.

The temperature T solves:

$$\varrho c \frac{\partial T}{\partial t} - k\Delta T = \lambda \frac{\partial \chi}{\partial t} \tag{1.1}$$

where ϱ is the density, c is the specific heat and k the thermal conductivity. λ is the latent heat of crystallization times the density and χ is the crystalline volume fraction. The term $\lambda \frac{\partial \chi}{\partial t}$ expresses the heat released by the crystallization process.

χ obeys the inequality $\chi \leq W_{eq}$ and when $\chi < W_{eq}$ the crystallization rate is expressed by:

$$\frac{\partial \chi}{\partial t} = 4\pi \int_0^t F(x, t) \left(\int_\tau^t G(x, s)\, ds \right)^2 d\tau \tag{1.2}$$

where:

$$F(T, \chi) = \left(1 - \frac{\chi}{W_{eq}(T)}\right)^p F_0(T) \tag{1.3}$$

$$G(T, \chi) = \left(1 - \frac{\chi}{W_{eq}(T)}\right)^q G_0(T) \tag{1.4}$$

are respectively the spherulite growth rate and nucleation rate and p, q are positive numbers (in the form proposed in [1]). Following [4], we write (1.2) as a system of differential equations:

$$\dot{M}_0(t) = F(t) \tag{1.5}$$

$$\dot{M}_1(t) = G(t)\, M_0(t) \tag{1.6}$$

$$\dot{M}_2(t) = 8\pi G(t)\, M_1(t) \tag{1.7}$$

$$\dot{\chi}(t) = G(t)\, M_2(t) \tag{1.8}$$

The variables $M_i(t)$ have the following physical meaning: $M_0(t)$ is the number of crystals (assumed to be spherical) per unit of volume at time t, $M_1(t)$ the sum of crystals radii, $M_2(t)$ the measure of the total surface of the crystals and $\chi(t)$ the crystal volume fraction.

The factors G_0 and F_0 are non-negative bell-shaped functions, vanishing outside a finite temperature range, so that crystallization can take place only if T belongs to such a range.

The corrective factor $\left(1 - \frac{\chi}{W_{eq}(T)}\right)^p$ slows down the crystal growth as χ approaches $\chi = W_{eq}$. The analogous factor in G has a similar meaning.

Such factors model various effects, particularly the so-called effect of impingement (due to the mutual collisions of expanding crystals), which obviously becomes more and more effective as the crystalline fraction increases, as well as segregation of the less crystallizable fraction of the system.

The identification of the values of p and q has been performed by means of comparison with the experimental data for isothermal crystallization, or by theoretical approach in several papers in the case of $W_{eq} = \text{const}$: the classical model of Avrami is approximated in the ideal case $W_{eq} = 1$ by $p = 0.7$ and $q = 1$. For a review about polymer crystallization see [2], [9], [10].

We investigate the model with $p = q \in (0, 1)$, with Dirichlet boundary conditions, and $p = q \geqslant 1$ with Neumann boundary conditions (it will be clear how to proceed if such conditions are interchanged).

If $p \geqslant 1$ we prove existence and uniqueness of the classical solution, if $p \in (0, 1)$ we find existence and continuous dependence on the data of the slightly weaker solution.

Equations (1.5)-(1.8) are called rate equations and (1.3), (1.4) were proposed in [1]. The well-posedness of the model has been proved in [1] for $W_{eq} = \text{const.}$, $p \geqslant 1$, $q \geqslant 1$. The case in which at least one between p, q is less than one has been studied in [3].

The dependence of W_{eq} on T introduces substantial changes in the proofs and the main scope of this paper is to extend the results of [1], [3] taking it into account.

It must be stressed that the temperature dependence of W_{eq} plays a crucial role in solidification of polymers. This fact has been clearly emphasized in [15] for the specific case of polypropylene.

2. – Existence and uniqueness ($\beta \geq 1$).

We consider the following problem, written in dimensionless variables:

$$\partial_t T - \Delta T = \lambda \partial_t \chi \quad \text{in} \quad \Omega \times (0, t^*) \tag{2.1}$$

$$T(x, 0) = 1 \quad \text{in} \quad \Omega \tag{2.2}$$

$$T(x, t) = \varphi(x, t) \quad \text{in} \quad \partial\Omega \times (0, t^*) \tag{2.3}$$

$$\frac{\partial M_0}{\partial t} = F_0(T)\left(1 - \frac{\chi}{W_{eq}(T)}\right)_+^\beta \quad \text{in} \quad \Omega \times (0, t^*) \tag{2.4}$$

$$\frac{\partial M_1}{\partial t} = 2G_0(T)\left(1 - \frac{\chi}{W_{eq}(T)}\right)_+^\beta M_0(x, t) \quad \text{in} \quad \Omega \times (0, t^*) \tag{2.5}$$

$$\frac{\partial M_2}{\partial t} = \pi G_0(T)\left(1 - \frac{\chi}{W_{eq}(T)}\right)_+^\beta M_1(x, t) \quad \text{in} \quad \Omega \times (0, t^*) \tag{2.6}$$

$$\frac{\partial \chi}{\partial t} = 4G_0(T)\left(1 - \frac{\chi}{W_{eq}(T)}\right)_+^\beta M_2(x, t) \quad \text{in} \quad \Omega \times (0, t^*) \tag{2.7}$$

with $\beta \geq 1$ and initial data:

$$M_i(x, 0) = \chi(x, 0) = 0 \qquad i = 0, 1, 2 \quad \text{in} \quad \Omega\,. \tag{2.8}$$

Here Ω is a bounded domain of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Throughout the paper we assume:

(i) $\varphi(x, t)$ trace on $\partial\Omega \times (0, t)$ of a function

$$\tilde{\varphi}(x, t) \in H^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [0, t])$$

such that:

$$\varphi(x, 0) = 1, \qquad \left.\frac{\partial \varphi}{\partial t}\right|_{t=0} = 0 \tag{2.9}$$

where 1 denotes a temperature above the melting point, and decreasing in t to a temperature below the glassy transition point,

(ii) $G_0, F_0 > 0$, $G_0, F_0 \in \mathrm{Lip}(\mathbb{R})$, $\frac{F_0(s)}{G_0(s)} \leq \mu$ $\forall s \geq 0$.

(iii) $W_{eq}(T)$ Lipschitz continuous, nonincreasing in T, such that:

$$W_{eq}(T) > 0 \quad \text{if} \quad T \in (T_g, T_m) \quad \text{and} \quad 0 < W_0 \leq W_{eq}(T) \leq 1\,.$$

We denote by L_f the Lipschitz constant of any function f and by C_f the L^∞ norm of f.

By means of a fixed point technique we show the following:

THEOREM 2.1. – *For any* $t^* > 0$ *the problem* (2.1)-(2.7) *has one unique classical solution:* $T \in H^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [0, t^*])$, M_i, χ *continuous with* $\frac{\partial \chi}{\partial t}$, $\frac{\partial M_i}{\partial t}$ *Hölder continuous.*

From the general theory of parabolic equation the following result is well known (see e.g. [12] to which we refer also for the symbols of functional spaces)

LEMMA 2.1. – *If* $\frac{\partial \chi}{\partial t} \in H^{\alpha, \frac{\alpha}{2}}(\overline{\Omega} \times [0, t^*])$, *the problem* (2.1, 2.2, 2.3) *has one unique solution* $T \in H^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{\Omega} \times [0, t^*])$.

An important step in the proof of Thm. 2.1 consists in the following extimates, which are simply obtained integrating sequentially (2.4)-(2.7) with respect to t:

LEMMA 2.2. – *In a given time interval* $(0, t^*)$ *we have:*

$$0 \leqslant \frac{\partial M_0}{\partial t} \leqslant C_F, \qquad 0 \leqslant M_0 \leqslant C_F t^* \tag{2.10}$$

$$0 \leqslant \frac{\partial M_1}{\partial t} \leqslant 2 C_F C_G t^*, \qquad 0 \leqslant M_1 \leqslant C_F C_G (t^*)^2 \tag{2.11}$$

$$0 \leqslant \frac{\partial M_2}{\partial t} \leqslant \pi C_G^2 C_F (t^*)^2, \quad 0 \leqslant M_2 \leqslant \frac{\pi}{3} C_G^2 C_F (t^*)^3 \tag{2.12}$$

$$0 \leqslant \frac{\partial \chi}{\partial t} \leqslant \frac{4}{3} \pi C_G^3 C_F (t^*)^3. \tag{2.13}$$

Now we set up a fixed point argument. Let:

$$X = \left\{ \chi \in K / \chi(x, 0) = \partial_t \chi(x, 0) = 0,\ 0 \leqslant \chi \leqslant R_1(t^*), \right. \\ \left. 0 \leqslant \frac{\partial \chi}{\partial t} \leqslant R_2(t^*),\ \left\| \frac{\partial \chi}{\partial t} \right\|^{\alpha} \leqslant \nu \right\} \tag{2.14}$$

be a closed and compact subset of:

$$K = \left\{ f(x,t) \in C(\Omega \times [0,t^*]) \,\middle|\, \frac{\partial f}{\partial t} \in C(\Omega \times [0,t^*]) \right\}$$

with norm: $\|f\|_K = \|f\|_\infty + \left\| \frac{\partial f}{\partial t} \right\|_\infty$.

In (2.14) $R_1(t^*)$, $R_2(t^*)$, ν are to be chosen, and:

$$\|f\|^\alpha := \sup_{(x,t),(y,\tau) \in \Omega \times [0,t^*]} \frac{|f(x,t) - f(y,t)|}{|x-y|^\alpha} + \sup_{(x,t),(y,\tau) \in \Omega \times [0,t^*]} \frac{|f(x,t) - f(x,\tau)|}{|t-\tau|^{\alpha/2}}.$$

Given $\chi \in X$ we can solve (2.1)-(2.3), according to Lemma 2.1, thus obtaining T. Next we compute M_i, $i = 0, 1, 2$ and finally we can integrate:

$$\partial_t \tilde{\chi} = 4 G_0(T) \left(1 - \frac{\chi}{W_{eq}(T)} \right)_+^\beta M_2(x,t)$$

with initial data $\tilde{\chi}(x,0) = 0$. When $\chi = \tilde{\chi}$, we get a solution to our problem.

In order to prove the existence of such a fixed point we show that, choosing the elements defining the set X in an appropriate way, we have:

(1) $\tilde{\chi} \in X$

(2) $\|\tau(\chi_1) - \tau(\chi_2)\|_K \leq \gamma \|\chi_1 - \chi_2\|_K$ with $0 < \gamma < 1$

which also implies uniqueness.

(1): we want to estimate: $\left\| \frac{\partial \tilde{\chi}}{\partial t} \right\|^\alpha \leq \nu$

We assume $\beta = 1$ (the proof is analogous if $\beta > 1$).

Using the estimates (2.10)-(2.13) we obtain:

$$\left\| \frac{\partial \tilde{\chi}}{\partial t} \right\|^\alpha \leq \|M_2\|^\alpha C_G + \|M_2\|_\infty \|G_0(T)\|^\alpha + \|M_2\|_\infty C_G \left\| \left(1 - \frac{\chi}{W_{eq}(T)} \right)_+ \right\|^\alpha + \leq h_1 \left(\|M_2\|^\alpha + \|T\|^\alpha (t^*)^3 + \left\| \left(1 - \frac{\chi}{W_{eq}(T)} \right)_+ \right\|^\alpha (t^*)^3 \right) \tag{2.15}$$

$$\left\| \frac{\partial M_2}{\partial t} \right\|^\alpha \leq h_2 \left(\|M_1\|^\alpha + (t^*)^2 \left(\|T\|^\alpha + \left\| \left(1 - \frac{\chi}{W_{eq}(T)} \right)_+ \right\|^\alpha \right) \right) \tag{2.16}$$

$$\left\| \frac{\partial M_1}{\partial t} \right\|^{\alpha} \leqslant h_3 \left(\| M_0 \|^{\alpha} + \| T \|^{\alpha}(t^*) + \left\| \left(1 - \frac{\chi}{W_{eq}(T)} \right)_+ \right\|^{\alpha}(t^*) \right) \tag{2.17}$$

$$\left\| \frac{\partial M_0}{\partial t} \right\|^{\alpha} \leqslant h_4 \left(\| T \|^{\alpha} + \left\| \left(1 - \frac{\chi}{W_{eq}(T)} \right)_+ \right\|^{\alpha} \right) \tag{2.18}$$

We notice that (see Thm 10.1, p. 204 of [12]):

$$\begin{aligned} \| T \|^{\alpha} &\leqslant h((t^*)^{1-\frac{\alpha}{2}} + R_1(t^*) + k(\Omega, \varphi)) \\ &\leqslant \bar{h}((t^*)^{\frac{1}{2}} + k(\Omega, \varphi)) \end{aligned} \tag{2.19}$$

$$\begin{aligned} \left\| \left(1 - \frac{\chi}{W_{eq}(T)} \right)_+ \right\|^{\alpha} &\leqslant \eta_0 \left(\| \chi \|^{\alpha} + \| \chi \|_{\infty} \left\| \frac{1}{W_{eq}(T(x, t))} \right\|^{\alpha} \right) \\ &\leqslant \eta_1 (\| \chi \|^{\alpha} + \| \chi \|_{\infty} \| T \|^{\alpha}) \\ &\leqslant \eta_2 (M t^* + (t^*)^4 \| T \|^{\alpha}) \\ &\leqslant \eta (M t^* + (t^*)^{\frac{9}{2}} + (t^*)^4 k(\Omega, \varphi)) \end{aligned} \tag{2.20}$$

(denoting by $h, h_i, \eta, \eta_i, \bar{h}$ constants dependent on C_G, C_F, π, L_F, L_G).

Finally we obtain:

$$\left\| \frac{\partial \tilde{\chi}}{\partial t} \right\|^{\alpha} \leqslant C(t^*)^3 (M t^* + (t^*)^{\frac{1}{2}} + (t^*)^{\frac{9}{2}} + t^* + 1) \tag{2.21}$$

then, $\forall \nu > 0$ we can find t^* such that:

$$\left\| \frac{\partial \tilde{\chi}}{\partial t} \right\|^{\alpha} \leqslant \nu \tag{2.22}$$

and, with the choice:

$$R_1(t^*) = \frac{4}{3} \pi C_G^3 C_F (t^*)^4 \tag{2.23}$$

$$R_2(t^*) = \frac{4}{3} \pi C_G^3 C_F (t^*)^3 \tag{2.24}$$

we have shown that: $\frac{\partial \tilde{\chi}}{\partial t} \in X$

(2) Let χ_1 and $\chi_2 \in X$. We show that:

$$\| \tilde{\chi}_1 - \tilde{\chi}_2 \|_K \leqslant \gamma \| \chi_1 - \chi_2 \|_K \tag{2.25}$$

with $0 < \gamma < 1$.

We define:

$$\delta_0 = M_{01}(x, t) - M_{02}(x, t),\ \delta_2 = M_{21}(x, t) - M_{22}(x, t),$$

$$\tilde{T} = T_1(x, t) - T_2(x, t),\ \delta_1 = M_{11}(x, t) - M_{12}(x, t),$$

$$\delta_3 = \chi_1 - \chi_2,\ \tilde{\delta}_3 = \tilde{\chi}_1 - \tilde{\chi}_2$$

and for $i = 1, 2$, $T_i(x, t)$ is solution of the problem:

$$\partial_t T_i - \Delta T_i = \lambda \partial_t \chi_i \text{ in } \Omega \times (0, t^*) \tag{2.26}$$

$$T_i(x, 0) = 1 \text{ in } \Omega \tag{2.27}$$

$$T_i(x, t) = \varphi(x, t) \text{ in } \partial\Omega \times (0, t^*) \tag{2.28}$$

We notice that:

$$|\tilde{T}(x, t)| \leq \lambda t^* \|\partial_t \delta_3\|_\infty \tag{2.29}$$

and that:

$$\begin{aligned} \left\| \left(\frac{\chi_1}{W_{eq}(T_1)} - \frac{\chi_2}{W_{eq}(T_2)} \right) \right\|_\infty &\leq C_1 \|\delta_3\|_\infty + C_2 \|W'_{eq}(\overline{T})\|_\infty \|T_1 - T_2\|_\infty \\ &\leq C\big(\|\delta_3\| + \|\tilde{T}\|_\infty\big) \\ &\leq C\big(t^* \|\partial_t \delta_3\|_\infty + \|\tilde{T}\|_\infty\big) \end{aligned} \tag{2.30}$$

Examining the differential system for δ_i, we obtain:

$$\|\partial_t \delta_0\| \leq c t^* \|\partial_t \delta_3\| \tag{2.31}$$

$$\|\partial_t \delta_1\| \leq (c_1 t^* + c_2 (t^*)^2) \|\partial_t \delta_3\| \tag{2.32}$$

$$\|\partial_t \delta_2\| \leq (c_1 (t^*)^2 + c_2 (t^*)^3 + c_3 t^* + c_4) \|\partial_t \delta_3\| \tag{2.33}$$

$$\|\partial_t \tilde{\delta}_3\| \leq \tilde{C} t^* ((t^*)^3 + (t^*)^2 + t^* + 1) \|\partial_t \delta_3\| \tag{2.34}$$

Thus for t^* sufficiently small we have a contraction in the chosen topology. The mapping $\chi \to \tilde{\chi}$ is completely continuous for any value of t^*, so we have a solution for all t^* applying Schauder's theorem.

If we assume that the solution can bifurcate at the time t_0, with a similar procedure, using the same fixed point technique, we reach a contradiction, since we show that the solution has in fact a unique continuation beyond t_0.

3. – Existence and uniqueness $(0 < \beta < 1)$.

Now we look at problem (2.1)-(2.7) when $0 < \beta < 1$, with initial data:

$$T(x, 0) = T_0(x) \text{ in } \Omega \tag{3.1}$$

$$M_i(x, 0) = \chi(x, 0) = 0 \text{ in } \Omega \tag{3.2}$$

and boundary data:

$$\frac{\partial T(x, t)}{\partial n} = -H(x)(T - T_a) \text{ in } \partial\Omega \times (0, t^*) \tag{3.3}$$

where n denotes the outer normal to $\partial\Omega$.

We assume:

$$T_0 \in W^2_\infty(\Omega) \qquad T_0 \geqslant 0 \tag{3.4}$$

$$T_a \in W^{2,1}_\infty(\partial\Omega \times (0, t^*) \qquad T_a \geqslant 0 \qquad H \in C^\infty(\partial\Omega) \tag{3.5}$$

with $H > 0$.

We consider the following weak formulation of the problem: find

$$T(x, t), \quad \chi(x, t), \quad M_i(x, t) \quad i = 0, 1, 2$$

such that:

$$\chi, M_i \in W^1_\infty(0, t^*, L^\infty(\Omega))$$

(3.1), (3.2) hold a.e.in Ω, (2.4)-(2.7) hold a.e. in $\Omega \times [0, t^*]$ and

$$T(x, t) \in W^{2,1}_2(\Omega \times [0, t^*]) \cap L^\infty(\Omega \times [0, t^*])$$

is such that $\forall \varphi \in H^1(\Omega)$ and a.e. in $(0, t^*)$:

$$\int_\Omega (\varphi \partial_t T + \nabla\varphi \cdot \nabla T - \varphi \partial_t \chi)\, dx + \int_{\partial\Omega} \varphi H(T - T_a)\, d\sigma = 0 \tag{3.6}$$

3.1. *Continuous dependence and uniqueness.*

Let:

$$(\chi_1, T_1, M_{01}, M_{11}, M_{21}) \qquad (\chi_2, T_2, M_{02}, M_{12}, M_{22})$$

be weak solutions of the problem (2.1)-(2.7) with respective data:

$$(T_{01}, T_{a1}) \qquad (T_{02}, T_{a2}).$$

We define:

$$\widetilde{T} = T_1 - T_2, \qquad \widetilde{\chi} = \chi_1 - \chi_2, \qquad \widetilde{M}_i = M_{i1} - M_{i2} \tag{3.7}$$

$$\widetilde{T}_0 = T_{01} - T_{02}, \qquad \widetilde{T}_a = T_{a1} - T_{a2} \tag{3.8}$$

In the following theorem we prove that the solution of our problem depends continuously on the data in the $L^{\infty}(0, t^*, L^1(\Omega))$ norm:

THEOREM 3.1. – *Let* (T_{0j}, T_{aj}), $j = 1, 2$, *be two sets of data, both satisfying* (3.4)-(3.5) *and* $(\chi_j, T_j, M_{0j}, M_{1j}, M_{2j})$ *be weak solutions of the corresponding problems* (2.1)-(2.7). *For t sufficiently small we have:*

$$\|T_1 - T_2\|_{L^{\infty}(0,t^*,L^1(\Omega))} + \|\chi_1 - \chi_2\|_{L^{\infty}(0,t^*,L^1(\Omega))} + \sum_{i=0}^{2}\|M_{i1} - M_{i2}\|_{L^{\infty}(0,t^*,L^1(\Omega))} \tag{3.9}$$
$$\leq \gamma(\|T_{01} - T_{02}\|_{L_1(\Omega)} + \|T_{a1} - T_{a2}\|_{L_1(\partial\Omega \times [0,\, t^*]})$$

where γ *depends on* λ, C_G, $C_F L_G$, L_F, C_H.

We use the following two lemmas, whose proof can be omitted:

LEMMA 3.1. – *Let* $(\chi_1, T_1, M_{01}, M_{11}, M_{21})$ *and* $(\chi_2, T_2, M_{02}, M_{12}, M_{22})$ *be weak solutions of* (2.1)-(2.7); *we have:*

$$\left|\left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^{\beta} - \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta}\right| \leq 4\alpha_1 |T_1 - T_2| + \tag{3.10}$$
$$\frac{8C_G t M_{21}}{W_0}\left|\left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^{\beta} - \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta}\right|.$$

LEMMA 3.2. – *For a pair of solutions* $(\chi_1, T_1, M_{01}, M_{11}, M_{21})$, $(\chi_2, T_2, M_{02}, M_{12}, M_{22})$, *we have:*

$$\left|\left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta} - \left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^{\beta}\right| \leq \frac{4}{W_0}|\chi_1 - \chi_2| +$$
$$\frac{8C_G t M_{21}}{W_0}\left|\left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta} - \left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^{\beta}\right|.$$

PROOF OF THEOREM 3.1. – We follow the technique used in [3].

We subtract from each other the differential equations (2.4)-(2.7) satisfied by M_{0i}, M_{1i}, M_{2i}, χ_i, obtaining:

$$\frac{\partial \widetilde{M}_0}{\partial t} = F_0(T_1)\left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta + \tag{3.12}$$
$$- F_0(T_2)\left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta$$

$$\frac{\partial \widetilde{M}_1}{\partial t} = 2G_0(T_1)\left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta M_{01}(x, t) + \tag{3.13}$$
$$- 2G_0(T_2)\left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta M_{02}(x, t)$$

$$\frac{\partial \widetilde{M}_2}{\partial t} = \pi G_0(T_1)\left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta M_{11}(x, t) + \tag{3.14}$$
$$- \pi G_0(T_2)\left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta M_{12}(x, t)$$

$$\frac{\partial \tilde{\chi}}{\partial t} = 4G_0(T_1)\left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta M_{21}(x, t) + \tag{3.15}$$
$$- 4G_0(T_2)\left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta M_{22}(x, t)$$

with initial data:

$$\tilde{\chi}(x, 0) = 0, \widetilde{M}_i(x, 0) = 0 .$$

We multiply the equations above by:

$$\text{sign } \widetilde{M}_0, \text{ sign } \widetilde{M}_1, \text{ sign } \widetilde{M}_2, \text{ sign } \tilde{\chi}$$

respectively, and we integrate (3.12)-(3.15) in $\Omega \times [0, t]$ for each $t \in [0, t^*]$. We obtain:

$$\|\tilde{\chi}(\cdot, t)\|_{L_1(\Omega)} \leq 4 L_G C_{M_{21}} \int_0^t \|\widetilde{T}(\cdot, \tau)\|_{L_1(\Omega)} d\tau + 4 C_G \int_0^t \|\widetilde{M}_2(\cdot, \tau)\|_{L_1(\Omega)} d\tau + \tag{3.16}$$
$$4 \int_{\Omega \times [0, t]} G_0(T_1) M_{21} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta - \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta \right| dx\, d\tau -$$
$$4 \int_{\Omega \times [0, t]} G_0(T_1) M_{21} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta - \left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta \right| dx\, d\tau$$

$$(3.17)\quad \|\widetilde{M}_2(\cdot,\,t)\|_{L_1(\Omega)} \leq \pi L_G C_{M_{11}} \int_0^t \|\widetilde{T}(\cdot,\,\tau)\|_{L_1(\Omega)}\,d\tau + \pi C_G \int_0^t \|\widetilde{M}_1(\cdot,\,\tau)\|_{L_1(\Omega)}\,d\tau +$$

$$\pi \int_{\Omega\times[0,\,t]} G_0(T_1)\,M_{11} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^{\beta} - \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta} \right| dx\,d\tau$$

$$\pi \int_{\Omega\times[0,\,t]} G_0(T_1)\,M_{11} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta} - \left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^{\beta} \right| dx\,d\tau$$

$$(3.18)\quad \|\widetilde{M}_1(\cdot,\,t)\|_{L_1(\Omega)} \leq 2 L_G C_{M_{01}} \int_0^t \|\widetilde{T}(\cdot,\,\tau)\|_{L_1(\Omega)}\,d\tau + 2 C_G \int_0^t \|\widetilde{M}_0(\cdot,\,\tau)\|_{L_1(\Omega)}\,d\tau +$$

$$2 \int_{\Omega\times[0,\,t]} G_0(T_1)\,M_{01} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^{\beta} - \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta} \right| dx\,d\tau$$

$$2 \int_{\Omega\times[0,\,t]} G_0(T_1)\,M_{01} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta} - \left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^{\beta} \right| dx\,d\tau$$

$$(3.19)\quad \|\widetilde{M}_0(\cdot,\,t)\|_{L_1(\Omega)} \leq L_F \int_0^t \|\widetilde{T}(\cdot,\,\tau)\|_{L_1(\Omega)}\,dx\tau +$$

$$\int_{\Omega\times[0,\,t]} F_0(T_1) \left| \left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^{\beta} - \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta} \right| dx\,d\tau$$

$$\int_{\Omega\times[0,\,t]} F_0(T_1) \left| \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^{\beta} - \left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^{\beta} \right| dx\,d\tau$$

Let $\mathrm{sign}_\varepsilon(\cdot)$ be a smooth function such that $s \cdot \mathrm{sign}_\varepsilon(s) \geq 0$, $\mathrm{sign}'_\varepsilon(s) \geq 0$ $\forall s \in \mathbb{R}$ and as $\varepsilon \downarrow 0$ $\mathrm{sign}_\varepsilon(s) \to \mathrm{sign}(s)$ uniformly over (0,1].

We subtract the weak formulations (3.6) for T_1 and for T_2 from each other, we multiply (3.6) by $\mathrm{sign}_\varepsilon(\widetilde{T})$ and integrate w.r.t. $\tau \in (0,\,t)$ with $t \in (0,\,t^*)$. We find:

$$\int_0^t \int_\Omega \left(\mathrm{sign}_\varepsilon(\widetilde{T})\,\partial_t \widetilde{T} + \mathrm{sign}'_\varepsilon(\widetilde{T})\,|\nabla \widetilde{T}\,|^2\right) dx\,d\tau + \int_0^t \int_{\partial\Omega} H(\widetilde{T} - \widetilde{T}_a)\ \mathrm{sign}_\varepsilon(\widetilde{T})\,d\sigma\,d\tau$$

$$= \lambda \int_0^t \int_\Omega \mathrm{sign}_\varepsilon(\widetilde{T})\,\partial_\tau \tilde{\chi}\,dx\,d\tau$$

Now, we integrate by parts and we let $\varepsilon \downarrow 0$:

$$\|\widetilde{T}(\cdot,t)\|_{L_1} \leqslant \|\widetilde{T}_0\|_{L_1(\Omega)} + \|\widetilde{T}_a\|_{L_1(\partial\Omega\times[0,t^*])} + \int_0^t 4C_G\|\widetilde{M}_2\| + 4L_G C_{M_{21}}\|\widetilde{T}\|\,d\tau + \int_{\Omega\times[0,t]} 4G_0(T_1)\,M_{21}\left|\left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta - \left(1-\frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta\right| dx\,d\tau - \int_{\Omega\times[0,t]} 4G_0(T_1)\,M_{21}\left|\left(1-\frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta - \left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta\right| dx\,d\tau\,. \tag{3.21}$$

We multiply equations (3.16) and (3.21) by a constant $\Lambda > 0$ to be specified, and adding (3.16)-(3.19) to (3.21) it follows that:

$$\Lambda\|\tilde{\chi}(\cdot,t)\|_{L_1(\Omega)} + \sum_{i=0}^{2}\|\widetilde{M}_i(\cdot,t)\|_{L_1(\Omega)} + \Lambda\|\widetilde{T}(\cdot,t)\|_{L_1(\Omega)} \leqslant \Lambda\|\widetilde{T}_0\|_{L_1(\Omega)} + \Lambda\|\widetilde{T}_a\|_{L_1(\partial\Omega)} + \gamma_1\int_0^t\|\widetilde{T}(\cdot,\tau)\|_{L_1(\Omega)}\,d\tau + \gamma_2\int_0^t\sum_{i=0}^{2}\|\widetilde{M}_i(\cdot,\tau)\|_{L_1(\Omega)}\,d\tau + \int_{\Omega\times[0,t]}\left|\left(1-\frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta - \left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta\right|(\gamma_3 - 4\Lambda M_{21})\,dx\,d\tau + \int_{\Omega\times[0,t]}\left|\left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta - \left(1-\frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta\right|(\gamma_3 - 4\Lambda M_{21})\,dx\,d\tau \tag{3.22}$$

where:

$$\gamma_3 = 4C_{M_{21}}\Lambda + \pi C_{M_{11}} + 2C_{M_{01}} + C_F\mu\,. \tag{3.23}$$

By means of lemma 4.1, 4.2 and choosing $\Lambda > 0$ such that:

$$t^* < \left(\frac{1}{\alpha_1}\right)^{\frac{1}{4}} \tag{3.24}$$

we find:

$$\Lambda\|\tilde{\chi}(\cdot,t)\|_{L_1(\Omega)} + \sum_{i=0}^{2}\|\widetilde{M}_i(\cdot,t)\|_{L_1(\Omega)} + \Lambda\|\widetilde{T}(\cdot,t)\|_{L_1(\Omega)} \leqslant \Lambda\|\widetilde{T}_0\|_{L_1(\Omega)} + \widetilde{\Lambda}\|\widetilde{T}_a\|_{L_1(\partial\Omega)} + \tilde{\gamma}_1\int_0^t\|\widetilde{T}(\cdot,\tau)\|\,d\tau + \sum_{i=0}^{2}\gamma_2\int_0^t\|\widetilde{M}_i(\cdot,\tau)\|_{L_1(\Omega)}\,d\tau + \frac{4}{W_0}G(T_1)\,\gamma_3\int_0^t\|\tilde{\chi}(\cdot,\tau)\|_{L_1(\Omega)}\,d\tau\,. \tag{3.25}$$

We may now apply Gronwall's lemma and we obtain the thesis. ■

COROLLARY 3.1. – *The theorem above guarantees uniqueness of the solution in* $[0, t^*]$ *with* $t^* < \left(\frac{1}{\alpha_1}\right)^{\frac{1}{4}}$.

3.2. *Existence.*

First of all, we prove some a priori regularity result. Let (χ, M_0, M_1, M_2, T) be a solution of our problem (2.1)-(2.7) with the conditions (3.4), (3.5) on initial data:

THEOREM 3.2. – *For each compact set* $k \subset \Omega$ *we have:*

$$\|T\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(K\times[0,t])} \leq \gamma(\alpha, K) < \infty \tag{3.26}$$

$$|\chi(x,t) - \chi(y,\tau)| + \sum_{i=0}^{2} |M_i(x,t) - M_i(y,\tau)| \tag{3.27}$$

$$\leq \omega(|x-y| + |x-y|^\beta + |t-\tau|; K)$$

with $\alpha \in (0,1)$, *for all* $x, y \in K$, $0 \leq t, \tau < \bar{t}$ *with* ω *modulus of continuity such that* $\omega(0, K) = 0$, $\omega(\cdot, K) \in C^0(R^+)$, *depending (as well as* γ*) on* $\|T_a\|_{L_\infty}$, $\|T_0\|_{L_\infty}$, *and on* C_G, C_F, L_G, L_F.

PROOF. – Since $\partial_t \chi \in L^\infty(\Omega \times [0, \bar{t}])$, (3.26) follows from [12]: chapt. 4 theorem 9.1 and chapt. 3 lemma 3.3.

Now we define:

$$\eta(t) = |\chi(x,t) - \chi(y,t)| + \sum_{i=0}^{2} |M_i(x,t) - M_i(y,t)|. \tag{3.28}$$

Reasoning as in (3.12)-(3.15), we find:

$$\partial_t |\tilde{\chi}| \leq 4 L_G C_{M_{21}} |\tilde{T}| + 4 C_G |\widetilde{M}_2| + \tag{3.29}$$

$$4 G_0(T_1) M_{21} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta - \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta \right| -$$

$$4 G_0(T_1) M_{21} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta - \left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta \right|$$

$$\partial_t |\widetilde{M}_2| \leq \pi L_G C_{M_{11}} |\tilde{T}| + \pi C_G |\widetilde{M}_1| + \tag{3.30}$$

$$\pi G_0(T_1) C_{M_{11}} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta - \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta \right| +$$

$$\pi G_0(T_1) C_{M_{11}} \left| \left(1 - \frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta - \left(1 - \frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta \right|$$

$$(3.31)\qquad \partial_t\,|\widetilde{M}_1| \leqslant 2L_G C_{M_{01}}\,|\widetilde{T}\,| + 2C_G\,|\widetilde{M}_0| +$$

$$2G_0(T_1)\,C_{M_{01}}\left|\left(1-\frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta - \left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta\right| +$$

$$2G_0(T_1)\,C_{M_{01}}\left|\left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta - \left(1-\frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta\right|$$

$$(3.32)\qquad \partial_t\,|\widetilde{M}_0| \leqslant L_F C_{M_{01}}\,|\widetilde{T}\,| +$$

$$F_0(T_1)\left|\left(1-\frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta - \left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta\right| +$$

$$F_0(T_1)\left|\left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta - \left(1-\frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta\right|.$$

We multiply (3.29) by a constant $\Lambda > 0$ and we add (3.29)-(3.32):

$$(3.33)\qquad \Lambda\partial_t\,|\tilde{\chi}\,| + \sum_{i=0}^{2}\partial_t\,|\widetilde{M}_i| \leqslant$$

$$c_1\,|\widetilde{T}\,| + c_2\sum_{i=0}^{2}|\widetilde{M}_i| + c_3\left|\left(1-\frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta - \left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta\right| +$$

$$(\mu C_F + 2C_{M_{01}} + \pi C_{M_{11}} - 4\Lambda M_{21})\,G_0(T_1)\left|\left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta - \left(1-\frac{\chi_2}{W_{eq}(T_2)}\right)_+^\beta\right|.$$

Using the estimate of lemma 3.2, and taking:

$$\Lambda = (\mu C_F + 2C_{M_{01}} + \pi C_{M_{11}})\,\frac{2}{W_0}\,C_G\bar{t}$$

we have:

$$(3.34)\qquad \partial_t\,\eta(t) \leqslant$$

$$c_0\left(\eta(t) + |T(x,t)-T(y,t)| + \left|\left(1-\frac{\chi_1}{W_{eq}(T_1)}\right)_+^\beta - \left(1-\frac{\chi_1}{W_{eq}(T_2)}\right)_+^\beta\right|\right) \leqslant$$

$$\tilde{c}\left(\eta(t) + |T(x,t)-T(y,t)| + \left|\frac{1}{W_{eq}(T_1)} - \frac{1}{W_{eq}(T_2)}\right|^\beta\right) \leqslant$$

$$c(\eta(t) + |T(x,t)-T(y,t)| + |T(x,t)-T(y,t)|^\beta)\,.$$

Now we use the Gronwall's Lemma, concluding that:

$$(3.35)\qquad \eta(t) \leqslant$$

$$\exp(\hat{c}t)\left(\sup_{\tau\in[0,t]}|T(x,\tau)-T(y,\tau)| + \sup_{\tau\in[0,t]}|T(x,\tau)-T(y,\tau)|^{\beta}\right) \leqslant$$

$$\exp(\hat{c}t)(\gamma(\alpha,K)|x-y| + \gamma^{\beta}|x-y|^{\beta}) \leqslant$$

$$\exp(\hat{c}t)(\gamma(\alpha,K)|x-y| + \gamma^{\beta}|x-y|^{\beta}) \qquad \forall t \geqslant 0$$

and:

$$(3.36)\qquad |c(x,t)-\chi(y,\tau)| + \sum_{i=0}^{2}|M_i(x,t)-M_i(y,\tau)| \leqslant$$

$$\eta(t) + |c(y,t)-\chi(y,\tau)| + \sum_{i=0}^{2}|M_i(y,t)-M_i(y,\tau)| \leqslant$$

$$\omega(|x-y| + |x-y|^{\beta} + |t-\tau|\,;K)$$

$$\forall\ 0 \leqslant t,\tau \leqslant \bar{t} \quad \text{and} \quad x,y \in K \subset \Omega\,. \qquad \blacksquare$$

THEOREM 3.3. – *Under the conditions* (3.1)-(3.5), *problem* (2.1)-(2.7) *has a weak solution.*

PROOF. – We look at a regularization of problem (2.1)-(2.7). We replace the factor $\left(1-\frac{\chi}{W_{eq}(T)}\right)_+^{\beta}$ with a sequence $\psi_{\varepsilon}(\eta) \in C^{\infty}(R)$ $\left(\text{with } \eta = \frac{\chi}{W_{eq}(T)}\right)$ such that:

$$\psi'_{\varepsilon}(\eta) \leqslant 0, \quad \psi_{\varepsilon}(0) = 1, \quad \psi_{\varepsilon}(\eta) = 0 \ \ \forall \eta > 1$$

and:

$$\psi_{\varepsilon}(\eta) = (1-\eta)^{\beta} \quad \text{with} \quad \eta \in [0, 1-\varepsilon)$$

so that:

$$\psi_{\varepsilon}(\eta) \to (1-\eta)^{\beta}$$

uniformly in [0,1].

The regularized problem, with $T_{a\varepsilon}$ and $T_{0\varepsilon}$ regular approximations of T_a

and of T_0, is:

$$\frac{\partial M_{0\varepsilon}}{\partial t} = F_0(T_\varepsilon)\,\psi_\varepsilon(\eta_\varepsilon) \tag{3.37}$$

$$\frac{\partial M_{1\varepsilon}}{\partial t} = 2G_0(T_\varepsilon)\,\psi_\varepsilon(\eta_\varepsilon)\,M_{0\varepsilon}(x,\,t) \tag{3.38}$$

$$\frac{\partial M_{2\varepsilon}}{\partial t} = \chi G_0(T)\,\psi_\varepsilon(\eta_\varepsilon)\,M_{1\varepsilon}(x,\,t) \tag{3.39}$$

$$\frac{\partial \chi_\varepsilon}{\partial t} = 4G_0(T)\,\psi_\varepsilon(\eta_\varepsilon)\,M_{2\varepsilon}(x,\,t) \tag{3.40}$$

and:

$$\int_\Omega (\varphi\partial_t T_\varepsilon + \nabla\varphi\cdot\nabla T_\varepsilon - \varphi\partial_t\chi_\varepsilon)\,dx + \int_{\partial\Omega} \varphi H(T_\varepsilon - T_{a\varepsilon})\,d\sigma = 0 \tag{3.41}$$

with:

$$T_{0\varepsilon}(x,\,0) = T_{0\varepsilon}(x),\;\; \chi_\varepsilon(x,\,0) = M_{i\varepsilon}(x,\,0) = 0\,.$$

This problem has a unique classical solution:

$$(\chi_\varepsilon,\,T_\varepsilon,\,M_{i\varepsilon})$$

to which we may apply the a priori extimates of Lemma 2.2. We apply Ascoli-Arzelà's theorem (possibily extracting subsequences) and if $\varepsilon\downarrow 0$ we have:

$$\chi_\varepsilon\to\chi \qquad M_{i\varepsilon}\to M_i$$

uniformly in each compact set $\Omega\times[0,\bar{t}]$. Moreover:

$$\partial_t\chi_\varepsilon\rightharpoonup\partial_t\chi, \qquad \partial_t M_{i\varepsilon}\rightharpoonup\partial_t M_i$$

weakly in $L_2(\Omega\times[0,\,t^*])$.

Now we consider the equation (3.41), taking $\varphi = \partial_t T_\varepsilon$, and we integrate over $[0,t]$:

$$\frac{1}{2}\int_{\Omega\times[0,\,t]}(\partial_\tau T_\varepsilon)^2\,dx\,d\tau + \frac{1}{2}\int_\Omega |\nabla T_\varepsilon(x,\,t)|^2\,dx + \frac{1}{2}\int_{\partial\Omega} HT_\varepsilon^2(x,\,t)\,d\sigma \leqslant \tag{3.42}$$

$$\leqslant \frac{1}{2}\int_\Omega |\nabla T_{0\varepsilon}|^2\,dx + \frac{1}{2}\int_{\partial\Omega} HT_{0\varepsilon}^2\,d\sigma +$$

$$+\int_{\Omega\times[0,t]}\left(G_0(T_\varepsilon)\,\psi_\varepsilon\left(\frac{\chi_\varepsilon}{W_{eq}(T_\varepsilon)}\right)M_{2\varepsilon}\right)^2 dx\,d\tau+$$

$$-\int_0^t\int_{\partial\Omega} HT_{a\varepsilon}\,\partial_t T_\varepsilon\,d\sigma\,d\tau\,.$$

From (3.42) we find:

(3.43) $$\|T_\varepsilon\|_{H^1(0,t^*,L^2(\Omega))\cap L^\infty(0,t^*,H^1(\Omega))}\leqslant\gamma$$

with γ depending on t^*, C_G, C_F, $\|T_0\|_{H_1(\Omega)}$, $\|T_a\|_{W_1^1(0,t^*,L^2(\partial\Omega))}$ and on C_H, but not on ε.

If $\varepsilon\downarrow 0$:

(3.44) $$T_\varepsilon\to T$$

weakly in $H^1(\Omega\times[0,t^*])$ and a.e. in $\Omega\times[0,t^*]$. Now we multiply the equation:

(3.45) $$\partial_t T_\varepsilon-\Delta T_\varepsilon=\lambda\partial_t\chi_\varepsilon$$

by a function $\Phi\in L^2(0,t^*,H^1(\Omega))$ and we integrate in $\Omega\times(0,t)$:

(3.46) $$\int_0^t\int_\Omega(\Phi\partial_t T_\varepsilon+\nabla\Phi\cdot\nabla T_\varepsilon-\Phi\partial_t\chi_\varepsilon)\,dx\,d\tau+\int_0^t\int_{\partial\Omega}\Phi H(T_\varepsilon-T_{a\varepsilon})\,d\sigma\,d\tau=0$$

Letting $\varepsilon\downarrow 0$ in (3.46), we have that $\forall\varphi\in H^1(\Omega)$ and a.e. in $(0,t^*)$:

(3.47) $$\int_\Omega(\varphi\partial_t T+\nabla\varphi\cdot\nabla T-\varphi\partial_t\chi)\,dx+\int_{\partial\Omega}\varphi H(T-T_a)\,d\sigma=0\,.\qquad\blacksquare$$

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