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A-Compactifications and A-Weight of Alexandroff Spaces (*).

A. Caterino - G. Dimov - M. C. Vipera

Sunto. – *Questo lavoro riguarda l'insieme ordinato $A\mathcal{K}(X, \alpha)$ delle A-compattificazioni di uno spazio di Alexandroff (X, α). Si definisce e si studia l'«A-peso» $aw(X, \alpha)$ dello spazio (X, α) e, sulla base di risultati in* [7], [5], *si presentano proprietà reticolari di $A\mathcal{K}(X, \alpha)$ e di $A\mathcal{K}_{aw}(X, \alpha)$, l'insieme delle A-compattificazioni (Y, t) di (X, α) tali che $w(Y) = aw(X, \alpha)$. Si caratterizzano le famiglie di funzioni continue limitate che generano una A-compattificazione di (X, α). In analogia con definizioni e risultati in* [3], *si introducono e si studiano la nozione di famiglia di funzioni che «A-determina» una A-compattificazione (Y, t) e l'invariante cardinale $a\delta(Y, t)$ (minima cardinalità di una famiglia che A-determina (Y, t)).*

Summary. – *The paper is devoted to the study of the ordered set $A\mathcal{K}(X, \alpha)$ of all, up to equivalence, A-compactifications of an Alexandroff space (X, α). The notion of A-weight (denoted by $aw(X, \alpha)$) of an Alexandroff space (X, α) is introduced and investigated. Using results in* ([7]) *and* ([5]), *lattice properties of $A\mathcal{K}(X, \alpha)$ and $A\mathcal{K}_{aw}(X, \alpha)$ are studied, where $A\mathcal{K}_{aw}(X, \alpha)$ is the set of all, up to equivalence, A-compactifications Y of (X, α) for which $w(Y) = aw(X, \alpha)$. A characterization of the families of bounded functions generating an A-compactification of (X, α) is obtained. The notion of A-determining family of functions, analogous to the one of determining family given in* ([3]), *is introduced and relations with the original notion are investigated. A characterization of the families of functions which A-determine a given A-compactification is found. The cardinal invariant $a\delta(Y, t)$, corresponding to the cardinal invariant $\delta(Y, t)$ defined in* ([3]), *is introduced and studied.*

1. – Introduction.

The notion of an *Alexandroff space* (briefly, *A-space*) was introduced by A. D. Alexandroff in [1] (under the name of *completely normal space*) as a foundation for a general theory of measures and linear functionals. It was rediscovered by H. Gordon [12] (under the name of *zero-set space*) and studied by many authors (see the excellent survey paper of A. Hager [14]). An A-space is a pair (X, α), where X is a set and α is a special subfamily of subsets of X, called *cozero field*. We shall be interested only in the *separated* cozero fields which, in turn, were rediscovered by E. F. Steiner ([20]) under the name of *separating nest-generated intersection rings*, and by R. Alò and H. L. Shapiro

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([2]) – as *strong delta normal bases* (in fact, in [12, 20, 2], the family $\{X\backslash U \mid U \in \alpha\}$ is regarded).

The notion of *A-compactification* of an A-space was introduced by A. D. Alexandroff in [1]. A-compactifications were studied in many papers (see, e.g., [1, 7, 8, 12, 13, 19]). The present paper was born as an attempt to answer the following three natural questions:

1. What does it mean «A-compactification which does not increase the weight»?

2. How can be defined «A-determining families of functions» and what can be proved about them?

3. Which families of functions generate an A-compactification?

The lattice properties of the ordered set $\mathcal{K}_w(X)$ of all, up to equivalence, Hausdorff compactifications of a Tychonoff space X which have the same weight as X were studied by A. Caterino and M. C. Vipera in [5]. The notion of a family of functions determining a compactification was introduced and studied by B. Ball and S. Yokura [3].

In this paper, we show that many results obtained in [5] and [3] for compactifications have their analogues for A-compactifications and we prove, as well, some other results about generation of A-compactifications and about the lattice properties of the ordered set $A\mathcal{K}(X, \alpha)$ of all, up to equivalence, A-compactifications of an A-space (X, α). The results analogous to those of [5] and [3] cannot be obtained automatically. For example, one obstacle is that, fixing a cozero field α on a space (X, τ), one can have that any A-compactification of (X, α) has weight strictly greater than $w(X, \tau)$. The appropriate notion of weight of an A-space is introduced in this paper. It is called *A-weight* and is denoted by $aw(X, \alpha)$. We make use of it in the whole paper. It is very surprising that it was not introduced till now (as far as we know). Further, the Hewitt realcompactification vX plays no role in the results of B. Ball and S. Yokura [3], but its analogue for A-spaces, the Wallman realcompactification $v(X, \alpha)$, takes part in the formulation of the corresponding results for the A-compactifications. The following curious fact (which can be derived from a result of A. Hager [13]) shows very clearly the difference between the notions of A-determining and determining family of functions: if X is a pseudocompact non-locally compact space then no compactification of X is determined by a constant function, but every compactification of X is A-determined (with respect to some compatible cozero field) by any constant function.

2. – Preliminaries.

We shall denote by **R** (resp., **Q**) the real line (resp., the rationals); $\mathcal{P}(X)$ will stand for the power set of the set X; by ω (resp., ω_1) it will be denoted the first infinite ordinal number (resp., the first uncountable ordinal number) and c will stand for the cardinality of $\mathbf{R}$.

Let X be a Tychonoff space.

NOTATION 2.1. – As usual, we put $C(X) = \{f : X \to \mathbf{R} \mid f \text{ is continuous}\}$, $C^*(X) = \{f \in C(X) \mid f \text{ is bounded}\}$. For $f \in C(X)$, $\mathrm{Coz}(f)$ denotes the cozero set of f. For $\mathcal{F} \subseteq C(X)$, we put $\mathrm{Coz}(\mathcal{F}) = \{\mathrm{Coz}(f) \mid f \in \mathcal{F}\}$ and we write $\mathrm{Coz}(X)$ instead of $\mathrm{Coz}(C(X))$.

We denote by $\mathcal{K}(X)$ the set of all compactifications of X (up to the natural equivalence). We will consider $\mathcal{K}(X)$ partially ordered in the usual way. The Stone-Čech compactification of X will be denoted by βX and, when X is locally compact, αX will stand for the Alexandroff one-point compactification of X.

The following is well known:

FACT 2.2. – *(a) $\mathcal{K}(X)$ is a complete upper semilattice and $\beta X = \max(\mathcal{K}(X))$;*

(b) $\mathcal{K}(X)$ is a complete lattice if and only if X is locally compact. In this case the smallest element of $\mathcal{K}(X)$ is the one-point compactification αX of X.

We will usually denote a compactification of X by a pair (Y, t), where t is the dense embedding of X into the compact space Y. We can suppose, up to homeomorphism, that X is a subspace of Y and t is the canonical injection.

NOTATION 2.3. – For a compactification (Y, t) of X, we put $\mathcal{F}_t = \{f \in C^*(X) \mid f \text{ can be continuously extended to } Y\}$.

Let us recall the following (see, e.g., [9, 6]):

FACT 2.4. – *For every $(Y, t) \in \mathcal{K}(X)$ one has:*

(a) $\mathcal{F}_t$ is a subalgebra of $C^*(X)$ which separates points from closed sets;

(b) $\mathcal{F}_t$ is closed with respect to the uniform convergence topology;

(c) If (Z, h) is also in $\mathcal{K}(X)$, one has $(Y, t) < (Z, h)$ if and only if $\mathcal{F}_t \subset \mathcal{F}_h$;

(d) If $(Y, t) = \beta X$, then $\mathcal{F}_t = C^*(X)$.

NOTATION 2.5. – Let $\mathcal{F} \subseteq C^*(X)$. Following [6], we denote by $e_{\mathcal{F}}$ the diagonal map of $\mathcal{F}$ from X into $\mathbf{R}^{|\mathcal{F}|}$. Choosing an interval $I_f \subseteq \mathbf{R}$ containing $f(X)$, for each $f \in \mathcal{F}$, we can consider $e_{\mathcal{F}}$ as a map from X to a cube $\prod_{f \in \mathcal{F}} I_f$. In case $e_{\mathcal{F}}$ is an embedding, in particular, if $\mathcal{F}$ separates points from closed sets, then $(cl(e_{\mathcal{F}}(X)), e_{\mathcal{F}})$ is a compactification of X, denoted by $e_{\mathcal{F}}X$.

DEFINITION 2.6. – ([3]) We say that $\mathcal{F}$ *generates* the compactification (Y, t) if (Y, t) is equivalent to $e_{\mathcal{F}}X$.

The following proposition is well known (see [3] and [6]).

PROPOSITION 2.7. – (*a*) *The family* $\mathcal{F}_t$ *always generates* (Y, t);

(*b*) *If* $\mathcal{F}$ *generates* (Y, t), *then* (Y, t) *is the smallest element of the set of all compactifications of* X *to which every element of* $\mathcal{F}$ *continuously extends. In particular,* $\mathcal{F} \subseteq \mathcal{F}_t$;

(*c*) *Let* $\{(Y_j, t_j)\}_{j \in J}$ *be a family of compactifications. If, for every* $j \in J$, (Y_j, t_j) *is generated by a family* $\mathcal{F}_j \subseteq C^*(X)$, *then* $\bigcup_{j \in J} \mathcal{F}_j$ *generates* $\sup_{\mathcal{K}(X)}\{(Y_j, t_j)\}_{j \in J}$.

Now we recall some definitions and known facts about A-spaces and A-compactifications.

DEFINITION 2.8. – ([1]) Let X be a set. A subfamily α of $\mathcal{P}(X)$ is called a *cozero field* if it satisfies the following conditions:

a) $\emptyset, X \in \alpha$ and α is closed under finite intersection and countable unions.

b) (normality) If $A, B \in \alpha$, $A \cup B = X$ then there exist disjoint $C, D \in \alpha$ such that $A \cup C = X$, $B \cup D = X$.

c) If $A \in \alpha$ then there exist a countable family $\{A_n\}_{n \in \omega}$, with $A_n \in \alpha$, such that $X \backslash A = \bigcap_{n \in \omega} A_n$.

A cozero field α is said to be *separated* if, for every two distinct points of X, there is $A \in \alpha$ which contains exactly one of them.

The pair (X, α), where α is a (separated) cozero field, is called a (*separated*) *Alexandroff space* (*A-space*, for short).

DEFINITION 2.9. – ([1]) Let (X, α) be an A-space. For every $Z \subseteq X$, the family $\alpha|_Z = \{A \cap Z | A \in \alpha\}$ is a cozero field on Z and the pair $(Z, \alpha|_Z)$ is called an *A-subspace* of (X, α).

DEFINITION 2.10. – ([1]) A subset D of X is said to be *A-dense* in (X, α) if every nonempty member of α meets D. We denote by $d(X, \alpha)$ the minimum of the cardinalities of all A-dense subsets of (X, α).

DEFINITION 2.11. – ([1]) If $(X, \alpha), (Y, \gamma)$ are A-spaces, an A-map $f : (X, \alpha) \to (Y, \gamma)$ is a map from X to Y such that $f^{-1}(U) \in \alpha$ for every $U \in \gamma$. An A-map f is called an *A-isomorphism* if it is bijective and f^{-1} is also an A-

map; f is called an *A-embedding* if the restriction of f to the image $f(X)$ is an A-isomorhism from (X, α) onto $(f(X), \gamma|_{f(X)})$.

Clearly, the composition of two A-maps is an A-map.

DEFINITION 2.12. – ([12]) Let $\mathcal{A} = \{(X_j, \alpha_j)\}_{j\in J}$ be a family of A-spaces. For each j let $p_j\colon \prod_{j\in J} X_j \to X_j$ be the projection. We put $\prod_{j\in J}(X_j, \alpha_j) = \left(\prod_{j\in J} X_j, \alpha\right)$, where α is the cozero field which we obtain by taking the countable unions of the finite intersections of all members of the family $\bigcup_{j\in J}\{p_j^{-1}(U) \mid U \in \alpha_j\}$. $\prod_{j\in J}(X_j, \alpha_j)$ is the *product* of the family $\mathcal{A}$ in the category of A-spaces and A-maps.

DEFINITION 2.13. – ([1]) (X, α) is said to be A-compact if every cover of X contained in α has a finite subcover.

DEFINITION 2.14. – ([1]) Let (X, α) be an A-space. An *A-compactification* of (X, α) is a pair $((Y, \gamma), t)$ where (Y, γ) is an A-compact A-space, $t\colon (X, \alpha) \to (Y, \gamma)$ is an A-embedding and $t(X)$ is A-dense in (Y, γ). Given two A-compactifications $((Y, \gamma), t)$ and $((Y_1, \gamma_1), h)$ we say that $((Y, \gamma), t) \leq ((Y_1, \gamma_1), h)$ if there is an A-map $g : (Y_1, \gamma_1) \to (Y, \gamma)$ such that $g \circ h = t$. If such a map g is an A-isomorphism then we also have $((Y_1, \gamma_1), h) \leq ((Y, \gamma), t)$. In this case we say that $((Y, \gamma), t)$ and $((Y_1, \gamma_1), h)$ are equivalent.

We denote by $A\mathcal{K}(X, \alpha)$ the set of all, up to equivalence, A-compactifications of (X, α). The relation $\leq$ induces a partial order on $A\mathcal{K}(X, \alpha)$.

If $((Y, \gamma), t)$ is an A-compactification of (X, α), we can always suppose that (X, α) is an A-subspace of (Y, γ) and t is the inclusion map.

PROPOSITION 2.15. – ([1, 12]) *Every cozero field α on X is a base for a topology τ_α on X. If α is separated, then the space (X, τ_α) is Tychonoff.*

From now on, all A-spaces will be supposed to be separated and, by the word «space», we will mean «Tychonoff topological space».

DEFINITION 2.16. – If (X, τ) is a space and α is a cozero field on the set X, we say that α is a *compatible cozero field* (or, α is a cozero field *on the space* (X, τ)) if $\tau = \tau_\alpha$.

PROPOSITION 2.17. – ([1, 12]) *For every space X,* $\mathrm{Coz}(X)$ *is a compatible cozero field. Every compatible cozero field on X is contained in* $\mathrm{Coz}(X)$.

THEOREM 2.18. – ([4, 15]) *Let X be a space. Then* $\mathrm{Coz}(X)$ *is the unique compatible cozero field on X if and only if X is Lindelöf or almost compact.*

PROPOSITION 2.19. – ([1]) *Let* (X, α) *be an A-space and let* $X = (X, \tau_\alpha)$. *Then:*

(a) *A subset D of X is A-dense in* (X, α) *if and only if it is dense in X. Hence* $d(X, \alpha) = d(X)$.

(b) (X, α) *is A-compact if and only if X is compact. In this case one has* $\alpha = \mathrm{Coz}(X)$.

PROPOSITION 2.20. – ([1]) *Let* (X, α), (Y, γ) *be A-spaces and let* $f:(X, \alpha) \to (Y, \gamma)$ *be an A-map. Then:*

(a) *f is a continuous map from* (X, τ_α) *to* (Y, τ_γ);

(b) *if f is an A-isomorphism then it is a homeomorphism;*

(c) *if f is an A-embedding, it is also a topological embedding.*

The converses hold in case X and Y admit a unique cozero field.

From 2.18, 2.19 and 2.20 it follows:

PROPOSITION 2.21. – *Let X be a space and let* α *be a compatible cozero field on X. If* $((Y, \gamma), t)$ *is an A-compactification of* (X, α), *then* (Y, t) *is a compactification of X. If* $((Y, \gamma), t)$ *and* $((Y_1, \gamma_1), h)$ *are A-compactifications of* (X, α), *then* $((Y, \gamma), t) \leq ((Y_1, \gamma_1), h)$ *if and only if* $(Y, t) \leq (Y_1, h)$. *In particular* $((Y, \gamma), t)$ *and* $((Y_1, \gamma_1), h)$ *are equivalent if and only if* (Y, t) *and* (Y_1, h) *are equivalent compactifications of X.*

Therefore, an A-compactification of (X, α) can be viewed as a compactification (Y, t) of X such that $\mathrm{Coz}(Y)|_X = \alpha$ or, equivalently, $\mathrm{Coz}(\mathcal{F}_t) = \alpha$. Moreover, $(A\mathcal{K}(X, \alpha), \leq)$ can be considered as a subset of the ordered set $(\mathcal{K}(X), \leq)$.

NOTATION 2.22. – We denote by $\mathcal{F}(\alpha)$ the set of all bounded A-maps from (X, α) to $(\mathbf{R}, \mathrm{Coz}(\mathbf{R}))$. One has $\mathcal{F}(\alpha) \subseteq C^*(X)$, where $X = (X, \tau_\alpha)$.

It is easy to see that the set of the complements of the elements of α forms a normal base on X (in the sense of [10]). The Wallman compactification induced by that base (see [10]) is denoted by $\beta(X, \alpha)$. It is well known that $\beta(X, \mathrm{Coz}(X)) = \beta X$.

THEOREM 2.23. – ([1]) *Let* (X, α) *be an A-space. Then:*

(a) $\beta(X, \alpha)$ *is an A-compactification;*

(b) *For every* $(Y, t) \in A\mathcal{K}(X, \alpha)$, *one has* $(Y, t) \leq \beta(X, \alpha)$, *that is* $\beta(X, \alpha)$ *is the maximum of* $A\mathcal{K}(X, \alpha)$;

(c) $\mathcal{F}(\alpha) = \{f \in C^*(X) \mid f$ *has a continuous extension to* $\beta(X, \alpha)\}$.

(d) *If* $(Y, t) \in A\mathcal{K}(X, \alpha)$ *then* $\mathcal{F}_t \subseteq \mathcal{F}(\alpha)$.

If X is any space and (Y, t) is a compactification of X, then $(Y, t) \in A\mathcal{K}(X, \alpha)$, where $\alpha = \mathrm{Coz}(Y)|_X = \mathrm{Coz}(\mathcal{F}_t)$ is a compatible cozero field on X. Therefore:

PROPOSITION 2.24. – *For every space X, one has $\mathcal{K}(X) = \bigcup_{\alpha \in \mathcal{CF}} A\mathcal{K}(X, \alpha)$, where $\mathcal{CF}$ is the set of all compatible cozero fields on X. The union is disjoint, so we have a partition of $\mathcal{K}(X)$.*

For all undefined here notions and notations see [9].

3. – A-weight of Alexandroff spaces and weight of A-compactifications.

DEFINITION 3.1. – Let (X, α) be an A-space. We will say that a subset $\mathcal{B}$ of α is an *A-base* for (X, α) if every element of α can be expressed as a countable union of members of $\mathcal{B}$. The *A-weight* of (X, α), denoted by $aw(X, \alpha)$, will be the minimum cardinality of an A-base.

DEFINITION 3.2. – Let (X, α) be an A-space. A subset $\mathcal{S}$ of α is said to be an *A-subbase* of α if the family of the finite intersections of the elements of $\mathcal{S}$ is an A-base for α. Clearly $aw(X, \alpha)$ is also the minimum cardinality of an A-subbase.

REMARK 3.3. – Every A-(sub)base of (X, α) is a (sub)base for the space (X, τ_α).

REMARK 3.4. – If $\mathcal{B}$ is an A-(sub)base of (X, α), then, for every cozero field γ on X containing $\mathcal{B}$, one has $\alpha \subseteq \gamma$.

The following proposition is obvious.

PROPOSITION 3.5. – *Let (X, α) be an A-space. Then:*
(a) $|\alpha| \leq (aw(X, \alpha))^\omega$.
(b) *For each $Z \subseteq X$, one has $aw(Z, \alpha|_Z) \leq aw(X, \alpha)$.*
(c) $w(X, \tau_\alpha) \leq aw(X, \alpha)$.

Let us show that the inequality in 3.5(c) can be strict.

EXAMPLE 3.6. – If D is a discrete space with $|D| = c$, then $|\mathrm{Coz}(D)| = |\mathcal{P}(D)| = 2^c$. Hence, by 3.5$(a)$ and the fact that $c^\omega = c$, we obtain $aw(D, \mathrm{Coz}(D)) > c = w(D)$.

The above example can be generalized as follows.

PROPOSITION 3.7. – *Let* μ *be a cardinal such that, for every cardinal* θ *satisfying* $\theta < 2^{\mu}$, *one has* $\theta^{\omega} < 2^{\mu}$. *Then, for the discrete space* $D(\mu)$ *of cardinaly* μ, *one has* $aw(D(\mu), \mathrm{Coz}(D(\mu))) > w(D(\mu))$.

Notice that, under GCH, every cardinal with uncountable cofinality satisfies the hypothesis of the above proposition (see, e.g., [17]). On the other hand, it is compatible with ZFC that ω_1 does not satisfy it.

THEOREM 3.8. – *Let* (X, α) *be an A-space and let* $X = (X, \tau_{\alpha})$ *be Lindelöf. Then* $aw(X, \alpha) = w(X)$.

PROOF. – Since α is a base for τ_{α}, there exists a base $\mathcal{B} \subseteq \alpha$ with $|\mathcal{B}| = w(X)$. Every element U of α, being an F_{σ}, is Lindelöf and so it is a countable union of members of $\mathcal{B}$. Therefore $\mathcal{B}$ is an A-base for α. Hence $aw(X, \alpha) \leq w(X, \tau_{\alpha})$. Now 3.5(*c*) finishes the proof. ∎

We will show later (see 3.18, 3.20) that there exist non-Lindelöf spaces X such that for every compatible cozero field α on X one has $aw(X, \alpha) = w(X)$.

PROPOSITION 3.9. – *For a family* $\{(X_j, \alpha_j)\}_{j \in J}$ *of A-spaces, one has*

$$aw\left(\prod_{j \in J}(X_j, \alpha_j)\right) = \max\left(|J|, \sup_{j \in J}\{aw(X_j, \alpha_j)\}\right).$$

PROOF. – It follows from the fact that the family $\bigcup_{j \in J}\{p_j^{-1}(U) \mid U \in \alpha_j\}$ is an A-subbase for $\prod_{j \in J}(X_j, \alpha_j)$ (see 2.12). ∎

The proof of the following proposition is essentially the same as the proof of the analogous result about weight and open continuous maps.

PROPOSITION 3.10. – *If* $f : (X, \alpha) \to (Y, \gamma)$ *is a surjective A-map such that, for every* $A \in \alpha$, $f(A) \in \gamma$, *then* $aw(Y, \gamma) \leq aw(X, \alpha)$.

The following result is analogous to the well-known theorem of P. Alexandroff and P. Urysohn.

PROPOSITION 3.11. – *For every A-base* $\mathcal{B}$ *of* (X, α), *there is an A-base* $\mathcal{B}_1 \subseteq \mathcal{B}$ *such that* $|\mathcal{B}_1| = aw(X, \alpha)$.

PROOF. – Let $\mathcal{B}_0$ be an A-base of (X, α) with $|\mathcal{B}_0| = aw(X, \alpha)$. We put $J = \{(B_1, B_2) \in \mathcal{B}_0 \times \mathcal{B}_0 \mid \exists A \in \mathcal{B} : B_1 \subseteq A \subseteq B_2\}$. For every $(B_1, B_2) \in J$ we choose $A_{(B_1, B_2)} \in \mathcal{B}$ with $B_1 \subseteq A_{(B_1, B_2)} \subseteq B_2$ and we put $\mathcal{B}_1 = \{A_{(B_1, B_2)} \mid (B_1, B_2) \in J\}$. One has $|\mathcal{B}_1| \leq aw(X, \alpha)$. We will prove that $\mathcal{B}_1$ is an A-base of (X, α). If $U \in \alpha$, then U is the union of a countable family $\mathcal{C} = \{V_n\}_{n \in \omega} \subseteq \mathcal{B}_0$. Every V_n is the union of a countable family $\{A_{n,m}\}_{m \in \omega} \subseteq \mathcal{B}$ and every $A_{n,m}$ is the union of a countable family $\{W_{n,m,k}\}_{k \in \omega} \subseteq \mathcal{B}_0$. Then $\mathcal{C}' = \{W_{n,m,k}\}_{n,m,k \in \omega}$ is a countable subfamily of $\mathcal{B}_0$ whose union is U. Put $I = (\mathcal{C}' \times \mathcal{C}) \cap J$. Then $|I| = \omega$. Clearly, for every $W \in \mathcal{C}'$, there is $V \in \mathcal{C}$ such that $(W, V) \in I$. Therefore, $\{A_{(W,V)}\}_{(W,V) \in I}$ is a countable subfamily of $\mathcal{B}_1$ whose union is U. ■

PROPOSITION 3.12. – *Let (X, α) be an A-space and let $X = (X, \tau_\alpha)$. If (Y, t) is an A-compactification of (X, α) then $w(X) \leq aw(X, \alpha) \leq w(Y)$.*

PROOF. – Since (X, α) is (A-isomorphic to) an A-subspace of $(Y, \mathrm{Coz}(Y))$, one has $aw(X, \alpha) \leq aw(Y, \mathrm{Coz}(Y)) = w(Y)$. ■

The second inequality in the above proposition can be strict. If X is any second countable space, then, by 3.8, $aw(X, \mathrm{Coz}(X)) = w(X) = \omega$. As we know, $\beta X \in A\mathcal{K}(X, \mathrm{Coz}(X))$ and it is easy to see that $w(\beta X) = c$.

COROLLARY 3.13. – *Every space X has a compatible cozero field α such that $aw(X, \alpha) = w(X)$.*

PROOF. – This follows from 3.12 and from the fact that for every space X there is a compactification (Y, t) such that $w(Y) = w(X)$ (see 2.3.23 of [9]).

REMARK 3.14. – The above corollary implies that, if X is (Lindelöf or) almost compact, then $aw(X, \alpha) = w(X)$, where α is the unique compatible cozero field.

COROLLARY 3.15. – *For every (X, α), $aw(X, \alpha) \leq 2^{d(X, \alpha)}$.*

PROOF. – One has $aw(X, \alpha) \leq w(\beta(X, \alpha)) \leq 2^{d(X, \tau_\alpha)} = 2^{d(X, \alpha)}$. ■

COROLLARY 3.16. – *For every A-space (X, α), one has $w(X, \tau_\alpha) \leq aw(X, \alpha) \leq 2^{w(X, \tau_\alpha)}$.*

PROOF. – It follows from 3.15, 3.5(c) and the fact that $d(X) \leq w(X)$. ■

COROLLARY 3.17. – *If X is a space such that $w(X) = 2^{d(X)}$, then for every compatible cozero field α one has $aw(X, \alpha) = w(X)$.*

EXAMPLE 3.18. – The Niemytzski plane satisfies the hypothesis of the above corollary and is not Lindelöf.

REMARK 3.19. – Notice that that both possibilities in the inequality stated in 3.15 can be realized. Indeed, Example 3.18 shows that there exist spaces (X, α) such that $aw(X, \alpha) = 2^{d(X, \alpha)}$. On the other hand, every space X for which $w(X) < 2^{d(X)}$ (e.g., every metrizable space) has, according to 3.13, a compatible cozero field α such that $aw(X, \alpha) = w(X)$ and hence $aw(X, \alpha) < 2^{d(X)}$.

COROLLARY 3.20. – *If X is a space such that $w(X) = w(\beta X)$, then $w(X) = aw(X, \alpha)$, for every compatible cozero field α on X.*

PROOF. – It follows from the inequalities $w(X) \leqslant aw(X, \alpha) \leqslant w(\beta(X, \alpha)) \leqslant w(\beta X) = w(X)$. ■

DEFINITION 3.21. – For an A-space (X, α), we put

$$A\mathcal{K}_{aw}(X, \alpha) = \{(Y, t) \in A\mathcal{K}(X, \alpha) \,|\, w(Y) = aw(X, \alpha)\}.$$

We will see that $A\mathcal{K}_{aw}(X, \alpha)$ is always nonempty.

LEMMA 3.22. – *Let $(Y, t) \in A\mathcal{K}(X, \alpha)$ and let $\mathcal{G} \subseteq \mathcal{F}_t$. If $\mathrm{Coz}(\mathcal{G})$ is an A-base of (X, α), then $\mathcal{G}$ generates an A-compactification (Z, h) of (X, α) with $(Z, h) \leqslant (Y, t)$.*

PROOF. – Since $\mathrm{Coz}(\mathcal{G})$ is a base of $X = (X, \tau_\alpha)$, we have that $\mathcal{G}$ separates points from closed sets. Therefore $\mathcal{G}$ generates a compactification (Z, h) of X (see 2.5, 2.6). Then, by 2.7(b), one has $\mathcal{G} \subseteq \mathcal{F}_h$ and $(Z, h) \leqslant (Y, t)$. Hence, by 2.4, $\mathcal{F}_h \subseteq \mathcal{F}_t$. Then $\mathrm{Coz}(\mathcal{G}) \subseteq \mathrm{Coz}(\mathcal{F}_h) \subseteq \mathrm{Coz}(\mathcal{F}_t)$. Since $\mathrm{Coz}(\mathcal{F}_h) = \mathrm{Coz}(Z)|_X$ is a cozero field on X, we obtain, by 3.4, that $\alpha \subseteq \mathrm{Coz}(\mathcal{F}_h) \subseteq \mathrm{Coz}(\mathcal{F}_t) = \alpha$, that is $\mathrm{Coz}(\mathcal{F}_t) = \alpha$. Hence $(Z, h) \in A\mathcal{K}(X, \alpha)$. ■

THEOREM 3.23. – *For every $(Y, t) \in A\mathcal{K}(X, \alpha)$, there exists $(Z, h) \in A\mathcal{K}_{aw}(X, \alpha)$ such that $(Z, h) \leqslant (Y, t)$.*

PROOF. – Let $\mathcal{B}$ be an A-base of (X, α) with $|\mathcal{B}| = aw(X, \alpha)$. Since $\alpha = \mathrm{Coz}(\mathcal{F}_t)$, for every $B \in \mathcal{B}$ we can choose $f_B \in \mathcal{F}_t$ such that $B = \mathrm{Coz}(f_B)$. Put $\mathcal{G} = \{f_B \,|\, B \in \mathcal{B}\}$. Then $|\mathcal{G}| = aw(X, \alpha)$. By the above lemma, $\mathcal{G}$ generates an A-compactification (Z, h) of (X, α) such that $(Z, h) \leqslant (Y, t)$. Since (Z, h) is (homeomorphic to) a subspace of $\mathbf{R}^{|\mathcal{G}|}$, one has $w(Z) \leqslant aw(X, \alpha)$. The reverse inequality always holds (see 3.12), so the conclusion follows. ■

COROLLARY 3.24. – *For every A-space (X, α), the following are equivalent:*

(a) $aw(X, \alpha) > w(X, \tau_\alpha)$;

(b) $w(Y) > w(X, \tau_\alpha)$, *for every* $(Y, t) \in A\mathcal{K}(X, \alpha)$.

PROOF. – It follows from 3.12 and from the above theorem. ∎

The following definition generalizes the notion of the Hewitt realcompactification υX of a space. Let us note, before stating it, that α is a compatible cozero field on a space X if and only if the family $\{X \setminus U \mid U \in \alpha\}$ is a *separating, nest-generated intersection ring* (in the sense of E. F. Steiner [20]) (also called *strong delta normal base* in [2]).

DEFINITION 3.25 ([16, 13, 19, 12]). – Let α be a compatible cozero field on the space X. Let us consider the following subspace of $\beta(X, \alpha)$:

$$\upsilon(X, \alpha) = \{u \in \beta(X, \alpha) \mid u \text{ has the countable intersection property}\}.$$

(We recall that $\beta(X, \alpha)$ is the space of all $\mathcal{Z}_\alpha$-ultrafilters, where $\mathcal{Z}_\alpha = \{X \setminus U \mid U \in \alpha\}$). The space $\upsilon(X, \alpha)$ is called the *Wallman realcompactification* of X with respect to α.

Let us recall some known facts about $\upsilon(X, \alpha)$. First of all, $\upsilon X = \upsilon(X, \mathrm{Coz}(X))$. Further, $\upsilon(X, \alpha)$ is always realcompact. For $\alpha \neq \mathrm{Coz}(X)$, $\upsilon(X, \alpha)$ can be different from X even when X is realcompact. An equivalent definition of $\upsilon(X, \alpha)$ is the following one:

$$\upsilon(X, \alpha) = \bigcap \{U \in \mathrm{Coz}(\beta(X, \alpha)) \mid X \subset U\}.$$

More generally one has:

THEOREM 3.26. – (Theorem 3.9 of [19], Theorem 4.2 of [13]) Let (Y, t) be an A-compactification of (X, α). Then the canonical map from $\beta(X, \alpha)$ onto Y maps homeomorphically $\upsilon(X, \alpha)$ onto its image. Hence $X \subseteq \upsilon(X, \alpha) \subseteq Y$ (up to homeomorphism). Moreover, $\upsilon(X, \alpha) = \bigcap \{U \in \mathrm{Coz}(Y) \mid X \subset U\}$.

COROLLARY 3.27. – *Let α be a compatible cozero field on the space X.*
(*a*) *If $aw(X, \alpha) = w(X)$ then $w(\upsilon(X, \alpha)) = w(X)$.*
(*b*) *If $\upsilon(X, \alpha)$ is Lindelöf, then $w(\upsilon(X, \alpha)) = aw(X, \alpha)$.*

PROOF. – (*a*) From 3.23 and 2.23 we have that $A\mathcal{K}_{aw}(X, \alpha) \neq \emptyset$. Let $Y \in A\mathcal{K}_{aw}(X, \alpha)$. Then, by 3.26, $w(X) \leq w(\upsilon(X, \alpha)) \leq w(Y) = aw(X) = w(X)$.

(*b*) We know from 2.18 that $\upsilon(X, \alpha)$ has a unique compatible cozero field $\gamma = \mathrm{Coz}(\upsilon(X, \alpha))$ and, by 3.8, $aw(\upsilon(X, \alpha), \gamma) = w(\upsilon(X, \alpha))$. Let $Y \in A\mathcal{K}_{aw}(X, \alpha)$. Then, by 3.26, $aw(X, \alpha) \leq w(\upsilon(X, \alpha)) \leq w(Y) = aw(X, \alpha)$. ∎

4. – Lattice properties.

Let (X, α) be an A-space and let $X = (X, \tau_\alpha)$.

LEMMA 4.1. – *Let* $(Y, t), (Z, h), (S, u) \in \mathcal{K}(X)$ *and suppose* $(Y, t) \leq (Z, h) \leq (S, u)$. *If* $(Y, t), (S, u) \in A\mathcal{K}(X, \alpha)$, *then* (Z, h) *is also in* $A\mathcal{K}(X, \alpha)$.

PROOF. – One has $\mathcal{F}_t \subseteq \mathcal{F}_h \subseteq \mathcal{F}_u$ and $\mathrm{Coz}(\mathcal{F}_t) = \mathrm{Coz}(\mathcal{F}_u) = \alpha$. ■

PROPOSITION 4.2. – (a) *For every* $\mathcal{S} \subseteq A\mathcal{K}(X, \alpha)$, *one has* $\sup_{\mathcal{K}(X)} \mathcal{S} \in A\mathcal{K}(X, \alpha)$ (*hence* $\sup_{\mathcal{K}(X)} \mathcal{S} = \sup_{A\mathcal{K}(X, \alpha)} \mathcal{S}$ *and* $A\mathcal{K}(X, \alpha)$ *is a complete upper subsemilattice of* $\mathcal{K}(X)$);

(b) *If* $\mathcal{S} \subseteq A\mathcal{K}(X, \alpha)$ *has an infimum in* $A\mathcal{K}(X, \alpha)$, *then* $\inf_{A\mathcal{K}(X, \alpha)} \mathcal{S} = \inf_{\mathcal{K}(X)} \mathcal{S}$.

(c) *If* $A\mathcal{K}(X, \alpha)$ *has a smallest element, then* $A\mathcal{K}(X, \alpha)$ *is a complete lattice.*

PROOF. – To prove (a), it suffices to observe that $\sup_{\mathcal{K}(X)} \mathcal{S} \leq \sup_{\mathcal{K}(X)} (A\mathcal{K}(X, \alpha)) = \max(A\mathcal{K}(X, \alpha)) = \beta(X, \alpha)$ and apply 4.1. The proof of (b) is an easy consequence of 4.1 and (c) follows from the fact that $A\mathcal{K}(X, \alpha)$ is a complete upper semilattice. ■

In Cor. 4.9 of [8], it was shown by a different proof that $A\mathcal{K}(X, \alpha)$ is a complete upper semilattice.

Let us note, in connection with 4.2(b), that if $\mathcal{S} \subseteq A\mathcal{K}(X, \alpha)$ has an infimum in $\mathcal{K}(X)$, then, in general, we cannot affirm that S has an infimum in $A\mathcal{K}(X, \alpha)$ (see Example 4.9 below).

If a space X has more than one compatible cozero field, the local compactness of X is not sufficient to ensure that $A\mathcal{K}(X, \alpha)$ has a smallest element. A necessary and sufficient condition is given in [7]. We need first some definitions.

DEFINITION 4.3. – ([7]) Let X be a space and let α be a compatible cozero field on X. X is said to be *realcompact with respect to* $\beta(X, \alpha)$ *(or with respect to any element of* $A\mathcal{K}(X, \alpha)$*) if* $v(X, \alpha) = X$. *X is said to be pseudocompact with respect to* $\beta(X, \alpha)$ if $v(X, \alpha) = \beta(X, \alpha)$. Clearly X is realcompact (pseudocompact) if and only if X is realcompact (resp. pseudocompact) with respect to $\beta(X, \mathrm{Coz}(X)) (= \beta X)$.

THEOREM 4.4. – (Theorems 3, 4 of [7]) (a) $|A\mathcal{K}(X, \alpha)| = 1$ *if and only if* X *is pseudocompact with respect to* $\beta(X, \alpha)$.

(*b*) *If* $|A\mathcal{K}(X, \alpha)| > 1$, *then* $A\mathcal{K}(X, \alpha)$ *has a smallest element if and only if* $\upsilon(X, \alpha)$ *is locally compact and Lindelöf.*

We will need the following fact obtained in the proof of Theorem 4 of [7]:

LEMMA 4.5. – [7] *Let* (Z, h) *be an A-compactification of* (X, α) *and let* $z_1, z_2 \in Z \setminus \upsilon(X, \alpha)$. *Then the compactification of* $X = (X, \tau_\alpha)$ *obtained by collapsing* z_1 *and* z_2 *to one point is still an A-compactification of* (X, α).

Now we can prove the following:

PROPOSITION 4.6. – *Let* X *be a space and let* α *be a compatible cozero field on* X. *Suppose that* X *is realcompact with respect to* $\beta(X, \alpha)$.

(*a*) *If* X *is locally compact, then* $\inf_{\mathcal{K}(X)}(A\mathcal{K}(X, \alpha))$ *is the one-point compactification* αX *of* X.

(*b*) *If* X *is not locally compact then* $A\mathcal{K}(X, \alpha)$ *does not have infimum in* $\mathcal{K}(X)$.

PROOF. – Suppose that $A\mathcal{K}(X, \alpha)$ has an infimum (Y, t) in $\mathcal{K}(X)$ and there are two distinct points y_1, y_2 in $Y \setminus X$. Let $(Z, h) \in A\mathcal{K}(X, \alpha)$ and let q be the unique map from Z onto Y which is the identity on X. Then there are $z_1, z_2 \in Z \setminus X$ such that $q(z_i) = y_i$, $i = 1, 2$. By the above lemma, the compactification of X obtained by collapsing z_1 and z_2 to one point is still in $A\mathcal{K}(X, \alpha)$, but it cannot be greater than or equal to (Y, t), which is a contradiction. Therefore, if $(Y, t) = \inf_{\mathcal{K}(X)}(A\mathcal{K}(X, \alpha))$, then $Y \setminus X$ must contain just one point. This proves both (*a*) and (*b*). ■

COROLLARY 4.7. – *Let* X *be a realcompact space. If* X *is locally compact, then*

$$\inf_{\mathcal{K}(X)} (A\mathcal{K}(X, \mathrm{Coz}(X))) = \alpha X .$$

If X *is not locally compact then* $A\mathcal{K}(X, \mathrm{Coz}(X))$ *does not have infimum in* $\mathcal{K}(X)$.

PROOF. – Put $\alpha = \mathrm{Coz}(X)$ in Proposition 4.6. ■

REMARK 4.8. – Suppose X is locally compact. Put $\alpha_{\min} = \mathrm{Coz}(\alpha X)|_X$ (where αX is the one-point compactification of X). One has $\alpha_{\min} \subseteq \alpha$ for every compatible cozero field (see 2.4(*c*)). Hence, unless X admits only a unique compatible cozero field, $\alpha_{\min} \neq \mathrm{Coz}(X)$. Therefore $\alpha X \in A\mathcal{K}(X, \mathrm{Coz}(X))$ (and is the smallest element in it) if and only if X is almost compact or Lindelöf (see 2.18). It is known that, if X is not Lindelöf, then one has $A\mathcal{K}(X, \alpha_{\min}) = \{\alpha X\}$ (Theorem 2.6 of [19]).

In [7] it is proved that, if X is not either Lindelöf or locally compact, then X does not admit a smallest compatible cozero field.

EXAMPLE 4.9. – The space $D = D(c)$ is locally compact but $vD = D$ is not Lindelöf. Then, by 4.4(b), $A\mathcal{K}(D, \mathrm{Coz}(D))$ does not have a smallest element. However, by Proposition 4.6 (or Corollary 4.7), $\inf_{\mathcal{K}(D)}(A\mathcal{K}(D, \mathrm{Coz}(D))) = \alpha D$.

Let (X, α) be an A-space and put $X = (X, \tau_\alpha)$. We will give some lattice properties of $A\mathcal{K}_{aw}(X, \alpha)$ regarded as subset of the partially ordered set $A\mathcal{K}(X, \alpha)$.

THEOREM 4.10. – (a) *If* $(Y, t) \in A\mathcal{K}_{aw}(X, \alpha)$, $(Z, h) \in A\mathcal{K}(X, \alpha)$ *and* $(Z, h) \leqslant (Y, t)$, *then* $(Z, h) \in A\mathcal{K}_{aw}(X, \alpha)$;

(b) *A subset* $\mathcal{S}$ *of* $A\mathcal{K}_{aw}(X, \alpha)$ *has a supremum in* $A\mathcal{K}_{aw}(X, \alpha)$ *if and only if the supremum of* $\mathcal{S}$ *in* $A\mathcal{K}(X, \alpha)$ *belongs to* $A\mathcal{K}_{aw}(X, \alpha)$;

(c) $A\mathcal{K}_{aw}(X, \alpha)$ *is a* μ-*complete upper subsemilattice of* $A\mathcal{K}(X, \alpha)$, *where* μ *is equal to* $aw(X, \alpha)$;

(d) *Let* $\mathcal{S} \subseteq A\mathcal{K}_{aw}(X, \alpha)$. *Then* $\mathcal{S}$ *has a supremum in* $A\mathcal{K}_{aw}(X, \alpha)$ *if and only if there is a subset* $\mathcal{N} \subseteq \mathcal{S}$, *with* $|\mathcal{N}| \leqslant \mu = aw(X, \alpha)$, *such that* $\sup_{A\mathcal{K}(X, \alpha)} \mathcal{N} = \sup_{A\mathcal{K}(X, \alpha)} \mathcal{S}$.

PROOF. – (a) follows from 3.12 and from the well known result about the weight of perfect images.

(b) easily follows from (a).

To prove (c), let $\{(Y_j, t_j)\}_{j \in J}$ be a subfamily of $A\mathcal{K}_{aw}(X, \alpha)$, with $|J| \leqslant \mu$. Every Y_j can be embedded in a Tychonoff cube of weight μ, that is, (Y_j, t_j) is generated by a family $\mathcal{F}_j \subseteq C^*(X)$, with $|\mathcal{F}_j| = \mu$. Then, by 2.7(c), the family $\mathcal{F} = \bigcup_{j \in J} \mathcal{F}_j$ generates $\sup_{\mathcal{K}(X)}\{(Y_j, t_j)\}_{j \in J} = \sup_{A\mathcal{K}(X, \alpha)} \{(Y_j, t_j)\}_{j \in J}$. Since $|\mathcal{F}| = \mu$, one has $\sup_{A\mathcal{K}(X, \alpha)} \{(Y_j, t_j)\}_{j \in J} \in A\mathcal{K}_{aw}(X, \alpha)$.

The «if» part of (d) easily follows from (c). Conversely, suppose that the family $\mathcal{S} = \{(Y_j, t_j)\}_{j \in J}$ has a supremum (Y, t) in $A\mathcal{K}_{aw}(X, \alpha)$ (or, equivalently, $\sup_{A\mathcal{K}(X, \alpha)} \mathcal{S} \in A\mathcal{K}_{aw}(X, \alpha)$). We know, by 4.2($a$) and by 2.7($c$), that $\mathcal{F} = \bigcup_{j \in J} \mathcal{F}_{t_j}$ generates (Y, t). By Proposition 2.7 of [5] and Theorem 4.2 of [3], $\mathcal{F}$ contains a family $\mathcal{G}$ with $|\mathcal{G}| = \mu$ which generates (Y, t). For every $g \in \mathcal{G}$, choose $j(g) \in J$ such that $g \in \mathcal{F}_{t_{j(g)}}$. Put $\mathcal{N} = \{(Y_{j(g)}, t_{j(g)})\}_{g \in \mathcal{G}}$. Then clearly $(Z, h) = \sup_{A\mathcal{K}(X, \alpha)} \mathcal{N} \leqslant \sup_{A\mathcal{K}(X, \alpha)} \mathcal{S} = (Y, t)$. On the other hand, $\mathcal{G} \subseteq \mathcal{F}_h$; hence $(Y, t) \leqslant (Z, h)$. ∎

PROPOSITION 4.11. – (a) $\sup\limits_{A\mathcal{K}(X,\,\alpha)} (A\mathcal{K}_{aw}(X, \alpha)) = \beta(X, \alpha)$;

(b) *$A\mathcal{K}_{aw}(X, \alpha)$ is a complete upper semilattice if and only if $\beta(X, \alpha) \in A\mathcal{K}_{aw}(X, \alpha)$.*

PROOF. – We know, by 4.2(a), that $A\mathcal{K}_{aw}(X, \alpha)$ has a supremum $(Y, t) \in A\mathcal{K}(X, \alpha)$. Thus we have only to prove $\beta(X, \alpha) \leq (Y, t)$, that is, that every $f \in \mathcal{F}(\alpha)$ extends to (Y, t) (see 2.23(c) and 2.4(c)). Let (Z, h) be any element of $A\mathcal{K}_{aw}(X, \alpha)$ and let $\mathcal{G} \subseteq C^*(X)$ be a generating family for (Z, h), with $|\mathcal{G}| = aw(X, \alpha)$. We can suppose that $\mathcal{G}$ separates points from closed sets ([9], 2.3.23). Since $(Z, h) \in A\mathcal{K}(X, \alpha)$, one has $\mathcal{G} \subseteq \mathcal{F}(\alpha)$. Let f be any element of $\mathcal{F}(\alpha)$. Then $\mathcal{G} \cup \{f\}$ generates a compactification (Z_1, h_1). Since $\mathcal{G} \cup \{f\} \subset \mathcal{F}(\alpha)$, one has $(Z, h) \leq (Z_1, h_1) \leq \beta(X, \alpha)$. Hence, by 4.1, $(Z_1, h_1) \in A\mathcal{K}(X, \alpha)$. Moreover, we have $w(Z_1) = |\mathcal{G} \cup \{f\}| = aw(X, \alpha)$, that is $(Z_1, h_1) \in A\mathcal{K}_{aw}(X, \alpha)$. This implies that $(Z_1, h_1) \leq (Y, t)$, i.e. $f \in \mathcal{F}_{h_1} \subseteq \mathcal{F}_t$ (see 2.4(c)). Therefore f extends to (Y, t) and this proves (a). (b) immediately follows. ■

EXAMPLE 4.12. – The condition «$\beta(X, \alpha) \in A\mathcal{K}_{aw}(X, \alpha)$» in 4·11($b$) is satisfied, for example, by:

(a) every A-space (X, α) which is pseudocompact with respect to $\beta(X, \alpha)$ (this follows from 4.4(a) and 3.23);

(b) every A-space (X, α) such that $w(X) = w(\beta X)$, where $X = (X, \tau_\alpha)$ (this follows from the proof of 3.20).

PROPOSITION 4.13. – (a) *A subset $\mathcal{S}$ of $A\mathcal{K}_{aw}(X, \alpha)$ has an infimum in $A\mathcal{K}_{aw}(X, \alpha)$ if and only if $\mathcal{S}$ has a lower bound in $A\mathcal{K}(X, \alpha)$; in this case one has* $\inf\limits_{A\mathcal{K}_{aw}(X,\,\alpha)} \mathcal{S} = \inf\limits_{A\mathcal{K}(X,\,\alpha)} \mathcal{S}$;

(b) $A\mathcal{K}_{aw}(X, \alpha)$ is a lattice if and only if $A\mathcal{K}(X, \alpha)$ is a lattice;

(c) *$A\mathcal{K}_{aw}(X, \alpha)$ is a complete lower semilattice if and only if $A\mathcal{K}(X, \alpha)$ has a smallest element.*

PROOF. – (a) follows from 4.2(a) and 4.10(a). (b) is a consequence of (a) and 4.10(c). Clearly (a) also implies (c). ■

Combining 4.13(c) with the Theorems 3, 4 of [7], mentioned above (see 4.4), we obtain:

COROLLARY 4.14. – *$A\mathcal{K}_{aw}(X, \alpha)$ is a complete lower semilattice if and only if either (X, α) is pseudocompact with respect to $\beta(X, \alpha)$ or $v(X, \alpha)$ is locally compact and Lindelöf.*

5. – A-determining families of functions.

The following lemma is essentially known.

LEMMA 5.1. – ([9], 2.3.D) *Let X be a space and let $\mathcal{F} \subseteq C^*(X)$. Then $\mathcal{F}$ generates a compactification of X if and only if the family $\{f^{-1}(a, +\infty) \mid f \in \mathcal{F}, a \in \mathbf{R}\} \cup \{g^{-1}(-\infty, b) \mid g \in \mathcal{F}, b \in \mathbf{R}\}$ is a subbase for (the open sets of) X.*

THEOREM 5.2. – *Let X be a space and let α be a compatible cozero field on X. A family $\mathcal{F} \subseteq C^*(X)$ generates an A-compactification of (X, α) if and only if the family*

$$\mathcal{S} = \{f^{-1}(a, +\infty) \mid f \in \mathcal{F}, a \in \mathbf{R}\} \cup \{g^{-1}(-\infty, b) \mid g \in \mathcal{F}, b \in \mathbf{R}\}$$

is an A-subbase of (X, α).

PROOF. – First suppose that $\mathcal{S}$ is an A-subbase of (X, α). This implies that $\mathcal{S}$ is a subbase for X. Hence, by the above lemma, $\mathcal{F}$ generates a compactification (Y, t) of X. Then $\mathcal{F} \subseteq \mathcal{F}_t$ and, clearly, $\mathcal{S} \subseteq \mathrm{Coz}(\mathcal{F}_t) = \mathrm{Coz}(Y)|_X$. On the other hand, one can easily see that $\mathcal{F} \subseteq \mathcal{H}(\alpha)$ and hence every member of $\mathcal{F}$ extends to $\beta(X, \alpha)$ (see 2.23(c)). Then, by 2.7(b), $(Y, t) \leq \beta(X, \alpha)$ and $\mathcal{F}_t \subseteq \mathcal{H}(\alpha)$ (see also 2.4). Therefore $\mathcal{S} \subseteq \mathrm{Coz}(\mathcal{F}_t) \subseteq \alpha$. Since $\mathcal{S}$ is an A-subbase for (X, α), we obtain $\mathrm{Coz}(\mathcal{F}_t) = \alpha$, that is, $(Y, t) \in A\mathcal{K}(X, \alpha)$.

Now suppose that $\mathcal{F}$ generates an A-compactification $(Y, t) = e_{\mathcal{F}}X$. Let us denote by p_f the projection from $K = \prod_{f \in \mathcal{F}} I_f$ onto I_f, for each $f \in \mathcal{F}$ (see 2.5 and 2.6). Put $\mathcal{N} = \{(a, +\infty) \mid a \in \mathbf{R}\} \cup \{(-\infty, b) \mid b \in \mathbf{R}\}$. For every $V \in \alpha$ one has $V = e_{\mathcal{F}}^{-1}(U)$, with $U \in \mathrm{Coz}(K)$. Since U is an F_σ, then U is Lindelöf, so it is a countable union of open sets of the form $p_{f_1}^{-1}(T_1) \cap \ldots \cap p_{f_n}^{-1}(T_n)$ with $f_i \in \mathcal{F}$, $T_i \in \mathcal{N}$ for every i. Since, for $f \in \mathcal{F}$, $p_f \circ e_{\mathcal{F}} = f$, V is a countable union of sets of the form $f_1^{-1}(T_1) \cap \ldots \cap f_n^{-1}(T_n)$. This means that $\mathcal{S}$ is an A-subbase of (X, α). ∎

REMARK 5.3. – The above theorem remains true if, in the definition of $\mathcal{S}$, we replace $\mathbf{R}$ by $\mathbf{Q}$ or by the set of the dyadic rationals.

Let X be a space. Following [3], we say that a subfamily $\mathcal{F}$ of $C^*(X)$ *determines* a compactification (Y, t) of X if $(Y, t) = \min\{(Z, h) \in \mathcal{K}(X) \mid \mathcal{F} \subseteq \mathcal{F}_h\}$. We put $\delta(Y, t) = \min\{|\mathcal{F}| \mid \mathcal{F} \text{ determines } (Y, t)\}$

DEFINITION 5.4. – Let (X, α) be an A-space. We say that a family $\mathcal{F} \subseteq \mathcal{H}(\alpha)$ *A-determines* an A-compactification (Y, t) of (X, α) if $(Y, t) = \min\{(Z, h) \in A\mathcal{K}(X, \alpha) \mid \mathcal{F} \subseteq \mathcal{F}_h\}$.

Clearly, if (Y, t) is an A-compactification of (X, α) then a family $\mathcal{F} \subseteq \mathcal{H}(\alpha)$

A-determines (Y, t) if and only if $\mathcal{F}_t = \cap\{\mathcal{F}_h \mid (Z, h) \in A\mathcal{K}(X, \alpha), \mathcal{F} \subseteq \mathcal{F}_h\}$.

Let X be a space and let α be a compatible cozero field. Every subfamily of $C^*(X)$ which generates a compactification $(Y, t) \in A\mathcal{K}(X, \alpha)$, also A-determines (Y, t). More generally, if $\mathcal{F} \subseteq \mathcal{F}(\alpha)$ determines $(Y, t) \in A\mathcal{K}(X, \alpha)$, then $\mathcal{F}$ also A-determines (Y, t). The converse is not true in general (see 5.5 below and take there X to be a pseudocompact, non-locally compact space).

EXAMPLE 5.5. – Let X be a space and let $\mathcal{C}$ be any family of constant real-valued functions on X. Clearly $\mathcal{C} \subseteq \mathcal{F}(\alpha)$ for every compatible cozero field α on X.

Notice that, when X is not locally compact, $\mathcal{C}$ does not determine any compactification of X (see Theorem 2.1 of [3]).

Let us consider the following three cases.

(a) Let X be locally compact. Then, clearly, $\mathcal{C}$ A-detemines (and determines) the A-compactification αX of $(X, \alpha_{\min})$.

(b) Let X be a realcompact, non-Lindelöf space. Then, by 4.4, $\mathcal{C}$ does not A-determine any A-compactification of $(X, \mathrm{Coz}(X))$.

(c) Let X be pseudocompact. Then X is pseudocompact with respect to $\beta(X, \alpha)$ for every compatible cozero field α (see Cor. 5.5 of [13]). Hence, as it follows from 3.26 (or 4.4(a)), $|A\mathcal{K}(X, \alpha)| = 1$ for every α. Therefore, for every α, $\mathcal{C}$ A-determines $\beta(X, \alpha)$. Now, 2.24 implies that every compactification (Y, t) of X is A-determined by $\mathcal{C}$ with respect to (X, α), where $\alpha = \mathrm{Coz}(Y)|_X$.

More generally, one clearly obtains:

PROPOSITION 5.6. – *Let (X, α) be an A-space. If $X = (X, \tau_\alpha)$ is pseudocompact with respect to $\beta(X, \alpha)$, then every $\mathcal{F} \subseteq \mathcal{F}(\alpha)$ A-determines $\beta(X, \alpha)$.*

REMARK 5.7. – It is possible to easily extend to A-determining families many properties of determining families. For instance, the easily formulated analogues of Theorems 3.2, 3.3 of [3] remain true for A-determining families. In particular, if we put $X = (X, \tau_\alpha)$ and consider $C^*(X)$ endowed with the topology of uniform convergence, then a family $\mathcal{F} \subseteq \mathcal{F}(\alpha)$ A-determines $(Y, t) \in A\mathcal{K}(X, \alpha)$ if and only if $cl_{\mathcal{F}(\alpha)}(\mathcal{F}) = cl_{C^*(X)}(\mathcal{F})$ A-determines (Y, t).

NOTATION 5.8. – If $(Y, t) \in A\mathcal{K}(X, \alpha)$, let us denote by q_t the unique map from $\beta(X, \alpha)$ to Y which is the identity on X.

If $\mathcal{F} \subseteq \mathcal{F}_t$, then, for every $f \in \mathcal{F}$, we denote by f^Y the unique extension of f to Y. Put $\mathcal{F}^Y = \{f^Y \mid f \in \mathcal{F}\}$. For every $\mathcal{F} \subseteq \mathcal{F}(\alpha)$, we denote by $\mathcal{F}^{v(X, \alpha)}$ the extensions of the elements of $\mathcal{F}$ to $v(X, \alpha)$ (see Definition 3.25).

REMARK 5.9. – Let $(Y, t) \in A\mathcal{K}(X, \alpha)$. We know that the restriction of q_t to $v(X, \alpha)$ is the identity map and q_t maps $\beta(X, \alpha)\setminus v(X, \alpha)$ onto $Y\setminus v(X, \alpha)$ (see the proof of Theorem 3.9 of [19]).

If $(Y, t) \leqslant (Z, h)$, where (Z, h) is also in $A\mathcal{K}(X, \alpha)$, then, clearly, the canonical map q from Z to Y is the identity on $v(X, \alpha)$ and maps $Z\setminus v(X, \alpha)$ onto $Y\setminus v(X, \alpha)$.

For every $(Y, t) \in A\mathcal{K}(X, \alpha)$, let us denote by t_v the embedding of $v(X, \alpha)$ into Y. Clearly, (Y, t_v) is a compactification of $v(X, \alpha)$. One has $(Y, t) \leqslant (Z, h)$ if and only if $(Y, t_v) \leqslant (Z, h_v)$.

THEOREM 5.10. – *Let (X, α) be an A-space and suppose that $X = (X, \tau_\alpha)$ is not pseudocompact with respect to $\beta(X, \alpha)$. Let $\mathcal{F}$ be a subset of $\mathcal{F}(\alpha)$ and let $(Y, t) \in A\mathcal{K}(X, \alpha)$. Then $\mathcal{F}$ A-determines (Y, t) if and only if $\mathcal{F} \subseteq \mathcal{F}_t$ and $\mathcal{F}^Y$ separates points of $Y\setminus v(X, \alpha)$.*

PROOF. – ($\Rightarrow$) Clearly, we have that $\mathcal{F} \subseteq \mathcal{F}_t$. Let us prove that $\mathcal{F}^Y$ separates points of $Y\setminus v(X, \alpha)$.

Suppose $y_1, y_2 \in Y\setminus v(X, \alpha)$ are not separated by $\mathcal{F}^Y$. From Lemma 4.5 it follows that the compactification (Z, h) of X obtained by collapsing y_1 and y_2 to one point, is still an A-compactification of (X, α). But, clearly, $\mathcal{F} \subseteq \mathcal{F}_h$ and this is a contradiction because $(Z, h) \leqslant (Y, t)$.

($\Leftarrow$) We shall show that $\mathcal{F}$ A-determines (Y, t).

Let $(Z, h) \in A\mathcal{K}(X, \alpha)$ be such that $\mathcal{F} \subseteq \mathcal{F}_h$. For $f \in \mathcal{F}$, one has $f^{\beta(X, \alpha)} = f^Z \circ q_h$. Hence $f^{\beta(X, \alpha)}$ is constant on the sets $q_h^{-1}(z)$, for $z \in Z$. We need to prove that there is a continuous map $q : Z \to Y$ which is the identity on X. Let us define q as follows: $q(z) = z$ if $z \in v(X, \alpha)$; $q(z) = q_t(u)$ if $z \in Z\setminus v(X, \alpha)$, where $u \in q_h^{-1}(z)$. We need to prove that $q_t(u)$ is independent on the choice of u. Suppose that, for $u, v \in q_h^{-1}(z)$ one has $y_1 = q_t(u) \neq q_t(v) = y_2$. There is $f \in \mathcal{F}$ such that $f^Y(y_1) \neq f^Y(y_2)$. Since $f^{\beta(X, \alpha)} = q_t \circ f^Y$, one has $f^{\beta(X, \alpha)}(u) \neq f^{\beta(X, \alpha)}(v)$, a contradiction. Therefore q is well defined and satisfies $q_t = q \circ q_h$. Since q_h is a quotient map, q is continuous. This completes the proof. ■

COROLLARY 5.11. – *Suppose (X, α) satisfies the hypotheses of the above theorem and let $(Y, t) \in A\mathcal{K}(X, \alpha)$. A subset $\mathcal{F}$ of $\mathcal{F}_t$ A-determines (Y, t) if and only if $\mathcal{F}^{v(X, \alpha)}$ determines the compactification (Y, t_v) of $v(X, \alpha)$.*

In particular, if X is realcompact with respect to $\beta(X, \alpha)$, then (Y, t) is A-determined by $\mathcal{F}$ if and only if it is determined by $\mathcal{F}$ as compactification of X.

PROOF. – It follows from the above theorem and Theorem 2.1 of [3]. ■

DEFINITION 5.12. – Let (X, α) be an A-space. For $(Y, t) \in A\mathcal{K}(X, \alpha)$, let us denote by $a\delta(Y, t)$ the minimum cardinality of a subfamily of $\mathcal{F}(\alpha)$ which A-determines (Y, t).

COROLLARY 5.13. – *Let* (X, α) *be as in* 5.10 *and let* $(Y, t) \in A\mathcal{K}(X, \alpha)$. *Then* $a\delta(Y, t) = \delta(Y, t_v)$.

PROOF. – From 5.11 one has $\delta(Y, t_v) \leq a\delta(Y, t)$. Let $\mathcal{G}$ be a subfamily of $C^*(v(X, \alpha))$ which determines (Y, t_v). Put $\mathcal{F} = \{g|_X \,|\, g \in \mathcal{G}\}$. Then $\mathcal{F} \subseteq \mathcal{F}_t$ and $\mathcal{G} = \mathcal{F}^{v(X, \alpha)}$. Hence, by 5.11, $\mathcal{F}$ A-determines (Y, t). Since $|\mathcal{F}| = |\mathcal{G}|$, we obtain $a\delta(Y, t) \leq \delta(Y, t_v)$. ■

PROPOSITION 5.14. – *Let* $(Y, t), (Z, h) \in A\mathcal{K}(X, \alpha)$. *Then:*

(a) $a\delta(Y, t) \leq w(Y \backslash v(X, \alpha)) \leq w(Y \backslash X)$.

(b) *If* $v(X, \alpha)$ *is locally compact and* $a\delta(Y, t)$ *is infinite, then* $a\delta(Y, t) = w(Y \backslash v(X, \alpha))$.

(c) *If* $(Y, t) \leq (Z, h)$, *and* $a\delta(Z, h)$ *is infinite, then* $a\delta(Y, t) \leq a\delta(Z, h)$.

PROOF. – (a) and (b) follow from 5.13 and Theorem 4.2 of [3].
(c) follows from 5.9, 5.13 and Theorem 4.3 of [3]. ■

PROPOSITION 5.15. – *If* $(Y, t) \in A\mathcal{K}(X, \alpha)$ *and* $w(Y) > aw(X, \alpha)$, *then* $a\delta(Y, t) = w(Y)$.

PROOF. – Suppose $\mathcal{F}$ A-determines (Y, t) and $|\mathcal{F}| < w(Y)$. Let $(Z, h) \in A\mathcal{K}_{aw}(X, \alpha)$ and $(Z, h) < (Y, t)$ (see 3.23). Then there is $\mathcal{G} \subseteq \mathcal{F}_h$, with $|\mathcal{G}| = aw(X, \alpha)$, which separates points from closed sets of X. Clearly $|\mathcal{G} \cup \mathcal{F}| < w(Y)$. Since $\mathcal{F} \subseteq \mathcal{G} \cup \mathcal{F} \subseteq \mathcal{F}_t$, clearly $\mathcal{G} \cup \mathcal{F}$ A-determines (Y, t). But $\mathcal{G} \cup \mathcal{F}$ separates points from closed sets and, hence, it also generates (Y, t). Then $w(Y) \leq |\mathcal{G} \cup \mathcal{F}|$, a contradiction. ■

A consequence of the Stone-Weierstrass theorem is that, for a compactification (Y, t) of X, $w(Y) = d(\mathcal{F}_t)$ (with respect to the topology of uniform convergence). So one has:

COROLLARY 5.16. – *Let* $(Y, t) \in A\mathcal{K}(X, \alpha)$ *and suppose* $w(Y) > aw(X, \alpha)$. *If* $\mathcal{F}$ *A-determines* (Y, t) *then there exists* $\mathcal{G} \subset \mathcal{F}$ *which A-determines* (Y, t) *with* $|\mathcal{G}| = a\delta(Y, t)$.

PROOF. – From $\mathcal{F} \subset \mathcal{F}_t$, we obtain $d(\mathcal{F}) \leq d(\mathcal{F}_t)$ (since the topology of uniform convergence is metrizable). Let $\mathcal{G}$ be a dense subset of $\mathcal{F}$ of cardinality $d(\mathcal{F})$. By 5.7, $\mathcal{G}$ A-determines (Y, t) and one has, using 5.15, that $a\delta(Y, t) \leq |\mathcal{G}| = d(\mathcal{F}) \leq d(\mathcal{F}_t) = w(Y) = a\delta(Y, t)$.

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