# Bollettino Unione Matematica Italiana



A. Mercaldo


# Existence and boundedness of minimizers of a class of integral functionals


*Bollettino dell'Unione Matematica Italiana, Serie 8*, Vol. **6-B** (2003), n.1, p. 125–139.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2003_8_6B_1_125_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2003.

Bollettino U. M. I.
(8) **6**-B (2003), 125-139

# Existence and Boundedness of Minimizers of a Class of Integral Functionals.

A. MERCALDO (*)

**Sunto.** – *In questo lavoro si considera una classe di funzionali integrali, il cui integrando verifica le seguenti condizioni*

$$f(x,\eta,\xi) \geqslant a(x)\frac{|\xi|^p}{(1+|\eta|)^\alpha} - b_1(x)|\eta|^{\beta_1} - g_1(x),$$

$$f(x,\eta,0) \leqslant b_2(x)|\eta|^{\beta_2} + g_2(x),$$

*dove* $0 \leqslant \alpha < p$, $1 \leqslant \beta_1 < p$, $0 \leqslant \beta_2 < p$, $\alpha + \beta_i \leqslant p$, $a(x)$, $b_i(x)$, $g_i(x)$ $(i=1,2)$ *sono funzioni non negative che soddisfano opportune ipotesi di sommabilità. Si dimostra l'esistenza e la limitatezza di minimi di tali funzionali nella classe di funzioni appartenenti allo spazio di Sobolev pesato* $W^{1,p}(a)$, *che assumono un assegnato dato al bordo* $u_0 \in W^{1,p}(a) \cap L^\infty(\Omega)$.

**Summary.** – *In this paper we consider a class of integral functionals whose integrand satisfies growth conditions of the type*

$$f(x,\eta,\xi) \geqslant a(x)\frac{|\xi|^p}{(1+|\eta|)^\alpha} - b_1(x)|\eta|^{\beta_1} - g_1(x),$$

$$f(x,\eta,0) \leqslant b_2(x)|\eta|^{\beta_2} + g_2(x),$$

*where* $0 \leqslant \alpha < p$, $1 \leqslant \beta_1 < p$, $0 \leqslant \beta_2 < p$, $\alpha + \beta_i \leqslant p$, $a(x)$, $b_i(x)$, $g_i(x)$ $(i=1,2)$ *are nonnegative functions satisfying suitable summability assumptions. We prove the existence and boundedness of minimizers of such a functional in the class of functions belonging to the weighted Sobolev space* $W^{1,p}(a)$, *which assume a boundary datum* $u_0 \in W^{1,p}(a) \cap L^\infty(\Omega)$.

## 1. – Introduction.

Let us consider functionals of Calculus of Variations of the type

$$F(v) = \int_\Omega f(x, v, \nabla v)\, dx, \tag{1.1}$$

(*) Work partially supported by MURST.

where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, having finite Lebesgue measure and $f: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, convex in $\xi$ which satisfies the following growth conditions

$$f(x, \eta, \xi) \geqslant a(x) \frac{|\xi|^p}{(1+|\eta|)^\alpha} - b_1(x)|\eta|^{\beta_1} - g_1(x), \tag{1.2}$$

$$f(x, \eta, 0) \leqslant b_2(x)|\eta|^{\beta_2} + g_2(x), \tag{1.3}$$

where $p > 1$, $0 \leqslant \alpha < p$, $1 \leqslant \beta_1 < p$, $0 \leqslant \beta_2 < p$, $\alpha + \beta_i \leqslant p$, $(i = 1, 2)$ and $a(x)$, $b_i(x)$, $g_i(x)$ $(i = 1, 2)$ are nonnegative functions, which belong to some Lebesgue space.

Our aim is to prove existence and boundedness of minimizers of $F$ in the class of functions $v$ belonging to the weighted Sobolev space $W^{1,p}(a)$, which assume a boundary datum $u_0 \in W^{1,p}(a) \cap L^\infty(\Omega)$ in a weak sense, i.e. $v - u_0 \in W_0^{1,p}(a)$.

Here we recall that the weighted Sobolev space $W^{1,p}(a)$ is the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p,a} = \|u\|_{1,p} + \||\nabla u|\|_{1,p},$$

where

$$\|u\|_{1,p} = \left(\int_\Omega |u|^p a(x)\, dx\right)^{1/p}.$$

Moreover $W_0^{1,p}(a)$ is the closure of $C_0^\infty(\overline{\Omega})$ in $W^{1,p}(a)$.

In [BO] existence and regularity results are proved for a class of functionals, whose model is $F(v)$ with $f(x, \eta, \xi)$ given by

$$f(x, \eta, \xi) = \frac{|\xi|^p}{(1+|\eta|)^\alpha} - b(x)\,\eta, \tag{1.4}$$

with $\alpha < p - 1$. Similar functionals are studied in [GP2]. The properties of solutions of equations related to functonals (1.1) are studied by many authors (see, e.g. [AFT], [BDO], [Tr], [GP1], [GP2]).

The difficulties which arise in studying functionals (1.1) are due to the fact that, in general, they are not coercive in the space $W^{1,p}(a)$ and then $F$ may not attain minimum on this space. As in [BO], in this paper we extend the functional $F$ to a functional $G$ defined on a larger space, that is the class of functions $v$ belonging to $W^{1,q}(\Omega)$ such that $v - u_0 \in W_0^{1,q}(\Omega)$, for a suitable $q$ less than $p$ and such that the inclusion of $W^{1,p}(a)$ in $W^{1,q}(\Omega)$ holds (see, e.g., [MS]). We prove that the functional $G$ is coercive and weakly lower semicontinuous in the above space, so that it admits a minimizer in such a class of functions. Roughly speaking, we show that the functional $G$ is coercive in the class of functions $v$

belonging to a $W^{1,q}(\Omega)$ such that $v - u_0 \in W_0^{1,q}(\Omega)$, if the growth of $f(x, \eta, \xi)$ with respect to $\eta$ is controlled from below, that is if we assume $\alpha + \beta_1 < p$ or if $\alpha + \beta_1 = p$ and the norm of $b_1$ is small enough.

In Section 2, we prove that any minimizer of $G$ is bounded under the following assumptions of summability of the coefficients

$$\frac{1}{a} \in L^{\frac{m}{p-1}}(\Omega), \qquad b_i \in L^{r_i}(\Omega), \qquad g_i \in L^{k_i}(\Omega),$$

with

$$\frac{1}{r_i} + \frac{p-1}{m} < \frac{p}{n}, \qquad \frac{1}{k_i} + \frac{p-1}{m} < \frac{p}{n}, \qquad i = 1, 2 \tag{1.5}$$

and under the conditions

$$\alpha + \beta_i \leq p, \qquad i = 1, 2. \tag{1.6}$$

We use, among other tools, a result proved by Talenti in [T] (see also [M]). Finally, since we have boundedness of minimizers, the growth conditions on $F$ allows to prove that the minimizers of $G$ belong to $W^{1,p}(a)$ and thus they are minimizers of $F$.

Let us observe that when $f$ is given by (1.4) and $a(x)$ is constant, the results which we obtain coincide with those proved in [BO].

Related results are also contained in [C1], [C2], [CS], [S].

## 2. – An existence result.

In the present Section we show that $F$, suitable extended, has a minimum in the class of functions $v$ belonging to $W^{1,q}(\Omega)$ and assuming the boundary datum $u_0$, that is $v - u_0 \in W_0^{1,q}(\Omega)$, where

$$q = \frac{mn(p-\alpha)}{m(n-\alpha) + n(p-1)}.$$

More precisely let us consider the functional (1.1) under the assumption (1.2) and

(2.1) $\frac{1}{a} \in L^{\frac{m}{p-1}}(\Omega)$, with

$$\frac{m}{p-1} \geq \frac{n}{p}, \qquad 1 + \frac{p-1}{m} + \alpha\left(1 - \frac{1}{n}\right) < p < n\left(1 + \frac{p-1}{m}\right);$$

(2.2) $b_1 \in L^{r_1}(\Omega)$, with

$$\frac{1}{r_1} \leqslant 1 - \frac{\beta_1}{q^*},$$

where $q^* = nq/(n-q)$;

(2.3) $g_1 \in L^1(\Omega)$;

(2.4) $\alpha + \beta_1 < p$.

Moreover let us assume that the boundary datum $u_0$ belongs to $W^{1,p}(a) \cap L^\infty(\Omega)$.

We define the following functional

$$G(v) = \begin{cases} F(v), & \text{if } F(v) \text{ is finite}, \\ +\infty, & \text{otherwise}, \end{cases}$$

where $v \in W^{1,q}(\Omega)$ is a function such that $v - u_0 \in W_0^{1,q}(\Omega)$ and we prove that $G$ has a minimizer $u \in W^{1,q}(\Omega)$ such that $u - u_0 \in W_0^{1,q}(\Omega)$.

Let us observe that the condition

$$1 + \frac{p-1}{m} + \alpha\left(1 - \frac{1}{n}\right) < p$$

ensures that $q > 1$. Furthermore (2.1) implies

$$1 + \frac{p-1}{m} < p < n\left(1 + \frac{p-1}{m}\right);$$

this condition on $p$ together with the summability assumption on $1/a$ imply that the weighted Sobolev space $W^{1,p}(a)$ is embedded in the Sobolev space $W^{1,p\tau}(\Omega)$ with $1/\tau = 1 + (p-1)/m$ (see, e.g., [MS]). Moreover it results $q < p\tau$, so that $W^{1,p\tau}(\Omega)$ is included into $W^{1,q}(\Omega)$. Thus the functional $G(v)$ is well defined.

We prove the following existence result (see also [BO])

THEOREM 2.1. – *Let us assume conditions* (1.2), (2.1)-(2.4). *Then $G$ has a minimizer $u \in W^{1,q}(\Omega)$ such that $u - u_0 \in W_0^{1,q}(\Omega)$.*

PROOF. – By classical results, it is sufficient to prove that $G$ is both coercive and weakly lower semicontinuous in the class of functions $v$ belonging to $W^{1,q}(\Omega)$ such that $v - u_0 \in W_0^{1,q}(\Omega)$.

We begin by proving the coerciveness of the functional $G$, i.e. we prove that, for every $v \in W^{1,q}(\Omega)$ such that $v - u_0 \in W_0^{1,q}(\Omega)$, it results

$$G(v) \geqslant c\|v\|_{1,q}^{p-\alpha} - c, \tag{2.5}$$

where $c$ is a positive constant depending only on $n$, $m$, $p$, $r_1$, $|\Omega|$, $\alpha$, $\beta_1$, $\left\|\frac{1}{a}\right\|_{m/(p-1)}$, $\|b_1\|_{r_1}$, $\|b_1\|_1$, $\|u_0\|_\infty$, $\|\nabla u_0\|_q$ and $\|g\|_1$.

From now on $c$ will denote a positive constant depending only on data, whose value may change at each appearence.

From assumption (1.2), we have

$$G(v) \geqslant \int_\Omega \frac{|\nabla v|^p a(x)}{(1+|v|)^\alpha} dx - \int_\Omega b_1(x)|v|^{\beta_1} dx - \int_\Omega g_1(x)\, dx. \tag{2.6}$$

Now we evaluate the integrals on the right-hand side in (2.6).

By Hölder inequality, we have

$$\int_\Omega |\nabla v|^q dx \leqslant \left(\int_\Omega \frac{|\nabla v|^p a(x)}{(1+|v|)^\alpha} dx\right)^{\frac{q}{p}} \left(\int_\Omega \frac{1}{a(x)^{m/(p-1)}} dx\right)^{\frac{q}{p}\frac{(p-1)}{m}} \times \tag{2.7}$$

$$\times \left(\int_\Omega (1+|v|)^{q^*} dx\right)^{\frac{q}{p}\frac{\alpha}{q^*}},$$

since

$$\frac{q}{p} + \frac{q}{p}\frac{(p-1)}{m} + \frac{q}{p}\frac{\alpha}{q^*} = 1\,.$$

On the other hand, since $v - u_0 \in W_0^{1,q}(\Omega)$, by Sobolev embedding theorem, we deduce

$$\begin{aligned} \int_\Omega (1+|v|)^{q^*} dx &\leqslant c(1+\|u_0\|_\infty)^{q^*}|\Omega| + c\|v-u_0\|_{q^*}^{q^*} \leqslant \\ &\leqslant c + c\|\nabla(v-u_0)\|_q^{q^*} \leqslant \\ &\leqslant c + c\|\nabla v\|_q^{q^*} + c\|\nabla u_0\|_q^{q^*}. \end{aligned} \tag{2.8}$$

From (2.8), if $\|\nabla v\|_q$ is large enough, we deduce

$$\int_\Omega (1+|v|)^{q^*} dx \leqslant c\|\nabla v\|_q^{q^*}. \tag{2.9}$$

Combining (2.7) and (2.9), we have

$$\int_\Omega \frac{|\nabla v|^p a(x)}{(1+|v|)^\alpha} dx \geqslant c\|\nabla v\|_q^{p-\alpha}. \tag{2.10}$$

Furthermore, since condition (2.2) holds true, we can use Hölder inequality and Sobolev embedding theorem obtaining

$$(2.11)\qquad \int_\Omega b_1(x)\,|v|^{\beta_1}\,dx + \int_\Omega g_1(x)\,dx \leqslant$$

$$\leqslant c\int_\Omega b_1(x)\,|v-u_0|^{\beta_1}\,dx + c\int_\Omega b_1(x)\,|u_0|^{\beta_1}\,dx + \|g_1\|_1 \leqslant$$

$$\leqslant c\|b_1\|_{r_1}\|v-u_0\|_{q^*}^{\beta_1}\,|\Omega|^{1-1/r_1-\beta_1/q^*} + \|\,|u_0|^{\beta_1}\|_\infty\|b_1\|_1 + \|g_1\|_1 \leqslant$$

$$\leqslant c\|\nabla(v-u_0)\|_q^{\beta_1} + c \leqslant$$

$$\leqslant c\|\nabla v\|_q^{\beta_1} + c\,.$$

Combining (2.6), (2.10) and (2.11), we have

$$G(v) \geqslant c\|\nabla v\|_q^{p-\alpha} - c\|\nabla v\|_q^{\beta_1} - c\,.$$

Since $p-\alpha > \beta_1$, if $\|\nabla v\|_q$ is large enough, we have

$$G(v) \geqslant c\|\nabla v\|_q^{p-\alpha} - c\,.$$

Finally, we get

$$(2.13)\qquad \|v\|_{1,q}^{p-\alpha} = (\|\nabla v\|_q + \|v\|_q)^{p-\alpha} \leqslant$$

$$\leqslant c\|\nabla v\|_q^{p-\alpha} + c\|v-u_0\|_q^{p-\alpha} + c\|u_0\|_q^{p-\alpha} \leqslant$$

$$\leqslant c\|\nabla v\|_q^{p-\alpha} + c \leqslant$$

$$\leqslant c(G(v)+1)\,,$$

from which we obtain (2.5).

Finally, assumption (1.2) on $f$ allows to apply classical semicontinuity theorems for integral functionals (see, e.g., [DG], [G]).

REMARK 2.1. – Let us observe that if $\alpha+\beta_1 = p$, then $G$ is coercive in the class of functions $v$ belonging to $W^{1,q}(\Omega)$ such that $v-u_0 \in W_0^{1,q}(\Omega)$ for every $\alpha$ satisfying (2.1) and $b_1$ satisfying (2.2) with $\|b_1\|_{r_1}$ small enough. Indeed, looking carefully at inequality (2.11), the following estimate holds

$$\int_\Omega b_1(x)\,|v|^{\beta_1}\,dx + \int_\Omega g_1(x)\,dx \leqslant c\|b_1\|_{r_1}\,|\Omega|^{p/n-1/r_1-(p-1)/m}\|\nabla v\|_q^{p-\alpha} + c_1,$$

where $c$ is a constant depending only on $\beta_1$ and $c_1$ is a constant depending only on $r_1$, $|\Omega|$, $\beta_1$, $\|b_1\|_{r_1}$, $\|b_1\|_{L^1}$, $\|u_0\|_\infty$, $\|\nabla u_0\|_q$ and $\|g\|_1$.

Hence, using (2.6) and (2.10), we have

$$G(v) \geqslant c(1-\|b_1\|_{r_1}\,|\Omega|^{p/n-1/r_1-(p-1)/m})\|\nabla v\|_q^{p-\alpha} - c_1.$$

In this way we again obtain (2.5), if we assume

$$\|b_1\|_{r_1} < \frac{1}{|\Omega|^{p/n - 1/r_1 - (p-1)/m}} .$$

REMARK 2.2. – If $p > n\left(1 + \frac{p-1}{m}\right)$, $W^{1,p}(a)$ is embedded in $L^\infty(\Omega)$ (see, e.g. [MS]), so that, if $\alpha + \beta_1 < p$, then $F$ is coercive on $W^{1,p}(a)$ for every $b_1 \in L^1(\Omega)$. Indeed using (1.2), for every $v \in W^{1,p}(a)$ such that $v - u_0 \in W_0^{1,p}(a)$, we get

$$F(v) \geq \frac{1}{(1 + \|v\|_\infty)^\alpha} \int_\Omega |\nabla v|^p a(x)\, dx - \int_\Omega b_1(x) |v|^{\beta_1} dx - \int_\Omega g_1(x)\, dx . \tag{2.15}$$

Moreover, it results

$$\begin{aligned} \|v\|_\infty &\leq \|v - u_0\|_\infty + \|u_0\|_\infty \leq \\ &\leq c\|\nabla(v - u_0)\|_{p,a} + \|u_0\|_\infty \leq \\ &\leq c\|\nabla v\|_{p,a} + c . \end{aligned} \tag{2.16}$$

Substituing (2.16) in (2.15), it results

$$\begin{aligned} F(v) &\geq \frac{c}{(\|\nabla v\|_{p,a} + 1)^\alpha} \|\nabla v\|_{p,a}^p - \|b_1\|_1 \|v\|_\infty^{\beta_1} - \|g_1\|_1 \geq \\ &\geq c\|\nabla v\|_{p,a}^{p-\alpha} - c\|b_1\|_1 \|\nabla v\|_{p,a}^{\beta_1} - \|g_1\|_1 , \end{aligned}$$

for every $v$ such that $\|\nabla v\|_{p,a}$ is large enough.

Since $p - \alpha > \beta_1$, the last inequality gives

$$F(v) \geq c\|\nabla v\|_{p,a}^p - c ,$$

for every $v$ such that $\|\nabla v\|_{p,a}$ is large enough. By proceeding as in the proof of Theorem 2.1, we get again (2.5).

## 3. – Main result.

In this Section we will assume that the functional $G$ has a minimizer $u \in W^{1,q}(\Omega)$ such that $u - u_0 \in W_0^{1,q}(\Omega)$ and we will prove that such a minimizer is bounded. From this result we will deduce that $u$ is in $W^{1,p}(a)$ and thus $u$ is a minimizer of $F$. We recall that conditions which assure the existence of $u$ are given by Theorem 2.1.

THEOREM 3.2. – *Let us assume that conditions* (1.2), (1.3), (2.1) *are satisfied and that* $u_0 \in W^{1,p}(a) \cap L^\infty$. *Moreover, assume*

$$b_i \in L^{r_i}(\Omega), \qquad r_i \geq 1 \tag{3.1}$$

*with*

$$\frac{1}{r_i} + \frac{p-1}{m} < \frac{p}{n} \qquad i = 1, 2\,;$$

$$g_i \in L^{k_i}(\Omega), \qquad k_i \geq 1 \tag{3.2}$$

*with*

$$\frac{1}{k_i} + \frac{p-1}{m} < \frac{p}{n}, \qquad i = 1, 2\,;$$

$$\alpha + \beta_i \leq p, \qquad i = 1, 2\,. \tag{3.3}$$

*Then any minimizer* $u$ *of* $G$ *on* $W^{1,q}(\Omega)$ *such that* $u - u_0 \in W_0^{1,q}(\Omega)$ *is bounded and belongs to* $W^{1,p}(a)$ . *Thus* $u$ *is a minimizer of* $F$ *in the class of functions belonging to* $W^{1,p}(a)$ *such that* $u - u_0 \in W_0^{1,p}(a)$.

PROOF. – Let $u$ be a minimizer of $G$ on $W^{1,q}(\Omega)$ such that $u - u_0 \in W_0^{1,q}(\Omega)$. We have

$$G(u) \leq G(v),$$

for any ammissible function $v$.

By the assumptions, the functions

$$v(x) = \begin{cases} t\,, & t \leq u(x)\,, \\ u(x)\,, & -t < u(x) < t\,, \\ -t\,, & u(x) \leq -t\,, \end{cases}$$

are ammissible, if the interval $]-t, t[$ with $t \geq 0$ includes the range of the boundary datum. Moreover, since $F(v) < +\infty$, then $G(v) = F(v)$.

In this way, we obtain

$$\int_{|u|>t} f(x, u, \nabla u)\,dx \leq \int_{|u|>t} f(x, t \operatorname{sign} u, 0)\,dx\,.$$

By assumptions (1.2) and (1.3)

$$\int_{|u|>t} \frac{|\nabla u|^p a(x)}{(1+|u|)^\alpha} dx \leqslant \int_{|u|>t} b_1(x)|u|^{\beta_1} dx + \int_{|u|>t} g_1(x)\, dx + \tag{3.4}$$

$$+ t^{\beta_2} \int_{|u|>t} b_2(x)\, dx + \int_{|u|>t} g_2(x)\, dx ,$$

for any $t$ such that $t > \operatorname{ess\,sup} |u_0|$.

Since

$$\frac{q}{p}\left(1 + \frac{p-1}{m} + \frac{\alpha}{q^*}\right) = 1 ,$$

by (3.4), using Hölder inequality, we get

$$\int_{|u|>t} |\nabla u|^q \leqslant \left(\int_{|u|>t} \frac{|\nabla u|^p a(x)}{(1+|u|)^\alpha} dx\right)^{\frac{q}{p}} \left(\int_{|u|>t} \frac{1}{a(x)^{m/(p-1)}} dx\right)^{\frac{q}{p}\frac{p-1}{m}} \times \tag{3.5}$$

$$\times \left(\int_{|u|>t} (1+|u|)^{q^*} dx\right)^{\frac{\alpha q}{q^* p}} \leqslant$$

$$\leqslant \left[\int_{|u|>t} b_1(x)|u|^{\beta_1} dx + \int_{|u|>t} g_1(x)\, dx + t^{\beta_2} \int_{|u|>t} b_2(x)\, dx + \right.$$

$$\left. + \int_{|u|>t} g_2(x)\, dx \right]^{q/p} \left\| \frac{1}{a(x)} \right\|_{m/(p-1)}^{q/p} \left(\int_{|u|>t} (1+|u|)^{q^*} dx\right)^{\frac{\alpha q}{q^* p}} .$$

Now, we evaluate each integral in the right-hand side of (3.5).

Observe that the condition $\frac{1}{r_1} + \frac{p-1}{m} < \frac{p}{n}$ is equivalent to $p - \alpha < \left(1 - \frac{1}{r_1}\right) q^*$, so that, from (3.3) it follows that

$$\beta_1 < \left(1 - \frac{1}{r_1}\right) q^* .$$

By Hölder inequality and Sobolev embedding theorem, we get

$$\int_{|u|>t} b_1(x)|u|^{\beta_1} dx \leqslant c \int_{|u|>t} b_1(x)|u-t|^{\beta_1} dx + ct^{\beta_1} \int_{|u|>t} b_1(x)\, dx \leqslant \tag{3.6}$$

$$\leqslant c\|b_1\|_{r_1} \left(\int_{|u|>t} |u-t|^{q^*} dx\right)^{\beta_1/q^*} \mu(t)^{1-1/r_1-\beta_1/q^*} +$$

$$+ c\|b_1\|_{r_1} t^{\beta_1}\mu(t)^{1-1/r_1} \leqslant$$

$$\leqslant c\left(\int_{|u|>t} |\nabla u|^q\right)^{\beta_1/q} \mu(t)^{1-1/r_1-\beta_1/q^*} + ct^{\beta_1}\mu(t)^{1-1/r_1},$$

where $c$ is a positive constant which depends only on $\beta_1$, $n$, $m$, $p$, $\alpha$ and $\|b_1\|_{r_1}$. Moreover

$$\int_{|u|>t} b_2(x)\,dx \leqslant \|b_2\|_{r_2}\mu(t)^{1-1/r_2}, \tag{3.7}$$

$$\int_{|u|>t} (1+|u|)^{q^*}\,dx \leqslant c(1+t)^{q^*}\mu(t) + c\int_{|u|>t} |u-t|^{q^*}\,dx \leqslant \tag{3.8}$$

$$\leqslant c(1+t)^{q^*}\mu(t) + c\left(\int_{|u|>t} |\nabla u|^q\,dx\right)^{q^*/q}.$$

Taking into account (3.6)-(3.8), from (3.5), we get

$$\int_{|u|>t} |\nabla u|^q\,dx \leqslant c\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r1}-\frac{\beta 1}{q^*}\right)}\left[\left(\int_{|u|>t} |\nabla u|^q\,dx\right)^{\frac{\beta 1}{p}} + t^{\frac{q\beta 1}{p}}\mu(t)^{\frac{q\beta 1}{pq^*}}\right]\times \tag{3.9}$$

$$\times\left[(1+t)^{\frac{q\alpha}{p}}\mu(t)^{\frac{q\alpha}{pq^*}} + \left(\int_{|u|>t} |\nabla u|^q\,dx\right)^{\frac{\alpha}{p}}\right]+$$

$$+ct^{\frac{\beta 2q}{p}}\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r2}\right)}\left[(1+t)^{\frac{q\alpha}{p}}\mu(t)^{\frac{q\alpha}{pq^*}} + \left(\int_{|u|>t} |\nabla u|^q\,dx\right)^{\frac{\alpha}{p}}\right]+$$

$$+c\left[\left(\int_{|u|>t} g_1(x)\,dx\right)^{\frac{q}{p}} + \left(\int_{|u|>t} g_2(x)\,dx\right)^{\frac{q}{p}}\right]\times$$

$$\times\left[(1+t)^{\frac{q\alpha}{p}}\mu(t)^{\frac{q\alpha}{pq^*}} + \left(\int_{|u|>t} |\nabla u|^q\,dx\right)^{\frac{\alpha}{p}}\right].$$

Now, we want to evaluate the terms

$$I_1 = c\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r1}-\frac{\beta 1}{q^*}\right)}\left[\left(\int_{|u|>t} |\nabla u|^q\,dx\right)^{\frac{\beta 1}{p}} + t^{q\beta_1/p}\mu(t)^{\frac{q\beta 1}{pq^*}}\right]\times$$

$$\times\left[(1+t)^{q\alpha/p}\mu(t)^{q\alpha/pq^*} + \left(\int_{|u|>t} |\nabla u|^q\,dx\right)^{\frac{\alpha}{p}}\right],$$

$$I_2 = t^{\frac{\beta_2 q}{p}} \mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_2}\right)}\left[(1+t)^{\frac{q\alpha}{p}}\mu(t)^{\frac{q\alpha}{pq^*}} + \left(\int_{|u|>t} |\nabla u|^q\, dx\right)^{\frac{\alpha}{p}}\right],$$

$$I_3 = \left(\int_{|u|>t} g_1(x)\, dx\right)^{\frac{q}{p}}\left[(1+t)^{\frac{q\alpha}{p}}\mu(t)^{\frac{q\alpha}{pq^*}} + \left(\int_{|u|>t} |\nabla u|^q\, dx\right)^{\frac{\alpha}{p}}\right],$$

$$I_4 = \left(\int_{|u|>t} g_2(x)\, dx\right)^{\frac{q}{p}}\left[(1+t)^{\frac{q\alpha}{p}}\mu(t)^{\frac{q\alpha}{pq^*}} + \left(\int_{|u|>t} |\nabla u|^q\, dx\right)^{\frac{\alpha}{p}}\right].$$

Let us consider $I_1$. We can write

$$\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_1}-\frac{\beta_1}{q^*}\right)} = \mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_1}-\frac{p-\alpha}{q^*}\right)}\mu(t)^{\frac{q}{q^*}\left(1-\frac{\alpha+\beta_1}{p}\right)},$$

and since

$$\frac{\alpha}{p} + \frac{\beta_1}{p} + \frac{p-(\alpha+\beta_1)}{p} = 1\,,$$

we can apply Young inequality

$$I_1 \leqslant c\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_1}-\frac{p-\alpha}{q^*}\right)}\left\{\left(\frac{\alpha}{p}+\frac{\beta_1}{p}\right)\int_{|u|>t} |\nabla u|^q\, dx + \right.$$
$$\left. + \left[\frac{p-(\alpha+\beta_1)}{p} + \frac{\beta_1}{p}t^q + \frac{\alpha}{p}(1+t)^q\right]\mu(t)^{q/q^*}\right\} \leqslant$$
$$\leqslant c\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_1}-\frac{p-\alpha}{q^*}\right)}\left[(1+t)^q\mu(t)^{q/q^*} + \int_{|u|>t} |\nabla u|^q\, dx\right]. \tag{3.10}$$

Now we evaluate $I_2$. Since $\alpha+\beta_2 \leqslant p$, then we can write

$$\mu(t)^{q\left(1-\frac{1}{r_2}\right)} = \mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_2}-\frac{p-\alpha}{q^*}\right)}\mu(t)^{\frac{q\beta_2}{pq^*}}\mu(t)^{\frac{q}{q^*}\left(1-\frac{\alpha+\beta_2}{p}\right)},$$

and we can apply Young inequality, that is

$$I_2 \leqslant c\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_2}-\frac{p-\alpha}{q^*}\right)}\left\{\frac{p-(\alpha+\beta_2)}{p}\mu(t)^{q/q^*} + \frac{\alpha}{p}\int_{|u|>t} |\nabla u|^q\, dx + \right.$$
$$\left. + \left[\frac{\beta_2}{p}t^q + \frac{\alpha}{p}(1+t)^q\right]\mu(t)^{q/q^*}\right\} \leqslant$$
$$\leqslant c\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_2}-\frac{p-\alpha}{q^*}\right)}\left[(1+t)^q\mu(t)^{q/q^*} + \int_{|u|>t} |\nabla u|^q\, dx\right]. \tag{3.11}$$

In analogous way, we get

$$I_3 \leqslant c\|g_1\|_{h_1}^{q/p}\mu(t)^{\frac{q}{p}\left(1-\frac{1}{k_1}-\frac{p-\alpha}{q^*}\right)}\left[(1+t)^q\mu(t)^{q/q^*}+\int_{|u|>t}|\nabla u|^q\,dx\right], \tag{3.12}$$

and

$$I_4 \leqslant c\|g_2\|_{h_1}^{q/p}\mu(t)^{\frac{q}{p}\left(1-\frac{1}{k_2}-\frac{p-\alpha}{q^*}\right)}\left[(1+t)^q\mu(t)^{q/q^*}+\int_{|u|>t}|\nabla u|^q\,dx\right]. \tag{3.13}$$

Therefore, combining (3.9)-(3.13), we have

$$\begin{aligned}\int_{|u|>t}|\nabla u|^q\,dx \leqslant c\Big[&\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_1}-\frac{p-\alpha}{q^*}\right)}+\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_2}-\frac{p-\alpha}{q^*}\right)}+\\ &+\mu(t)^{\frac{q}{p}\left(1-\frac{1}{k_1}-\frac{p-\alpha}{q^*}\right)}+\mu(t)^{\frac{q}{p}\left(1-\frac{1}{k_2}-\frac{p-\alpha}{q^*}\right)}\Big]\left[(1+t)^q\mu(t)^{q/q^*}+\int_{|u|>t}|\nabla u|^q\,dx\right].\end{aligned} \tag{3.14}$$

Let us set $h=\min\{r_1, r_2, k_1, k_2\}$. We can assume that

$$\mu(t)<1\,, \qquad t\geqslant t_0, \tag{3.15}$$

for a suitable $t_0$. In this way it results

$$\begin{aligned}&\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_1}-\frac{p-1}{m}\right)}+\mu(t)^{\frac{q}{p}\left(1-\frac{1}{r_2}-\frac{p-1}{m}\right)}+\mu(t)^{\frac{q}{p}\left(1-\frac{1}{k_1}-\frac{p-1}{m}\right)}+\\ &\qquad+\mu(t)^{\frac{q}{p}\left(1-\frac{1}{k_2}-\frac{p-1}{m}\right)}\leqslant c\mu(t)^{\frac{q}{p}\left(1-\frac{1}{h}-\frac{p-1}{m}\right)}.\end{aligned}$$

Hence, from (3.14) we get

$$\int_{|u|>t}|\nabla u|^q\,dx \leqslant c\mu(t)^{\frac{q}{p}\left(1-\frac{1}{h}-\frac{p-\alpha}{q^*}\right)}\left[(1+t)^q\mu(t)^{q/q^*}+\int_{|u|>t}|\nabla u|^q\,dx\right].$$

Now, for $\bar t$ such that $\operatorname{ess\,sup}|u_0|\leqslant \bar t<\operatorname{ess\,sup}|u|$, we have

$$M\equiv 1-c\mu(\bar t)^{\frac{q}{p}\left(1-\frac{1}{h}-\frac{p-\alpha}{q^*}\right)}>0\,. \tag{3.16}$$

Therefore, we get

$$M\int_{|u|>t}|\nabla u|^q\,dx\leqslant c(1+t)^q\mu(t)^{\frac{q}{p}\left(1-\frac{1}{h}-\frac{p-\alpha}{q^*}\right)+\frac{q}{q^*}},$$

that is

$$\frac{1}{\mu(t)^{1/q}}\left(\int_{|u|>t}|\nabla u|^q\,dx\right)^{1/q}\leqslant\frac{c}{M}(1+t)\,\mu(t)^{-\frac{1}{p}\left(\frac{1}{h}+\frac{p-1}{m}\right)}, \tag{3.17}$$

for every $t \geqslant L$, where $L$ is the greatest lower bound of levels greater then 1 satisfying (3.15) and (3.16).

On the other hand, the following inequality holds true ([T]; see also [M], Lemma 4.1 and proof of Theorem 2.1)

$$q^{1/q}\left(1-\frac{q'}{k'}\right)^{1/q'} \frac{n\omega_n^{1/n}}{\mu(t)^{1/k}} \int_t^{+\infty} \mu(\tau)^{1/k-1/n}\, d\tau \leqslant \frac{1}{\mu(t)^{1/q}} \left( \int_{|u|>t} |\nabla u|^q\, dx \right)^{1/q}, \tag{3.18}$$

for some $k<q$, where $\omega_n$ denotes the measure of the ball of $\mathbb{R}^n$ having radius equal to 1, $q'$ and $k'$ denote the Hölder congiugate exponent of $q$ and $k$, respectively.

Combining (3.17) and (3.18), we get

$$\frac{1}{1+t} \leqslant \frac{c}{M} \frac{\mu(t)^{\frac{1}{k}-\frac{1}{p}\left(\frac{1}{h}+\frac{p-1}{m}\right)}}{\int_t^{+\infty} \mu(\tau)^{1/k-1/n}\, d\tau}, \tag{3.19}$$

for every $t \geqslant L$.

Now, let us denote

$$\delta = \frac{\dfrac{1}{k}-\dfrac{1}{n}}{\dfrac{1}{k}-\dfrac{1}{p}\left(\dfrac{1}{h}+\dfrac{p-1}{m}\right)}.$$

Since (3.1) holds true, it results $\delta<1$. Moreover, from (3.19) we get

$$\int_L^{\operatorname{ess\,sup}|u|} \frac{1}{(1+t)^\delta}\, dt \leqslant \frac{c}{M(1-\delta)} \int_L^{\operatorname{ess\,sup}|u|} \frac{d}{d\tau}\left( \int_t^{+\infty} \mu(\tau)^{1/k-1/n}\, d\tau \right)^{1-\delta} dt\,. \tag{3.20}$$

Using (3.19) we can majorize the right hand-side in (3.20) obtaining (see also [T])

$$\int_L^{\operatorname{ess\,sup}|u|} \frac{1}{(1+t)^\delta}\, dt \leqslant \left(\frac{c}{M}\right)^\delta \frac{1}{(1-\delta)} \mu(L)^{\frac{1}{n}-\frac{1}{p}\left(\frac{1}{h}+\frac{p-1}{m}\right)}. \tag{3.21}$$

Since

$$\int_L^{+\infty} \frac{1}{(1+t)^\delta}\, dt = +\infty\,,$$

(3.21) yields that $u$ belongs to $L^\infty(\Omega)$.

From (1.2) and (1.3) we deduce that $u$ belongs to $W^{1,p}(a)$. Indeed

$$\int_\Omega a(x)\,|u|^p\,dx \leqslant \|u\|_\infty^p\,\|a\|_1,$$

$$\frac{1}{(1+\|u\|_\infty)^\alpha}\int_\Omega a(x)\,|\nabla u|^p\,dx \leqslant \int_\Omega a(x)\,\frac{|\nabla u|^p\,dx}{(1+|u|)^\alpha} \leqslant$$

$$\leqslant F(u) + \int_\Omega b_1(x)\,|u|^{\beta_1} + \int_\Omega g_1(x)\,dx \leqslant$$

$$\leqslant G(u) + \|b_1\|_{r_1}\|u\|_\infty^{\beta_1} + \|g_1\|_1 \leqslant c\,.$$

Finally, we get that $u$ is a minimizer of $F$. Indeed

$$F(u) \geqslant \inf\{F(v)\colon v\in W^{1,p}(a)\ \text{ s.t. }\ v-u_0\in W_0^{1,p}(a)\} \geqslant$$

$$\geqslant \min\{G(v)\colon v\in W^{1,p}(a)\ \text{ s.t. }\ v-u_0\in W_0^{1,p}(a)\} \geqslant$$

$$\geqslant G(u) = F(u).$$

REMARK 3.1. – Let us observe that, if $|\Omega|$ is small enough, i.e. $|\Omega| < \min\{1, 1/2c\}$, then (3.14) and (3.16) hold true for every $t \geqslant \text{ess sup}\,|u_0|$ and (3.20) gives the following apriori bound for $|u|$

$$\text{ess } sup\,|u| \leqslant \text{ess sup}\,|u_0| + (c\cdot 1 - c|\Omega|^{\frac{q}{p}\left(1-\frac{1}{h}-\frac{p-\alpha}{q^*}\right)})^{\frac{\delta}{1-\delta}}\,|\Omega|^{\frac{1}{p}\left(\frac{1}{n}-\left(\frac{1}{h}+\frac{p-1}{m}\right)\right)}.$$

REMARK 3.2. – If we choose $\alpha = 0$ and $a(x)$ constant in $\Omega$, Theorem 3.1 gives the classical results for coercive functionals on $W_0^{1,p}(\Omega)$ (see, for example, [LU]).

## REFERENCES


[AFT] A. ALVINO - V. FERONE - G. TROMBETTI, *A priori estimates for a class of non uniformly elliptic equations,* Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 381-391.

[BO] L. BOCCARDO - L. ORSINA, *Existence and regularity of minima for integral functionals noncoercive in the energy space,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **25** (1997), 95-130.

[BDO] L. BOCCARDO - A. DALL'AGLIO, L. ORSINA, *Existence and regularity results for some elliptic equations with degenerate coercivity,* Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 51-58.

[C1] A. CIANCHI, *Local minimizers and rearrangements,* Appl. Math. Optim., **27** (1993), 261-274.

[C2] A. CIANCHI, *Boundedness of solutions to variational problems under general growth conditions,* Comm. Partial Differential Equations, **22** (1997), 1629-1646.

[CS] A. CIANCHI - R. SCHIANCHI, *A priori sharp estimates for minimizers,* Boll. Un. Mat. Ital., **7-B** (1993), 821-831.

[DG] E. DE GIORGI, *Teoremi di semicontinuità nel calcolo delle variazioni*, Lectures Notes, INDAM 1968.

[GP1] D. GIACHETTI - M. M. PORZIO, *Existence results for some non uniformly elliptic equations with irregular data,* J. Math. Anal. Appl., **257** (2001), 100-130.

[GP2] D. GIACHETTI - M. M. PORZIO, *Regularity results for some elliptic equations with degenerate coercivity,* Preprint.

[G] E. GIUSTI, *Metodi diretti nel calcolo delle Variazioni*, UMI, 1994.

[LU] O. A. LADYZENSKAYA - N. N. URAL'CEVA, *Equations aux dérivées partielles de type elliptic*, Dunod, Paris, 1968.

[M] V. MAZ'JA, *Sobolev spaces*, Springer-Verlag, Berlin (1985).

[Me] A. MERCALDO, *Boundedness of minimizers of degenerate functionals,* Differential Integral Equations, **9** (1996), 541-556.

[MS] M. K. V. MURTHY - G. STAMPACCHIA, *Boundary value problems for some degenerate elliptic operators,* Ann. Mat. Pura Appl., **90** (1971), 1-122.

[S] R. SCHIANCHI, *An $L^\infty$-estimates for the minima of functionals of the calculus of variations.,* Differential Integral Equations, **2** (1989), 383-421.

[T] G. TALENTI, *Boundedeness of minimizers,* Hokkaido Math. Jour., **19** (1990), 259-279.

[Tr] C. TROMBETTI, *Existence and regularity for a class of non uniformly elliptic equations in two dimensions,* Differential Integral Equations, **13** (2000), 687-706.

Dipartimento di Matematica e Applicazioni «R. Caccioppoli»
Università di Napoli «Federico II», Via Cintia-80126 Napoli, Italia
e-mail: mercaldo@unina.it

*Pervenuta in Redazione*
*il 21 gennaio 2002*