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Star-invertible Ideals of Integral Domains.

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Sunto. – *Sia* $*$ *uno star-operatore su* R *e* $*_s$ *lo star-operatore di carattere finito indotto da* $*$. *Lo scopo di questo lavoro è studiare quando* $* = v$ *o* $*_s = t$. *In particolare, proviamo che se ogni ideale primo di* R *è* $*$*-invertibile, allora* $* = v$ *e che se* R *è un dominio a* $*$*-fattorizzazione unica, allora* R *è un dominio di Krull.*

Summary. – *Let* $*$ *be a star-operation on* R *and* $*_s$ *the finite character star-operation induced by* $*$. *The purpose of this paper is to study when* $* = v$ *or* $*_s = t$. *In particular, we prove that if every prime ideal of* R *is* $*$*-invertible, then* $* = v$, *and that if* R *is a unique* $*$*-factorable domain, then* R *is a Krull domain.*

1. – Introduction.

Let R be a commutative integral domain with identity and K the quotient field of R. Let $F(R)$ be the set of nonzero fractional ideals of R. A mapping $A \to A_*$ of $F(R)$ into $F(R)$ is called a *star-operation* on R if the following conditions hold for all $a \in K - \{0\}$ and $A, B \in F(R)$.

1. $(aR)_* = aR$, $(aA)_* = aA_*$;
2. $A \subseteq A_*$, if $A \subseteq B$ then $A_* \subseteq B_*$; and
3. $(A_*)_* = A_*$.

It is easy to show that for all $A, B \in F(R)$, $(AB)_* = (AB_*)_* = (A_* B_*)_*$. A fractional ideal A of R is called a $*$*-ideal* if $A = A_*$. An integral ideal A of R is said to be a *maximal* $*$*-ideal* if A is maximal among proper integral $*$-ideals. Given any $*$-operation on R, we can construct another $*$-operation $*_s$ defined by $A_{*_s} = \cup \{J_* \mid 0 \neq J \subseteq A$ is finitely generated$\}$ for $A \in F(R)$. We say that $*_s$ is the *finite character star-operation induced by* $*$ and that $*$ is of *finite character* if $* = *_s$. It is well-known that the set of maximal $*_s$-ideals, denoted by $*_s$-Max(R), is nonempty and a maximal $*_s$-ideal is a prime ideal. $A \in F(R)$ is said to be $*$*-invertible* if $(AA^{-1})_* = R$. Note that A is $*$-invert-

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ible if and only if (1) $A_* = (a_1, \ldots, a_n)_*$ for some finite subset of elements $\{a_1, \ldots, a_n\}$ and (2) A_P is principal for each maximal $*$-ideal P. One can easily show that if A is $*_s$-invertible, then there is a finitely generated ideal J so that $A_* = J_*$.

The most important star-operations are (1) the d-operation $A_d = A$, (2) the v-operation $A_v = (A^{-1})^{-1}$ for $A \in F(R)$, and (3) the finite character star-operation induced by the v-operation, which is called the t-operation, i.e., $A_t = \cup \{J_v \mid 0 \neq J \subseteq A$ is finitely generated$\}$.

For any star-operation $*$ on R and for any $A \in F(R)$, we have $A \subseteq A_* \subseteq A_v$, and hence $(A_*)_v = A_v$. In particular, a v-ideal is a $*$-ideal for any $*$, and if A is $*$-invertible, then A is v-invertible (cf., [2, Corollary 3.4(a)]).

This paper is divided into three sections including the introduction. In Section 2, we study some properties of finite character star-operations. As in [3], we say that an ideal I of R is *$*$-nonfactorable* if it is a proper $*$-ideal and $I = (AB)_*$ implies that either $A_* = R$ or $B_* = R$, where A, B are ideals of R. Then R is called a *$*$-factorable domain* if each proper $*$-ideal is a $*$-product of $*$-nonfactorable ideals. Also, R is called a *unique $*$-factorable domain* if each proper $*$-ideal can be factored uniquely into a $*$-product of $*$-nonfactorable ideals. Let R be a unique $*$-factorable domain and let A be a proper $*$-invertible $*$-ideal. Then A is $*$-nonfactorable if and only if A is prime [3, Lemma 11]. Moreover, R is a unique t-factorable domain if and only if R is a Krull domain [3, Theorem 12]. In Section 3, we show that if R is a unique $*$-factorable domain, then R is a Krull domain.

2. – Finite character star-operations.

LEMMA 2.1. – *Let $*$ be a star-operation on R and let $*_s$ be the finite character star-operation induced by $*$. Let I be a nonzero fractional ideal of R. Then*

1. $I_{*_s} \subseteq I_t$.
2. $I_{*_s} = \bigcap_{P \in *_s - \mathrm{Max}(R)} I_{*_s} R_P$ *and* $I_* = \bigcap_{P \in *_s - \mathrm{Max}(R)} I_* R_P$.
3. *If I is $*$-invertible, then I is v-invertible and $I_* = I_v$.*
4. *If I is $*_s$-invertible, then I is t-invertible and $I_* = I_{*_s} = I_t = I_v$.*

PROOF. – (1): Since for each nonzero fractional ideal J of R, $J_* \subseteq J_v$, we have $I_{*_s} \subseteq I_t$.

(2): Let $x \in \bigcap_{P \in *_s - \mathrm{Max}(R)} I_{*_s} R_P$ and $J = [I_{*_s} \colon xR]_K$. Note that I_{*_s} is a $*_s$-ideal, and hence J is also a $*_s$-ideal [4, Ex.1, Section 32]. For all $P \in *_s$-$\mathrm{Max}(R)$, there is some $a \in R - P$ such that $ax \in I_{*_s}$ since $x \in I_{*_s} R_P$. Hence $J \nsubseteq P$ for all $P \in *_s$-$\mathrm{Max}(R)$. It follows that $J = J_{*_s} = R$, and hence $x \in I_{*_s}$. Thus $I_{*_s} =$

$\bigcap_{P \in *_s - \mathrm{Max}(R)} I_{*_s} R_P$. The proof of the second part is similar, or see [8, Proposition 2.8(3)].

(3): Let I be $*$-invertible. Then I is v-invertible, and hence

$$I_v = (II^{-1})_* I_v \subseteq ((II^{-1})_* I_v)_*$$

$$= (II^{-1} I_v)_* = (I(I^{-1} I_v)_*)_* = I_* .$$

Hence $I_* = I_v$.

(4): Let I be $*_s$-invertible. Then I is t-invertible [2, Corollary 3.4(b)], and hence

$$I_t \subseteq I_v = (II^{-1})_{*_s} I_v \subseteq ((II^{-1})_{*_s} I_v)_{*_s}$$

$$= (II^{-1} I_v)_{*_s} = (I(I^{-1} I_v)_{*_s})_{*_s} = I_{*_s}.$$

This completes the proof. ■

LEMMA 2.2. – *Let $*$ be a finite character star-operation on R and let P be a $*$-invertible prime ideal such that $P_* \subsetneq R$. Then*

1. *If A is a $*$-invertible $*$-ideal containing P, then $A = P = P_*$.*
2. *P is a t-invertible maximal t-ideal.*

PROOF. – (1): Suppose that $P \subsetneq A$ and let Q be a minimal prime ideal of A. Since $*$ has finite character, Q is a $*$-ideal [6, Theorem 9, p. 30], and hence PR_Q and AR_Q are principal ideals. Also, since Q is minimal over A, $QR_Q = \sqrt{AR_Q} = \sqrt{aR_Q}$ for some $a \in A$.

Let $PR_Q = pR_Q$. Then $p^n \in aR_Q$ and $p^{n-1} \notin aR_Q$ for some $n \geqslant 1$. Let $r \in R_Q$ such that $p^n = ar$. Since $a \notin pR_Q$ and pR_Q is a prime ideal, $r \in pR_Q$, and thus $r = pr'$ for some $r' \in R_Q$. Hence $p^{n-1} = ar' \in aR_Q$, a contradiction. Thus $P = A$, and hence $P = A = A_* = P_*$.

(2): Since P is $*$-invertible, P is t-invertible and $P_t = P_*$ by Lemma 2.1(4). Thus by (1), P is a t-invertible prime t-ideal, and hence P is a maximal t-ideal [5, Proposition 1.3]. ■

Recall from [1, Corollary 4] that for an ideal I of R, every minimal prime ideal of I_t is t-invertible if and only if $I_t = (P_1^{e_1} \dots P_n^{e_n})_t$ for some t-invertible prime t-ideals P_i and positive integers e_i. In this case, I is t-invertible. This result cannot be generalized to an arbitrary star-operation. For example, let K be a field and let X, Y be indeterminates over R. Consider a d-operation on $K[X, Y]$. Then (X^2, XY) is a d-ideal whose minimal prime ideals (X) and (Y) are d-invertible. But (X^2, XY) cannot be presented in the form $(X)^p (Y)^q = (X^p Y^q)$ for all positive integers p, q. If I is a $*$-invertible ideal, we have a star-operation analog of [1, Corollary 4].

THEOREM 2.3. – *Let $*$ be a finite character star-operation on R and let I be a $*$-invertible ideal of R. Then every minimal prime ideal of I_* is $*$-invertible if and only if $I_* = (P_1^{e_1} \dots P_n^{e_n})_*$ for some prime $*$-ideals P_i and positive integers e_i. In this case, $I_t = (P_1^{e_1} \dots P_n^{e_n})_t$.*

PROOF. – ($\Rightarrow$) Assume that every minimal prime ideal of I_* is $*$-invertible. Then, by [1, Theorem 1], the number of minimal prime ideals of I_* is finite. Let $P_1, \dots, P_n$ be the minimal prime ideals of I_*. Note that each P_i is a $*$-invertible $*$-ideal. Thus each P_i is a t-invertible maximal t-ideal by lemma 2.2(2). Hence $I_t = (P_1^{e_1} \dots P_n^{e_n})_t$ for some positive integers e_i [1, Theorem 3]. Moreover, since I and $P_1^{e_1} \dots P_n^{e_n}$ are $*$-invertible, $I_* = I_t$ and $(P_1^{e_1} \dots P_n^{e_n})_* = (P_1^{e_1} \dots P_n^{e_n})_t$ by Lemma 2.1 (3). Thus $I_* = (P_1^{e_1} \dots P_n^{e_n})_*$. ($\Leftarrow$) Let P be a minimal prime ideal of I_*. Then P is a $*$-ideal and contains some P_i. Since P_i is a $*$-ideal and $I \subseteq P_i$, we have $P_i = P$ by the minimality. Moreover, since I_* is $*$-invertible, $P = P_i$ is also $*$-invertible. ∎

THEOREM 2.4. – *Let $*$ be a finite character star-operation on R and let I be a nonzero proper ideal of R. If every prime ideal containing I is $*$-invertible, then every $*$-ideal containing I is a t-ideal.*

PROOF. – Let J be an ideal containg I. Then every prime ideal containg J is also $*$-invertible. Replacing J with I, it suffices to show that $I_* = I_t$. Note that since every prime $*$-ideal containing I is $*$-invertible, every prime $*$-ideal containing I is a maximal $*$-ideal by Lemma 2.2(1); whence P is a prime $*$-ideal containing I if and only if P is a minimal prime ideal of I_*. Let P be a minimal prime ideal of I_*. Then P is a $*$-invertible prime $*$-ideal, and hence the number of minimal prime ideals of I_* is finite [1, Theorem 1]. Let $\{P_1, \dots, P_n\}$ be the set of such prime ideals. Since each P_i is a $*$-invertible maximal $*$-ideal, P_i is a t-invertible maximal t-ideal by Lemma 2.2(2). Moreover, $\{P_1, \dots, P_n\}$ is the set of prime t-ideals containing I. For if Q is a prime t-ideal containing I, then Q contains a minimal prime ideal P of I_*. Since I_* is a $*$-ideal, P is also a $*$-ideal, and hence $P = P_i$ for some i. Also, since P_i is a maximal t-ideal, $P = Q$.

Thus, by Lemma 2.1(2), we have

$$\begin{aligned} I_* &= \bigcap_{P \in *\text{-Max}(R)} I_* R_P = \left(\bigcap_{i=1}^{n} I_* R_{P_i} \right) \cap R \\ &\supseteq \left(\bigcap_{i=1}^{n} IR_{P_i} \right) \cap R = \left(\bigcap_{i=1}^{n} (IR_{P_i})_t \right) \cap R \\ &= \left(\bigcap_{i=1}^{n} I_t R_{P_i} \right) \cap R = \bigcap_{P \in t\text{-Max}(R)} I_t R_P = I_t \supseteq I_*. \end{aligned}$$

Hence $I_* = I_t$, where the fourth equality holds because each $P_i R_{P_i}$ is a principal ideal. ∎

REMARK. – In [1, Theorem 3], Anderson-Zafrullah showed that every ideal containing I is $*$-invertible if and only if every prime $*$-ideal minimal over I_* is a $*$-invertible maximal $*$-ideal if and only if $I_* = (P_1^{e_1} \dots P_n^{e_n})_*$ for some $*$-invertible prime $*$-ideals P_i and positive integers e_i.

LEMMA 2.5. – ([4, Ex. 22, p. 52]) *Let Λ be a set of prime ideals of R. Then each proper ideal of the form $aR : bR \subsetneq R$ is contained in some $P \in \Lambda$ if and only if $R = \bigcap_{P \in \Lambda} R_P$.*

THEOREM 2.6. – *For an integral domain R, the followings are equivalent.*

1. *R is a Krull domain.*
2. *Every prime ideal of R is t-invertible.*
3. *Every prime ideal of R contains a t-invertible prime ideal.*
4. *Every prime ideal of R contains a $*$-invertible prime ideal for some finite character star-operation $*$ on R.*

PROOF. – (1) $\Rightarrow$ (2): [7, Theorem 3.6].

(2)$\Rightarrow$(3)$\Rightarrow$(4): These are clear.

(4)$\Rightarrow$(1): Suppose that $I := aR : bR \subsetneq R$ for some $a, b \in R$. Then I is a $*$-ideal [4, Section 32, Ex. 1]. Let P be a minimal prime ideal of I. Then P is a $*$-ideal [6, Theorem 9, p. 30] and PR_P is a t-ideal of R_P (note that PR_P is minimal over $IR_P = aR_P : bR_P$). Since P is a $*$-ideal, every prime ideal of R_P contains a invertible (and so principal) prime ideal. Hence R_P is a UFD [9, Theorem 5]. Also, since PR_P is a prime t-ideal, $\mathrm{ht}PR_P = 1$ and R_P is a rank one DVR. Hence, by Lemma 2.5, $R = \bigcap_{P \in X^1(R)} R_P$, where $X^1(R)$ is the set of height-one prime ideals of R. Moreover, since each height-one prime ideal is $*$-invertible and $*$ is of finite type, we also have that the intersection $R = \bigcap_{P \in X^1(R)} R_P$ has finite character. Hence R is a Krull domain. ∎

REMARK. – (1) Let $* = d$ and let R be a UFD of $\dim R \geqslant 2$. Then every prime ideal contains a $*$-invertible prime ideal, but $* \neq t$.

(2) The equivalent conditions (1), (2) and (3) of Theorem 2.6 appear in [7, Theorem 3.6]. (3) $\Rightarrow$ (4) is clear and (4) $\Rightarrow$ (3) follows from the fact that $A_* \subseteq A_t$ for $A \in F(R)$.

COROLLARY 2.7. – *Let $*$ be a finite character star-operation on R. If every prime ideal of R is $*$-invertible, then $* = t$. In particular, R is a Krull domain.*

PROOF. – Let I be a nonzero ideal of R. Then every prime ideal of R containing I is $*$-invertible and hence $I_* = I_t$ by Theorem 2.4. Thus $* = t$. Hence R is a Krull domain by Theorem 2.6. ■

An integral domain R is called a *Prüfer v-multiplication domain* (PVMD) if every finitely generated ideal of R is t-invertible, and R is called a *v-domain* if every finitely generated ideal of R is v-invertible.

THEOREM 2.8. – *Let $*$ be a star-operation on R and $*_s$ the star-operation induced by $*$.*

1. *If every ideal of R is $*$-invertible, then $* = v$. In particular, R is completely integrally closed.*

2. *If every finitely generated ideal of R is $*$-invertible, then $*_s = t$ and R is a v-domain.*

3. *If every finitely generated ideal of R is $*_s$-invertible, then $*_s = t$. In particular, R is a PVMD.*

PROOF. – (1): Let A be an ideal of R. Then

$$\begin{aligned} A_v &= (AA^{-1})_* A_v \subseteq ((AA^{-1})_* A_v)_* \\ &= (AA^{-1}A_v)_* = (A(A^{-1}A_v)_*)_* \\ &= (AR)_* = A_*. \end{aligned}$$

Thus $A_* = A_v$, and hence $* = v$.

(2): Let A be a finitely generated ideal of R. Then A is $*$-invertible, and so v-invertible (Lemma 2.1(3)). Thus R is a v-domain. Moreover, since $A_* = A_v$ (by (1)), for an ideal I of R we have that $I_{*_s} = \cup\{J_* \mid 0 \neq J \subseteq I$ is finitely generated$\} = \cup\{J_v \mid 0 \neq J \subseteq I$ is finitely generated$\} = I_t$. Hence $*_s = t$.

(3): Let I be a proper ideal of R. Then

$$\begin{aligned} I_{*_s} &= \bigcap_{P \in *_s - \mathrm{Max}(R)} I_{*_s} R_P \supseteq \bigcap_{P \in *_s - \mathrm{Max}(R)} IR_P \\ &= \bigcap_{P \in *_s - \mathrm{Max}(R)} (IR_P)_t = \bigcap_{P \in *_s - \mathrm{Max}(R)} I_t R_P \\ &= I_t \supseteq I_{*_s}, \end{aligned}$$

where the third equality follows from the fact that R_P is a valuation domain. Thus $I_* = I_t$, and hence $*_s = t$. ■

REMARK. – (1) An integral domain R is called a *Prüfer $*$-multiplication domain* if every finitely generated ideal of R is $*_s$-invertible. Theorem 2.8 shows that a Prüfer $*$-multiplication domain is a PVMD.

(2) If R is a PVMD that is not a Prüfer domain, then R is not a Prüfer d-multiplication domain.

3. – Unique $*$-factorable domains.

Recall that R is called a $*$-factorable domain if each proper $*$-ideal is a $*$-product of $*$-nonfactorable ideals. Also, R is called a unique $*$-factorable domain if each proper $*$-ideal can be factored uniquely into a $*$-product of $*$-nonfactorable ideals. For example, let $R = D + XL[\![X]\!]$, where D is a subring of a field L. Let $*$ be a star operation on R. Then R is a $*$-factorable domain if and only if D is a field. Moreover, R is a unique $*$-factorable domain if and only if $D = L$ [3, Corollary 8]. A t-factorable domain R is a Krull domain if and only if R is a PVMD [3, Theorem 9]. Moreover, R is a unique t-factorable domain if and only if R is a Krull domain [3, Theorem 12].

The purpose of this section is to prove that if R is a unique $*$-factorable domain, then R is a Krull domain.

LEMMA 3.1. – *Let R be a unique $*$-factorable domain. Then*

1. *Every $*$-invertible $*$-nonfactorable ideal P is a height-one prime ideal and R_P is a rank one DVR.*

2. *Every nonzero element of R is contained in a finite number of height-one prime ideals of R.*

PROOF. – (1): Let P be a $*$-invertible $*$-nonfactorable ideal. Then P is a prime ideal by [3, Lemma 11]. Suppose that ht$P \geqslant 2$. Then P contains a $*$-invertible $*$-nonfactorable ideal P_0 such that $P_0 \subsetneq P$. By [3, Lemma 11] P_0 is also a prime ideal; whence $P_0 R_P \subsetneq PR_P$ are principal prime ideals, a contradiction. Thus ht$P = 1$. Moreover, since P is a $*$-invertible $*$-ideal, $PP^{-1} \supsetneq P$, and hence R_P is a rank one DVR.

(2): Let a be a nonzero nonunit element of R. Since R is a unique $*$-factorable domain, there are $*$-nonfactorable ideals $A_1, \ldots, A_n$ such that $aR = (A_1 \ldots A_n)_*$. Since aR is $*$-invertible, each A_i is $*$-invertible and A_i is a height-one prime ideal (by (1)), hence a is contained in a finite number of height-one prime ideals $A_1, \ldots, A_n$. ■

THEOREM 3.2. – *If R is a unique $*$-factorable domain, then R is a Krull domain.*

PROOF. – Suppose that $I := aR : bR \subsetneq R$ for some $a, b \in R$. Since R is a unique $*$-factorable domain, there are some $*$-nonfactorable ideals $A_1, \ldots, A_k$ and $B_1, \ldots, B_n$ such that $aR = (A_1 \ldots A_k)_*$ and $bR = (B_1 \ldots B_n)_*$. Since aR and bR are $*$-invertible, A_i, B_j are $*$-invertible, and hence height-one prime ideals by Lemma 3.1(1). Note that

$$B_1 \ldots B_n I \subseteq (B_1 \ldots B_n I)_* = bI \subseteq aR = (A_1 \ldots A_k)_* \subseteq A_1$$

and assume that $I \nsubseteq A_i$ for $i = 1, \ldots, n$. Then since A_1 is a prime ideal, A_1 contains some B_i. We may assume that A_1 contains B_1, and hence $A_1 = B_1$ (note that $\mathrm{ht}A_1 = \mathrm{ht}B_1 = 1$). Since A_1, B_1 are $*$-invertible,

$$B_2 \ldots B_n I \subseteq (B_2 \ldots B_n I)_* \subseteq (A_2 \ldots A_k)_* .$$

By induction and the assumption that $I \nsubseteq A_i$, we have that $k \leqslant n$ and

$$(B_1 \ldots B_k)_* = (A_1 \ldots A_k)_* = aR \ ,$$

whence

$$bR = (B_1 \ldots B_n)_* = (A_1 \ldots A_k B_{k+1} \ldots B_n)_* = a(B_{k+1} \ldots B_n)_*$$

or

$$\frac{b}{a} R = (B_{k+1} \ldots B_n)_* \subseteq R \ .$$

Thus $aR : bR = R$, a contradiction. Hence $I \subseteq A_i$ for some A_i. Recall that A_i is a height-one prime ideal; whence $R = \bigcap_{P \in X^1(R)} R_P$ by Lemma 2.5. By Lemma 3.1(2), the intersection $R = \bigcap_{P \in X^1(R)} R_P$ has finite character, and R_P is a rank one DVR for each $P \in X^1(R)$. Hence R is a Krull domain. ■

COROLLARY 3.3. – *If R is a unique $*$-factorable domain with $*_s$-$\dim R = 1$, then $*_s = * = v = t$, where $*_s$ is the finite character star-operation induced by $*$.*

PROOF. – Let I be a proper ideal of R. Since R is a Krull domain (Theorem 3.2), $I_t = I_v = \bigcap_{P \in X^1(R)} IR_P$ [4, Theorem 44.2]. Moreover, by Lemma 2.1(1), since

$$I_* = \bigcap_{P \in X^1(R)} I_* R_P \supseteq \bigcap_{P \in X^1(R)} I_* R_P = I_{*_s} \supseteq \bigcap_{P \in X^1(R)} IR_P,$$

we have that $I_* = I_{*_s} = I_v = I_t$. ■

For a nonzero polynomial $f \in R[X]$, let A_f be the ideal of R generated by the coefficients of f.

LEMMA 3.4. – *If R is a unique $*$-factorable domain, then for any elements $a, b \in R$, $((a, b)^2)_* = (a^2, b^2)_*$.*

PROOF. – Let $f = aX + b$ and $g = aX - b$ be polynomials in $R[X]$. Then there is a positive integer $m \geq 1$ such that $A_f^{m+1} A_g = A_f^m A_{fg}$ [4, Theorem 28.1]. Since R is a unique $*$-factorable domain, we have that $(A_f A_g)_* = (A_{fg})_*$, and hence $((a, b)^2)_* = (a^2, b^2)_*$. ■

THEOREM 3.5. – *Let $\{P_\alpha\}$ be a set of prime ideals of R with $R = \cap R_{P_\alpha}$. For $A \in F(R)$, define $A_* = \cap AR_{P_\alpha}$ and $*_s$ the finite character star-operation induced by $*$. If R is a unique $*$-factorable domain, then $*_s = * = v = t$.*

PROOF. – It suffices to show that $*_s$-dim$R = 1$ by Corollary 3.3. Suppose that there exists a prime $*_s$-ideal Q with ht$Q \geq 2$. Since R is a Krull domain, we can take elements $a, b \in Q$ such that $(a, b)_v = R$. By Lemma 3.4, $(a^2, b^2)_* = ((a, b)^2)_*$. Thus $(a^2, b^2) R_Q = (a, b)^2 R_Q$, and hence $(a, b) R_Q$ is invertible [4, Proposition 24.2]. Since Q is a $*_s$-ideal, each height one prime ideal of R_Q is invertible and so is principal. Thus R_Q is a UFD [9, Theorem 5]. Note that $(a, b) R_Q$ is not contained in a height-one prime ideal of R_Q. Thus $R_Q = (a, b) R_Q \subseteq QR_Q \subsetneq R_Q$, a contradiction. Hence $*_s$-dim $R = 1$. This completes the proof. ■

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REFERENCES

[1] D. D. ANDERSON - M. ZAFRULLAH, *On t-invertibility, IV*, Factorization in Integral Domains, Lecture Notes in Pure and Applied Math., Marcel Dekker, **189** (1997), 221-225.
[2] D. F. ANDERSON, *A general theory of class groups*, Comm. Algebra, **16** (4) (1988), 805-847.
[3] D. F. ANDERSON - H. KIM - J. PARK, *Factorable domains*, to appear in Comm. Algebra.
[4] R. GILMER, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.
[5] E. HOUSTON - M. ZAFRULLAH, *On t-invertibility, II*, Comm. Algebra, **17** (1989), 1955-1969.

[6] P. Jaffard, *Les systèmes d'Idéaux*, Dunod, Paris, 1960.
[7] B. G. Kang, *On the converse of a well-known fact about Krull domains*, J. Algebra, **124** (1989), 284-299.
[8] B. G. Kang, *Prüfer v-multiplication domains and the ring* $R[X]_{N_v}$, J. Algebra, **123** (1989), 151-170.
[9] I. Kaplansky, *Commutative Rings*, revised edition, Univ. of Chicago Press, 1974.

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