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**Existence and Boundedness of Minimizers of a Class of Integral Functionals.**

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**Summary.** – In this paper we consider a class of integral functionals whose integrand satisfies growth conditions of the type

\[
f(x, \eta, \xi) \geq a(x) \frac{|\xi|^p}{(1 + |\eta|)^a} - b_1(x)|\eta|^\beta_1 - g_1(x),
\]

\[
f(x, \eta, 0) \leq b_2(x)|\eta|^\beta_2 + g_2(x),
\]

dove \(0 \leq \alpha < p, 1 \leq \beta_1 < p, 0 \leq \beta_2 < p, \alpha + \beta_1 \leq p\), \(a(x), b_1(x), g_1(x) (i = 1, 2)\) sono funzioni non negative che soddisfano opportune ipotesi di sommabilità. Si dimostra l’esistenza e la limitatezza di minimi di tali funzionali nella classe di funzioni appartenenti allo spazio di Sobolev pesato \(W^{1,p}(\alpha)\), che assumono un assegnato dato al bordo \(u_0 \in W^{1,p}(\alpha) \cap L^\infty(\Omega)\).

1. – Introduction.

Let us consider functionals of Calculus of Variations of the type

\[
F(v) = \int_\Omega f(x, v, \nabla v) \, dx,
\]

(*) Work partially supported by MURST.
where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, having finite Lebesgue measure and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, convex in $\xi$ which satisfies the following growth conditions

\begin{equation}
 f(x, \eta, \xi) \geq a(x) \frac{|\xi|^p}{(1 + |\eta|)^\alpha} - b_1(x) |\eta|^{\beta_1} - g_1(x),
\end{equation}

\begin{equation}
 f(x, \eta, 0) \leq b_2(x) |\eta|^{\beta_2} + g_2(x),
\end{equation}

where $p > 1$, $0 \leq \alpha < p$, $1 \leq \beta_1 < p$, $0 \leq \beta_2 < p$, $\alpha + \beta_i \leq p$, $(i = 1, 2)$ and $a(x)$, $b_i(x)$, $g_i(x)$ ($i = 1, 2$) are nonnegative functions, which belong to some Lebesgue space.

Our aim is to prove existence and boundedness of minimizers of $F$ in the class of functions $v$ belonging to the weighted Sobolev space $W^{1, p}(\alpha)$, which assume a boundary datum $u_0 \in W^{1, p}(\alpha) \cap L^\infty(\Omega)$ in a weak sense, i.e. $v - u_0 \in W^{1, p}_0(\alpha)$.

Here we recall that the weighted Sobolev space $W^{1, p}(\alpha)$ is the closure of $C^\infty(\Omega)$ with respect to the norm

$$
\|u\|_{1, p, \alpha} = \|u\|_{1, p} + \|\nabla u\|_{1, p},
$$

where

$$
\|u\|_{1, p} = \left( \int_\Omega |u|^p a(x) \, dx \right)^{1/p}.
$$

Moreover $W^{1, p}_0(\alpha)$ is the closure of $C^\infty_0(\Omega)$ in $W^{1, p}(\alpha)$.

In [BO] existence and regularity results are proved for a class of functionals, whose model is $F(v)$ with $f(x, \eta, \xi)$ given by

\begin{equation}
 f(x, \eta, \xi) = \frac{|\xi|^p}{(1 + |\eta|)^\alpha} - b(x) \eta,
\end{equation}

with $\alpha < p - 1$. Similar functionals are studied in [GP2]. The properties of solutions of equations related to functionals (1.1) are studied by many authors (see, e.g. [AFT], [BDO], [Tr], [GP1], [GP2]).

The difficulties which arise in studying functionals (1.1) are due to the fact that, in general, they are not coercive in the space $W^{1, p}(\alpha)$ and then $F$ may not attain minimum on this space. As in [BO], in this paper we extend the functional $F$ to a functional $G$ defined on a larger space, that is the class of functions $v$ belonging to $W^{1, q}(\Omega)$ such that $v - u_0 \in W^{1, q}_0(\Omega)$, for a suitable $q$ less than $p$ and such that the inclusion of $W^{1, p}(\alpha)$ in $W^{1, q}(\Omega)$ holds (see, e.g., [MS]). We prove that the functional $G$ is coercive and weakly lower semicontinuous in the above space, so that it admits a minimizer in such a class of functions. Roughly speaking, we show that the functional $G$ is coercive in the class of functions $v$
belonging to a $W^{1,q}(\Omega)$ such that $v-u_0 \in W^{1,q}_0(\Omega)$, if the growth of $f(x, \eta, \xi)$ with respect to $\eta$ is controlled from below, that is if we assume $\alpha + \beta_1 < p$ or if $\alpha + \beta_1 = p$ and the norm of $b_1$ is small enough.

In Section 2, we prove that any minimizer of $G$ is bounded under the following assumptions of summability of the coefficients

$$\frac{1}{a} \in L^{\frac{m}{p-1}}(\Omega), \quad b_i \in L^{r_i}(\Omega), \quad g_i \in L^{k_i}(\Omega),$$

with

$$\frac{1}{r_i} + \frac{p-1}{m} < \frac{p}{n}, \quad \frac{1}{k_i} + \frac{p-1}{m} < \frac{p}{n}, \quad i = 1, 2$$

and under the conditions

$$\alpha + \beta_i \leq p, \quad i = 1, 2.$$  

We use, among other tools, a result proved by Talenti in [T] (see also [M]).

Finally, since we have boundedness of minimizers, the growth conditions on $F$ allows to prove that the minimizers of $G$ belong to $W^{1,p}(\Omega)$ and thus they are minimizers of $F$.

Let us observe that when $f$ is given by (1.4) and $a(x)$ is constant, the results which we obtain coincide with those proved in [BO].

Related results are also contained in [C1], [C2], [CS], [S].

2. - An existence result.

In the present Section we show that $F$, suitable extended, has a minimum in the class of functions $v$ belonging to $W^{1,q}(\Omega)$ and assuming the boundary datum $u_0$, that is $v-u_0 \in W^{1,q}_0(\Omega)$, where

$$q = \frac{mn(p-\alpha)}{m(n-\alpha) + n(p-1)}.$$  

More precisely let us consider the functional (1.1) under the assumption (1.2) and

$$\frac{1}{a} \in L^{\frac{m}{p-1}}(\Omega), \quad \text{with} \quad \frac{m}{p-1} \geq \frac{n}{p}, \quad 1 + \frac{p-1}{m} + \alpha \left(1 - \frac{1}{n}\right) < p < n \left(1 + \frac{p-1}{m}\right);$$
(2.2) $b_1 \in L^{r_1}(\Omega)$, with
\[ \frac{1}{r_1} \leq 1 - \frac{\beta_1}{q^*}, \]
where $q^* = nq/(n - q)$;
(2.3) $g_1 \in L^1(\Omega)$;
(2.4) $\alpha + \beta_1 < p$.

Moreover let us assume that the boundary datum $u_0$ belongs to $W^{1, p}(a) \cap L^\infty(\Omega)$.

We define the following functional
\[ G(v) = \begin{cases} F(v), & \text{if } F(v) \text{ is finite,} \\ + \infty, & \text{otherwise,} \end{cases} \]
where $v \in W^{1, q}(\Omega)$ is a function such that $v - u_0 \in W^{1, q}_0(\Omega)$ and we prove that $G$ has a minimizer $u \in W^{1, q}(\Omega)$ such that $u - u_0 \in W^{1, q}_0(\Omega)$.

Let us observe that the condition
\[ 1 + \frac{p - 1}{m} + \alpha \left(1 - \frac{1}{n}\right) < p \]
ensures that $q > 1$. Furthermore (2.1) implies
\[ 1 + \frac{p - 1}{m} < p < n \left(1 + \frac{p - 1}{m}\right); \]
this condition on $p$ together with the summability assumption on $1/a$ imply that the weighted Sobolev space $W^{1, p}(a)$ is embedded in the Sobolev space $W^{1, \tau}(\Omega)$ with $1/\tau = 1 + (p - 1)/m$ (see, e.g., [MS]). Moreover it results $q < p\tau$, so that $W^{1, \tau}(\Omega)$ is included into $W^{1, q}(\Omega)$. Thus the functional $G(v)$ is well defined.

We prove the following existence result (see also [BO])

**Theorem 2.1.** – Let us assume conditions (1.2), (2.1)-(2.4). Then $G$ has a minimizer $u \in W^{1, q}(\Omega)$ such that $u - u_0 \in W^{1, q}_0(\Omega)$.

**Proof.** – By classical results, it is sufficient to prove that $G$ is both coercive and weakly lower semicontinuous in the class of functions $v$ belonging to $W^{1, q}(\Omega)$ such that $v - u_0 \in W^{1, q}_0(\Omega)$.

We begin by proving the coerciveness of the functional $G$, i.e. we prove that, for every $v \in W^{1, q}(\Omega)$ such that $v - u_0 \in W^{1, q}_0(\Omega)$, it results
\[ G(v) \geq c\|v\|_{1, q}^{-\alpha} - c, \]
where $c$ is a positive constant depending only on $n$, $m$, $p$, $r_1$, $|\Omega|$, $\alpha$, $\beta_1$, 
\[ \left\| \frac{1}{a} \right\|_{m/(p-1)}, \|b_1\|_{r_1}, \|b_1\|_1, \|u_0\|_x, \|\nabla u_0\|_q \text{ and } \|g\|_1. \]

From now on $c$ will denote a positive constant depending only on data, whose value may change at each appearance.

From assumption (1.2), we have
\[ G(v) \geq \int_\Omega \frac{|\nabla v|^p a(x)}{(1 + |v|)^{\alpha}} dx - \int_\Omega b_1(x) |v|^\beta dx - \int_\Omega g_1(x) dx. \]

Now we evaluate the integrals on the right-hand side in (2.6).

By Hölder inequality, we have
\[ \int_\Omega |\nabla v|^q dx \leq \left( \int_\Omega \frac{|\nabla v|^p a(x)}{(1 + |v|)^{\alpha}} dx \right)^{\frac{q}{p}} \left( \int_\Omega \frac{1}{a(x)^{m/(p-1)}} dx \right)^{\frac{q(p-1)}{m}} \times \]
\[ \times \left( \int_\Omega (1 + |v|)^{q^*} dx \right)^{\frac{q}{p} \frac{\alpha}{q^*}}, \]

since
\[ \frac{q}{p} + \frac{q(p-1)}{p} + \frac{q}{p} \frac{\alpha}{q^*} = 1. \]

On the other hand, since $v - u_0 \in W^{1,q}_0(\Omega)$, by Sobolev embedding theorem, we deduce
\[ \int_\Omega (1 + |v|)^{q^*} dx \leq c(1 + \|u_0\|_x)^{q^*} |\Omega| + c\|v - u_0\|_{q^*} \leq \]
\[ \leq c + c\|\nabla(v - u_0)\|_{q^*} \leq \]
\[ \leq c + c\|\nabla v\|_{q^*} + c\|\nabla u_0\|_{q^*}. \]

From (2.8), if $\|\nabla v\|_q$ is large enough, we deduce
\[ \int_\Omega (1 + |v|)^{q^*} dx \leq c\|\nabla v\|_{q^*}. \]

Combining (2.7) and (2.9), we have
\[ \int_\Omega \frac{|\nabla v|^p a(x)}{(1 + |v|)^{\alpha}} dx \geq c\|\nabla v\|_{q}^{\alpha - \alpha}. \]

Furthermore, since condition (2.2) holds true, we can use Hölder inequality
and Sobolev embedding theorem obtaining
\[
\int_{\Omega} b_1(x) |v|^{\beta_1} \, dx + \int_{\Omega} g_1(x) \, dx \leq \]
\[
\leq c \int_{\Omega} b_1(x) |v - u_0|^{\beta_1} \, dx + c \int_{\Omega} b_1(x) |u_0|^{\beta_1} \, dx + \|g_1\|_1 \leq \]
\[
\leq c \|b_1\|_{r_1} |v - u_0|^{\beta_1/\gamma} |\Omega|^{1-1/r_1 - \beta_1/\gamma} + \|u_0\|^{\beta_1} \|b_1\|_1 + \|g_1\|_1 \leq \]
\[
\leq c \|\nabla(v - u_0)\|^{\beta_1/\gamma} + c \leq \]
\[
\leq c \|\nabla v\|^{\beta_1/\gamma} + c .
\]
Combining (2.6), (2.10) and (2.11), we have
\[
G(v) \geq c \|\nabla v\|^{p - \alpha} - c \|\nabla v\|^{\beta_1} - c .
\]
Since \( p - \alpha > \beta_1 \), if \( \|\nabla v\|_q \) is large enough, we have
\[
G(v) \geq c \|\nabla v\|^{p - \alpha} - c .
\]
Finally, we get
\[
\|v\|_{1,q}^{p - \alpha} = (\|\nabla v\|_q + \|v\|_q)^{p - \alpha} \leq \]
\[
\leq c \|\nabla v\|^{p - \alpha} + c \|v - u_0\|^{p - \alpha} + c \|u_0\|^{p - \alpha} \leq \]
\[
\leq c \|\nabla v\|^{p - \alpha} + c \leq \]
\[
\leq c(G(v) + 1) ,
\]
from which we obtain (2.5).

Finally, assumption (1.2) on \( f \) allows to apply classical semicontinuity theorems for integral functionals (see, e.g., [DG], [G]).

**Remark 2.1.** – Let us observe that if \( \alpha + \beta_1 = p \), then \( G \) is coercive in the class of functions \( v \) belonging to \( W^{1,q}(\Omega) \) such that \( v - u_0 \in W^{1,q}_0(\Omega) \) for every \( a \) satisfying (2.1) and \( b_1 \) satisfying (2.2) with \( \|b_1\|_{r_1} \) small enough. Indeed, looking carefully at inequality (2.11), the following estimate holds
\[
\int_{\Omega} b_1(x) |v|^{\beta_1} \, dx + \int_{\Omega} g_1(x) \, dx \leq c \|b_1\|_{r_1} |\Omega|^{\frac{q}{n} - \frac{1}{r_1} - \frac{(p-1)/m}{q}} \|\nabla v\|^{p - \alpha} + c_1 ,
\]
where \( c \) is a constant depending only on \( \beta_1 \) and \( c_1 \) is a constant depending only on \( r_1 , |\Omega| , \beta_1 , \|b_1\|_{r_1} , \|b_1\|_{L^1} , \|u_0\|_{\infty} , \|\nabla u_0\|_q \), and \( \|g\|_1 \).

Hence, using (2.6) and (2.10), we have
\[
G(v) \geq c(1 - \|b_1\|_{r_1} |\Omega|^{\frac{q}{n} - \frac{1}{r_1} - \frac{(p-1)/m}{q}})\|\nabla v\|^{p - \alpha} - c_1 .
\]
In this way we again obtain (2.5), if we assume
\[ \|b_1\|_{r_1} < \frac{1}{|\Omega|^{\frac{1}{\rho_n-1/r_1-(\rho-1)/m}}}. \]

**Remark 2.2.** – If \( p > n \left( 1 + \frac{p-1}{m} \right) \), \( W^{1,p}(a) \) is embedded in \( L^\infty(\Omega) \) (see, e.g. [MS]), so that, if \( \alpha + \beta_1 < p \), then \( F \) is coercive on \( W^{1,p}(a) \) for every \( b_1 \in L^1(\Omega) \). Indeed using (1.2), for every \( v \in W^{1,p}(a) \) such that \( v - u_0 \in W^{1,p}_0(a) \), we get
\[
F(v) \geq \frac{1}{(1 + \|v\|_\infty)^\alpha} \int_\Omega |\nabla v|^p a(x) \, dx - \int_\Omega b_1(x) |v|^\beta_1 \, dx - \int_\Omega g_1(x) \, dx.
\]
Moreover, it results
\[
\|v\|_\infty \leq \|v - u_0\|_\infty + \|u_0\|_\infty \leq c\|\nabla (v - u_0)\|_{p, a} + \|u_0\|_\infty \leq c\|\nabla v\|_{p, a} + c.
\]
Substituing (2.16) in (2.15), it results
\[
F(v) \geq \frac{c}{(\|\nabla v\|_{p, a} + 1)^\alpha} \|\nabla v\|_{p, a}^\alpha - \|b_1\|_{r_1} \|v\|_{r_2}^\beta - \|g_1\|_{l_1} \geq c\|\nabla v\|_{p, a}^\alpha - c\|b_1\|_{l_1} \|\nabla v\|_{p, a}^\beta - \|g_1\|_{l_1},
\]
for every \( v \) such that \( \|\nabla v\|_{p, a} \) is large enough.
Since \( p - \alpha > \beta_1 \), the last inequality gives
\[
F(v) \geq c\|\nabla v\|_{p, a}^p - c,
\]
for every \( v \) such that \( \|\nabla v\|_{p, a} \) is large enough. By proceeding as in the proof of Theorem 2.1, we get again (2.5).

3. – Main result.

In this Section we will assume that the functional \( G \) has a minimizer \( u \in W^{1,q}(\Omega) \) such that \( u - u_0 \in W^{1,q}_0(\Omega) \) and we will prove that such a minimizer is bounded. From this result we will deduce that \( u \) is in \( W^{1,p}(a) \) and thus \( u \) is a minimizer of \( F \). We recall that conditions which assure the existence of \( u \) are given by Theorem 2.1.
THEOREM 3.2. – Let us assume that conditions (1.2), (1.3), (2.1) are satisfied and that $u_0 \in W^{1, p}(a) \cap L^\infty$. Moreover, assume

$$b_i \in L^{r_i}(\Omega), \quad r_i \geq 1$$

with

$$\frac{1}{r_i} + \frac{p - 1}{m} < \frac{p}{n} \quad i = 1, 2;$$

$$g_i \in L^{k_i}(\Omega), \quad k_i \geq 1$$

with

$$\frac{1}{k_i} + \frac{p - 1}{m} < \frac{p}{n}, \quad i = 1, 2;$$

$$\alpha + \beta_i \leq p, \quad i = 1, 2.$$ 

Then any minimizer $u$ of $G$ on $W^{1, q}(\Omega)$ such that $u - u_0 \in W^{1, q}_0(\Omega)$ is bounded and belongs to $W^{1, p}(a)$. Thus $u$ is a minimizer of $F$ in the class of functions belonging to $W^{1, p}(a)$ such that $u - u_0 \in W^{1, p}_0(a)$.

PROOF. – Let $u$ be a minimizer of $G$ on $W^{1, q}(\Omega)$ such that $u - u_0 \in W^{1, q}_0(\Omega)$. We have

$$G(u) \leq G(v),$$

for any ammissible function $v$.

By the assumptions, the functions

$$v(x) = \begin{cases} 
  t, & t \leq u(x), \\
  u(x), & -t < u(x) < t, \\
  -t, & u(x) \leq -t
\end{cases}$$

are ammissible, if the interval $]-t, t[$ with $t \geq 0$ includes the range of the boundary datum. Moreover, since $F(v) < + \infty$, then $G(v) = F(v)$.

In this way, we obtain

$$\int_{|u| > t} f(x, u, \nabla u) \, dx \leq \int_{|u| > t} f(x, t \text{ sign } u, 0) \, dx.$$
By assumptions (1.2) and (1.3)

\[
\int_{|u| > t} |\nabla u|^p a(x) \, dx \leq \int_{|u| > t} b_1(x) |u|^\beta_1 \, dx + \int_{|u| > t} g_1(x) \, dx + t^{\beta_2} \int_{|u| > t} b_2(x) \, dx + \int_{|u| > t} g_2(x) \, dx,
\]

for any \( t \) such that \( t > \text{ess sup} |u_0| \).

Since

\[
\frac{q}{p} \left( 1 + \frac{p-1}{m} + \frac{\alpha}{q^*} \right) = 1,
\]

by (3.4), using Hölder inequality, we get

\[
\int_{|u| > t} |\nabla u|^q \leq \left( \int_{|u| > t} |\nabla u|^p a(x) \, dx \right)^{\frac{q}{p}} \left( \int_{|u| > t} \frac{1}{a(x)^{m(p-1)}} \, dx \right)^{\frac{q(p-1)}{m}} \times \left( \int_{|u| > t} (1 + |u|)^{q^*} \, dx \right)^{\frac{mq}{q^*}} \leq
\]

\[
\left[ \int_{|u| > t} b_1(x) |u|^\beta_1 \, dx + \int_{|u| > t} g_1(x) \, dx + t^{\beta_2} \int_{|u| > t} b_2(x) \, dx + \int_{|u| > t} g_2(x) \, dx \right]^{\frac{q}{p}} \left\| \frac{1}{a(x)} \right\|_{m/(p-1)^p}^{\frac{q(p-1)}{m}} \left( \int_{|u| > t} (1 + |u|)^{q^*} \, dx \right)^{\frac{mq}{q^*}}.
\]

Now, we evaluate each integral in the right-hand side of (3.5).

Observe that the condition \( \frac{1}{r_1} + \frac{p-1}{m} < \frac{p}{n} \) is equivalent to \( p - \alpha < \left( 1 - \frac{1}{r_1} \right) q^* \), so that, from (3.3) it follows that

\[
\beta_1 < \left( 1 - \frac{1}{r_1} \right) q^*.
\]

By Hölder inequality and Sobolev embedding theorem, we get

\[
\int_{|u| > t} b_1(x) |u|^\beta_1 \, dx \leq c \int_{|u| > t} b_1(x) |u - t|^\beta_1 \, dx + ct^{\beta_1} \int_{|u| > t} b_1(x) \, dx \leq
\]

\[
\leq c \| b_1 \|_{r_1} \left( \int_{|u| > t} |u - t|^{q^*} \, dx \right)^{\frac{\beta_1}{q^*}} \mu(t)^{1-1/r_1 - \beta_1/q^*} +
\]
\[ \leq c \left( \int \mid \nabla u \mid |u|^{-1/q} \right)^{\beta_1/q} \mu(t)^{1-1/r_1-\beta_1/q^*} + c t^{\beta_1} \mu(t)^{1-1/r_1}, \]

where \( c \) is a positive constant which depends only on \( \beta_1, \eta, m, p, \alpha \) and \( \|b_1\|_{r_1}. \) Moreover

\[ \int_{|u| > t} b_2(x) \, dx \leq \|b_2\|_{r_2} \mu(t)^{1-1/r_2}, \quad (3.7) \]

\[ \int_{|u| > t} (1 + |u|)^{q^*} \, dx \leq c(1 + t)^{q^*} \mu(t) + c \int_{|u| > t} |u - t|^{q^*} \, dx \leq \]

\[ \leq c(1 + t)^{q^*} \mu(t) + c \left( \int_{|u| > t} \mid \nabla u \mid^{q^*} \, dx \right)^{q^*/q}. \]

Taking into account (3.6)-(3.8), from (3.5), we get

\[ \int_{|u| > t} \mid \nabla u \mid^{q^*} \, dx \leq c \mu(t)^{\frac{q}{p}} \left( 1 - \frac{1}{\eta} - \frac{\beta_1}{q^*} \right) \left[ \left( \int_{|u| > t} \mid \nabla u \mid^{q} \, dx \right)^{\frac{\beta_1}{p}} + t^{\frac{\beta_1}{p}} \mu(t)^{\frac{\eta}{pq^*}} \right] \times \]

\[ \times \left[ (1 + t)^{\frac{\eta}{p}} \mu(t)^{\frac{\eta}{pq^*}} + \left( \int_{|u| > t} \mid \nabla u \mid^{q} \, dx \right)^{\frac{\eta}{p}} \right] + c t^{\beta_2} \mu(t)^{\frac{q}{p}} \left( 1 - \frac{1}{\eta} \right) \left[ (1 + t)^{\frac{\eta}{p}} \mu(t)^{\frac{\eta}{pq^*}} + \left( \int_{|u| > t} \mid \nabla u \mid^{q} \, dx \right)^{\frac{\eta}{p}} \right] + c \left[ \left( \int_{|u| > t} g_1(x) \, dx \right)^{\frac{q}{p}} + \left( \int_{|u| > t} g_2(x) \, dx \right)^{\frac{q}{p}} \right] \times \]

\[ \times \left[ (1 + t)^{\frac{\eta}{p}} \mu(t)^{\frac{\eta}{pq^*}} + \left( \int_{|u| > t} \mid \nabla u \mid^{q} \, dx \right)^{\frac{\eta}{p}} \right]. \]

Now, we want to evaluate the terms

\[ I_1 = c \mu(t)^{\frac{q}{p}} \left( 1 - \frac{1}{\eta} - \frac{\beta_1}{q^*} \right) \left[ \left( \int_{|u| > t} \mid \nabla u \mid^{q} \, dx \right)^{\frac{\beta_1}{p}} + t^{\frac{\beta_1}{p}} \mu(t)^{\frac{\eta}{pq^*}} \right] \times \]

\[ \times \left[ (1 + t)^{\frac{\eta}{p}} \mu(t)^{\frac{\eta}{pq^*}} + \left( \int_{|u| > t} \mid \nabla u \mid^{q} \, dx \right)^{\frac{\eta}{p}} \right], \]
\[ I_2 = t \frac{\beta_2 q}{r} \mu(t)^{\frac{q}{q} \left(1 - \frac{1}{r} - \frac{\beta_1 - \beta_2}{q^*}\right)} \left[ (1 + t)^{\frac{pq}{q}} \mu(t)^{\frac{pq}{q^*}} + \left( \int_{|u| > t} |\nabla u|^q \, dx \right)^{\frac{\alpha}{p}} \right], \]

\[ I_3 = \left( \int_{|u| > t} g_1(x) \, dx \right)^{\frac{q}{p}} \left[ (1 + t)^{\frac{pq}{q}} \mu(t)^{\frac{pq}{q^*}} + \left( \int_{|u| > t} |\nabla u|^q \, dx \right)^{\frac{\alpha}{p}} \right], \]

\[ I_4 = \left( \int_{|u| > t} g_2(x) \, dx \right)^{\frac{q}{p}} \left[ (1 + t)^{\frac{pq}{q}} \mu(t)^{\frac{pq}{q^*}} + \left( \int_{|u| > t} |\nabla u|^q \, dx \right)^{\frac{\alpha}{p}} \right]. \]

Let us consider \( I_1 \). We can write

\[
\mu(t)^{\frac{q}{p} \left(1 - \frac{1}{r^2} - \frac{\beta_1}{q^*}\right)} = \mu(t)^{\frac{q}{p} \left(1 - \frac{1}{r^2} - \frac{\beta_1}{q^*}\right)} \mu(t)^{\frac{q}{p} \left(1 - \frac{\alpha + \beta_1}{p}\right)},
\]

and since

\[
\frac{\alpha}{p} + \frac{\beta_1}{p} + \frac{p - (\alpha + \beta_1)}{p} = 1,
\]

we can apply Young inequality

\begin{equation}
I_1 \leq c \mu(t)^{\frac{q}{p} \left(1 - \frac{1}{r^2} - \frac{\beta_1}{q^*}\right)} \left[ \left( \frac{\alpha}{p} + \frac{\beta_1}{p} \right) \int_{|u| > t} |\nabla u|^q \, dx + \left( \frac{p - (\alpha + \beta_1)}{p} + \frac{\beta_1}{p} t^q + \frac{\alpha}{p} (1 + t)^q \right) \mu(t)^{\frac{q}{q^*}} \right] \leq c \mu(t)^{\frac{q}{p} \left(1 - \frac{1}{r^2} - \frac{\beta_1}{q^*}\right)} \left[ (1 + t)^q \mu(t)^{\frac{q}{q^*}} + \int_{|u| > t} |\nabla u|^q \, dx \right].
\end{equation}

Now we evaluate \( I_2 \). Since \( \alpha + \beta_2 \leq p \), then we can write

\[
\mu(t)^{\frac{q}{p} \left(1 - \frac{1}{r^2} - \frac{\beta_1 - \beta_2}{q^*}\right)} \mu(t)^{\frac{q}{q^*} \left(1 - \frac{\alpha + \beta_2}{p}\right)},
\]

and we can apply Young inequality, that is

\begin{equation}
I_2 \leq c \mu(t)^{\frac{q}{p} \left(1 - \frac{1}{r^2} - \frac{\beta_1 - \beta_2}{q^*}\right)} \left[ \frac{p - (\alpha + \beta_2)}{p} \mu(t)^{\frac{q}{q^*}} + \frac{\alpha}{p} \int_{|u| > t} |\nabla u|^q \, dx + \left( \frac{\beta_2}{p} t^q + \frac{\alpha}{p} (1 + t)^q \right) \mu(t)^{\frac{q}{q^*}} \right] \leq c \mu(t)^{\frac{q}{p} \left(1 - \frac{1}{r^2} - \frac{\beta_1}{q^*}\right)} \left[ (1 + t)^q \mu(t)^{\frac{q}{q^*}} + \int_{|u| > t} |\nabla u|^q \, dx \right].
\end{equation}
In analogous way, we get
\[
I_3 \leq c \left\| g_1 \right\|_{k_1^p} \mu(t)^{\frac{a}{p} \left(1 - \frac{1}{k_1} - \frac{p-a}{q^*} \right)} \left[ (1 + t)^q \mu(t)^{q g^{q^*}} + \int_{|u| > t} |\nabla u|^q \,dx \right],
\]
and
\[
I_4 \leq c \left\| g_2 \right\|_{k_1^p} \mu(t)^{\frac{a}{p} \left(1 - \frac{1}{k_2} - \frac{p-a}{q^*} \right)} \left[ (1 + t)^q \mu(t)^{q g^{q^*}} + \int_{|u| > t} |\nabla u|^q \,dx \right].
\]
Therefore, combining (3.9)-(3.13), we have
\[
\int_{|u| > t} |\nabla u|^q \,dx \leq c \left[ \mu(t)^{\frac{a}{p} \left(1 - \frac{1}{k} - \frac{p-a}{q^*} \right)} + \mu(t)^{\frac{a}{p} \left(1 - \frac{1}{k_2} - \frac{p-a}{q^*} \right)} \right] \left[ (1 + t)^q \mu(t)^{q g^{q^*}} + \int_{|u| > t} |\nabla u|^q \,dx \right].
\]
Let us set \( h = \min \{ r_1, r_2, k_1, k_2 \} \). We can assume that
\[
\mu(t) < 1, \quad t \geq t_0,
\]
for a suitable \( t_0 \). In this way it results
\[
\mu(t)^{\frac{a}{p} \left(1 - \frac{1}{k} - \frac{p-a}{q^*} \right)} + \mu(t)^{\frac{a}{p} \left(1 - \frac{1}{k_2} - \frac{p-a}{q^*} \right)} \leq c \mu(t)^{\frac{a}{p} \left(1 - \frac{1}{k} - \frac{p-a}{q^*} \right)}.
\]
Hence, from (3.14) we get
\[
\int_{|u| > t} |\nabla u|^q \,dx \leq c \mu(t)^{\frac{a}{p} \left(1 - \frac{1}{k} - \frac{p-a}{q^*} \right)} \left[ (1 + t)^q \mu(t)^{q g^{q^*}} + \int_{|u| > t} |\nabla u|^q \,dx \right].
\]
Now, for \( \tilde{t} \) such that \( \text{ess sup} \ |u_0| \leq \tilde{t} < \text{ess sup} \ |u| \), we have
\[
M \equiv 1 - c \mu(\tilde{t})^{\frac{a}{p} \left(1 - \frac{1}{k} - \frac{p-a}{q^*} \right)} > 0.
\]
Therefore, we get
\[
M \int_{|u| > t} |\nabla u|^q \,dx \leq c (1 + t)^q \mu(t)^{\frac{a}{p} \left(1 - \frac{1}{k} - \frac{p-a}{q^*} \right) + \frac{q}{q^*}} ,
\]
that is
\[
\frac{1}{\mu(t)^{1/q}} \left( \int_{|u| > t} |\nabla u|^q \,dx \right)^{\frac{1}{q}} \leq \frac{c}{M} (1 + t)^{\frac{1}{p} \left( \frac{q}{q^*} \right) + \frac{q}{q^*}} ,
\]
for every $t \geq L$, where $L$ is the greatest lower bound of levels greater than 1 satisfying (3.15) and (3.16).

On the other hand, the following inequality holds true ([T]; see also [M], Lemma 4.1 and proof of Theorem 2.1)

\[(3.18)\quad q^{1/q} \left(1 - \frac{q'}{k'}\right)^{1/q'} \frac{n\omega_n^{1/n}}{\mu(t)^{1/k}} \int_{t}^{+\infty} \mu(\tau)^{1/k - 1/n} d\tau \leq \frac{1}{\mu(t)^{1/q}} \left(\int_{|u| > t} |\nabla u|^q \, dx\right)^{1/q},\]

for some $k < q$, where $\omega_n$ denotes the measure of the ball of $\mathbb{R}^n$ having radius equal to 1, $q'$ and $k'$ denote the Hölder conjugate exponent of $q$ and $k$, respectively.

Combining (3.17) and (3.18), we get

\[(3.19)\quad \frac{1}{1 + t} \leq \frac{c}{M} \frac{\mu(t)^{\frac{1}{p} - \frac{1}{n} + \frac{p-1}{m}}}{\int_{t}^{+\infty} \mu(\tau)^{1/k - 1/n} d\tau},\]

for every $t \geq L$.

Now, let us denote

\[\delta = \frac{1}{k} - \frac{1}{n} - \frac{1}{p} \left(\frac{1}{h} + \frac{p-1}{m}\right).\]

Since (3.1) holds true, it results $\delta < 1$. Moreover, from (3.19) we get

\[(3.20)\quad \int_{L} \frac{1}{(1 + t)^{\delta}} \, dt \leq \frac{c}{M(1 - \delta)} \int_{L} \frac{d}{d\tau} \left(\int_{t}^{+\infty} \mu(\tau)^{1/k - 1/n} d\tau\right)^{1-\delta} \, dt.\]

Using (3.19) we can majorize the right hand-side in (3.20) obtaining (see also [T])

\[(3.21)\quad \int_{L} \frac{1}{(1 + t)^{\delta}} \, dt \leq \left(\frac{c}{M}\right)\delta \frac{1}{(1 - \delta)} \mu(L)^{\frac{1}{p} - \frac{1}{n} + \frac{p-1}{m}}.\]

Since

\[\int_{L} \frac{1}{(1 + t)^{\delta}} \, dt = +\infty,\]

(3.21) yields that $u$ belongs to $L^\infty(\Omega)$.
From (1.2) and (1.3) we deduce that $u$ belongs to $W^{1, p}(a)$. Indeed
\[ \int_{\Omega} a(x) |u|^p \, dx \leq \|u\|_p^p \|a\|_1, \]
\[ \frac{1}{(1 + \|u\|_s)^a} \int_{\Omega} a(x) |\nabla u|^p \, dx \leq \int_{\Omega} a(x) \frac{|\nabla u|^p \, dx}{(1 + |u|)^a} \leq F(u) + \int_{\Omega} b_1(x) |u|^\beta_1 + \int_{\Omega} g_1(x) \, dx \leq \]
\[ \leq G(u) + \|b_1\|_\gamma \|u\|_s^{\beta_1} + \|g_1\|_1 \leq c. \]

Finally, we get that $u$ is a minimizer of $F$. Indeed
\[ F(u) \geq \inf \{ F(v) : v \in W^{1, p}(a) \text{ s.t. } v - u_0 \in W_0^{1, p}(a) \} \geq \]
\[ \geq \min \{ G(v) : v \in W^{1, p}(a) \text{ s.t. } v - u_0 \in W_0^{1, p}(a) \} \geq \]
\[ \geq G(u) = F(u). \]

**Remark 3.1.** – Let us observe that, if $|\Omega|$ is small enough, i.e. $|\Omega| < \min \{ 1, 1/2c \}$, then (3.14) and (3.16) hold true for every $t \geq \text{ess sup } |u_0|$ and (3.20) gives the following apriori bound for $|u|$ \[ \text{ess sup } |u| \leq \text{ess sup } |u_0| + (c \ 1 - c \ |\Omega|^{\frac{q}{p}} (1 - \frac{1}{p - \frac{p - n}{q}}) \frac{1}{1 - \delta} |\Omega|^{\frac{1}{p}} (1 - \frac{1}{p - \frac{p - 1}{m}})). \]

**Remark 3.2.** – If we choose $\alpha = 0$ and $a(x)$ constant in $\Omega$, Theorem 3.1 gives the classical results for coercive functionals on $W_0^{1, p}(\Omega)$ (see, for example, [LU]).

**References**


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