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Duplicate Inverse Series Relations and Hypergeometric Evaluations with $z = 1/4$.

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Sunto. – *Le relazioni inverse di Gould-Hsu (1973) sono state applicate sistematicamente alla ricerca di identità ipergeometriche. La loro versione duplicata è stabilita ed utilizzata per dimostrare numerose formule della ${}_3F_2[1/4]$-serie ipergeometrica. Le ulteriori valutazioni ipergeometriche sono ottenute per mezzo di relazioni ricorrenti.*

Summary. – *The Gould-Hsu (1973) inverse series relations have been systematically applied to the research of hypergeometric identities. Their duplicate version is established and used to demonstrate several terminating ${}_3F_2[1/4]$-summation formulas. Further hypergeometric evaluations with the same variable are obtained through recurrence relations.*

For a complex number c and a nonnegative integer n, denote the rising shifted-factorial by

$$(0.1a) \qquad (c)_0 = 1, \quad (c)_n = c(c+1)\dots(c+n-1), \qquad n = 1, 2, \dots .$$

Following Bailey [1], the hypergeometric series, for an indeterminate z and two nonnegative integers p and q, is defined by

$$(0.1b) \qquad {}_{1+p}F_q\left[\begin{matrix} a_0, & a_1, \dots, a_p \\ & b_1, \dots, b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \dots (a_p)_n}{n!(b_1)_n \dots (b_q)_n} z^n$$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summands on the right hand side.

When working on an enumerative problem of combinatorics, I happened to confirm three binomial convolution formulas

$$(0.2a) \qquad \sum_{k=0}^{2n} (-1)^k \binom{\lambda+2k}{k}\binom{\lambda+3n}{2n-k} = \binom{\lambda+3n}{n}$$

$$(0.2b) \qquad \sum_{k=0}^{1+2n} (-1)^k \binom{\lambda+2k}{k}\binom{2+\lambda+3n}{1+2n-k} = 0$$

$$(0.2c) \qquad \sum_{k=0}^{1+2n} (-1)^k \binom{\lambda+2k}{k}\binom{1+\lambda+3n}{1+2n-k} = -\binom{1+\lambda+3n}{n}$$

or their hypergeometric counterparts

$$(0.3a)\qquad {}_3F_2\left[\begin{matrix}-2n, & \frac{u}{2}, & \frac{1+u}{2} \\ & u, & u+n\end{matrix}; 4\right] = \frac{(2n)!\,(u)_n}{n!\,(u)_{2n}}$$

$$(0.3b)\qquad {}_3F_2\left[\begin{matrix}-1-2n, & \frac{u}{2}, & \frac{1+u}{2} \\ & u, & 1+u+n\end{matrix}; 4\right] = 0$$

$$(0.3c)\qquad {}_3F_2\left[\begin{matrix}-1-2n, & \frac{u}{2}, & \frac{1+u}{2} \\ & u, & u+n\end{matrix}; 4\right] = -\frac{(1+2n)!\,(u)_n}{n!\,(u)_{1+2n}}.$$

It doesn't seem that there exists an obvious linear relation among them. However, from the reversals of the binomial convolutions just displayed

$$(0.4a)\qquad \sum_{k=0}^{2n}(-1)^k\binom{\mu}{k}\binom{\mu+n-2k}{2n-k} = \binom{\mu}{n}$$

$$(0.4b)\qquad \sum_{k=0}^{1+2n}(-1)^k\binom{\mu}{k}\binom{\mu+n-2k}{1+2n-k} = 0$$

$$(0.4c)\qquad \sum_{k=0}^{1+2n}(-1)^k\binom{\mu}{k}\binom{1+\mu+n-2k}{1+2n-k} = \binom{\mu}{n}$$

it becomes trivial that there does exist such a relation. In fact, the last one is simply equal to the sum of the first two binomial convolutions. In terms of hypergeometric series, we can write down equivalently the following evaluations:

$$(0.5a)\qquad {}_3F_2\left[\begin{matrix}-2n, & v, & v+n \\ & \frac{v-n}{2}, & \frac{1+v-n}{2}\end{matrix}; \frac{1}{4}\right] = \frac{(2n)!}{n!\,(1-v)_n}$$

$$(0.5b)\qquad {}_3F_2\left[\begin{matrix}-1-2n, & v, & 1+v+n \\ & \frac{v-n}{2}, & \frac{1+v-n}{2}\end{matrix}; \frac{1}{4}\right] = 0$$

$$(0.5c)\qquad {}_3F_2\left[\begin{matrix}-1-2n, & v, & v+n \\ & \frac{v-n}{2}, & \frac{v-1-n}{2}\end{matrix}; \frac{1}{4}\right] = \frac{(1+2n)!}{n!\,(1-v)_{1+n}}.$$

The object of this paper is to demonstrate these results and the related hypergeometric evaluations. Based on the Gould-Hsu inverse series relations, the first section will construct their duplicate analogue. The dual formulas of the Saalschütz summation theorem will be established in Section 2. By means of the duplicate inversions, Section 3 will derive several hypergeometric evaluations with $z = 1/4$. Then the recurrence relations will be exploited in section 4 to obtain further hypergeometric identities. Duplicate inversions on 2-balanced hypergeometric identities are commented additionally in the last section.

1. – Inverse series relations.

For two complex variables x, y and four complex sequences $\{a_k, b_k, c_k, d_k\}_{k \geqslant 0}$, define two polynomial sequences by

$$\phi(x;0) \equiv 1, \quad \phi(x;m) = \prod_{i=0}^{m-1} (a_i + xb_i), \quad m = 1, 2, \ldots \tag{1.1a}$$

$$\psi(y;0) \equiv 1, \quad \psi(y;n) = \prod_{j=0}^{n-1} (c_j + yd_j), \quad n = 1, 2, \ldots. \tag{1.1b}$$

Then there is a useful pair of inverse series relations.

LEMMA. – (Gould-Hsu [12]).

$$f(m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \quad \phi(k;m) \quad g(k) \tag{1.2a}$$

$$g(m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{a_k + kb_k}{\phi(m;k+1)} f(k). \tag{1.2b}$$

For an alternative derivation and its multivariate generalization, refer to [2] and [3] respectively. Its application to binomial identities and hypergeometric evaluations may be found in [5, 6, 8, 9].

In order to adapt the Gould-Hsu inversions to the problem posed at the begining of the paper, here we present its duplicate form as follows.

THEOREM. – (Duplicate inverse series relations). *With ϕ and ψ-polynomials defined respectively by* (1.1a) *and* (1.1b), *the system of equations*

$$g(n) = \sum_{k \geqslant 0} \binom{n}{2k} \frac{a_k + 2kb_k}{\phi(n;k+1)\,\psi(n;k)} f(k) \tag{1.3a}$$

$$- \sum_{k \geqslant 0} \binom{n}{1+2k} \frac{c_k + (1+2k)\,d_k}{\phi(n;k+1)\,\psi(n;1+k)} f'(k) \tag{1.3b}$$

$$(1.3c)\qquad g(n) = \sum_{k\geqslant 0}\binom{n}{2k}\frac{c_k+2kd_k}{\phi(n;\,k)\,\psi(n;\,1+k)}\,f(k)$$

$$(1.3d)\qquad -\sum_{k\geqslant 0}\binom{n}{1+2k}\frac{a_k+(1+2k)b_k}{\phi(n;\,k+1)\,\psi(n;\,1+k)}\,f''(k)$$

is equivalent to the system of equations

$$(1.4a)\qquad f(n) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\,\phi(k;\,n)\,\psi(k;\,n)\,g(k)$$

$$(1.4b)\qquad f'(n) = \sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\phi(k;\,1+n)\,\psi(k;\,n)\,g(k)$$

$$(1.4c)\qquad f''(n) = \sum_{k=0}^{1+2n}(-1)^k\binom{1+2n}{k}\phi(k;\,n)\,\psi(k;\,1+n)\,g(k).$$

Moreover, there hold the following relations

$$(1.5a)\qquad f(n) = \frac{f'(n)\{c_n+d_n(1+2n)\}-f''(n)\{a_n+b_n(1+2n)\}}{(1+2n)\{a_nd_n-b_nc_n\}}$$

$$(1.5b)\qquad f'(n) = f(n)\,\frac{(1+2n)\{a_nd_n-b_nc_n\}}{c_n+d_n(1+2n)}+f''(n)\,\frac{a_n+b_n(1+2n)}{c_n+d_n(1+2n)}$$

$$(1.5c)\qquad f''(n) = f(n)\,\frac{(1+2n)\{b_nc_n-a_nd_n\}}{a_n+b_n(1+2n)}+f'(n)\,\frac{c_n+d_n(1+2n)}{a_n+b_n(1+2n)}.$$

PROOF. – For an inverse pair of infinite upper triangular matrices

$$A = (a_{ij})_{0\leqslant i\leqslant j<\infty}$$

$$B = (b_{ij})_{0\leqslant i\leqslant j<\infty}$$

the system of equations

$$f(m) = \sum_{j=0}^{m} a_{jm}\,g(j),\qquad m = 0,1,2,\ldots$$

is equivalent to the system

$$g(m) = \sum_{j=0}^{m} b_{jm}\,f(j),\qquad m = 0,1,2,\ldots.$$

Therefore, to prove the equivalence between two systems of equations, it suffices to substitute one system into another and then verify the desired result.

Now substituting (1.3a) and (1.3b) into the right hand side of (1.4a), we have

$$S(n) - S'(n) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \phi(k;\, n)\, \psi(k;\, n)\, g(k)$$

where

$$S(n) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \phi(k;\, n)\, \psi(k;\, n) \times \sum_m \binom{k}{2m} \frac{a_m + 2mb_m}{\phi(k;\, m+1)\, \psi(k;\, m)} f(m)$$

$$S'(n) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \phi(k;\, n)\, \psi(k;\, n) \times \sum_m \binom{k}{1+2m} \frac{c_m + (1+2m)\, d_m}{\phi(k;\, m+1)\, \psi(k;\, 1+m)} f'(m).$$

The resulting double sums $S(n) - S'(n)$ should reduce to $f(n)$. This can be accomplished by means of finite difference method.

For $S(n)$, we may rewrite it by interchanging the summantion order as

$$S(n) = \sum_{m=0}^{n} \binom{2n}{2m} \{a_m + 2mb_m\}\ f(m) \times \sum_{k=2m}^{2n} (-1)^k \binom{2n-2m}{k-2m} \frac{\phi(k;\, n)\, \psi(k;\, n)}{\phi(k;\, m+1)\, \psi(k;\, m)}.$$

When $n > m$, the fraction $\phi(k;\, n)\, \psi(k;\, n)/\phi(k;\, m+1)\, \psi(k;\, m)$ is in fact a polynomial of degree $2n - 1 - 2m$ in k. Therefore its divided differences of order $2(n - m)$ vanishes. This implies that the last sum with respect to k equals zero except for $m = n$ and so $S(n) \equiv f(n)$.

Following the same procedure, we assert that

$$S'(n) = \sum_{m=0}^{n-1} \binom{2n}{1+2m} \{c_m + (1+2m)\, d_m\}\ f'(m) \times \sum_{k=1+2m}^{2n} (-1)^k \binom{2n-1-2m}{k-1-2m} \frac{\phi(k;\, n)\, \psi(k;\, n)}{\phi(k;\, m+1)\, \psi(k;\, 1+m)}$$

is identical with zero, i.e., $S'(n) \equiv 0$. In conclusion, we have $S(n) - S'(n) \equiv f(n)$.

Similarly, replacing $\{g(k)\}$ in (1.4b) by (1.3a) and (1.3b), we can demonstrate that the resulting double sums reduce to $f'(n)$.

For three sequences $\{f(n)\}$, $\{f'(n)\}$ and $\{f''(n)\}$ determined respectively by (1.4a), (1.4b) and (1.4c), it is not hard to check the relation (1.5c), which is equivalent to other two relations (1.5a) and (1.5b). Therefore, there are only two independent sequences among (1.4a), (1.4b) and (1.4c). In fact, substitute (1.3c) and (1.3d) into (1.4a) and (1.4c), then the same process used before can justify that the resulting double sums reduce to $f(n)$ and $f''(n)$ respectively. This means that the system of equations (1.3a) and (1.3b) is in turn equivalent to the system determined by (1.3c) and (1.3d). Hence, if one of two equation systems (1.3a)-(1.3b) and (1.3c)-(1.3d) is properly specified as a known relation, we can invert it to get three dual relations determined by (1.4a), (1.4b) and (1.4c). That is the philosophy of the «so-called» inversion techniques [5, 6]. It will be developed in Section 4 for obtaining terminating hypergeometric summation formulas. ■

2. – The Saalschütz theorem and its dual formulas.

Recalling the Saalschütz theorem (cf. [1, p. 9])

$$(2.1)\qquad {}_3F_2\left[\begin{matrix}-n, & a, & b \\ & c, & 1+a+b-c-n\end{matrix};1\right]=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$$

we may specify it as

$$(2.2)\qquad {}_3F_2\left[\begin{matrix}-\frac{m}{2}, & \frac{1-m}{2}, & u+m \\ & \frac{1+u}{2}, & \frac{2+u}{2}\end{matrix};1\right]=\frac{(u)_{2m}}{(u)_m(1+u)_m}$$

which may be rewritten, in accordance with (1.2a), as

$$\sum_{k\geqslant 0}\binom{m}{2k}(u+k)_m\frac{(1/2)_k(u)_k}{\left(\frac{1+u}{2}\right)_k\left(\frac{2+u}{2}\right)_k}=\frac{(u)_{2m}}{(1+u)_m}.$$

Then (1.2b) gives its dual relation

$$\sum_{k\geqslant 0}(-1)^k\binom{m}{k}\frac{u+\frac{3k}{2}}{\left(u+\frac{m}{2}\right)_{1+k}}\frac{(u)_{2k}}{(1+u)_k}=\begin{cases}\dfrac{(1/2)_n(u)_n}{\left(\frac{1+u}{2}\right)_n\left(\frac{2+u}{2}\right)_n}, & m=2n\\[2ex] 0, & m=1+2n\end{cases}$$

which may be reformulated, according to the parity of m, as binomial convolution formulas

$$\sum_{k\geqslant 0}(-1)^k \frac{u+3k/2}{u+2k}\binom{u+2k}{k}\binom{u+3n}{2n-k}=\binom{u+3n}{n} \tag{2.3a}$$

$$\sum_{k\geqslant 0}(-1)^k \frac{u+3k/2}{u+2k}\binom{u+2k}{k}\binom{\frac{3}{2}+u+3n}{1+2n-k}=0\,. \tag{2.3b}$$

From one side, (2.3a) is quite similar to the convolution (0.2a) stated at the begining of this paper except for a rational factor in the summand. This fact stimulated me to persist in working with inversion techniques on the problem. But just for this slight difference, I have been keeping in search for a long time without success. That is the reason that I have to modify the Gould-Hsu inverse series relations to construct their duplicate version in the Theorem.

Let $\delta=0,\ 1$ be the usual Kronecker delta. Then the Saalschütz formula may be specified in another way

$$ {}_3F_2\left[\begin{matrix}-m, & \frac{x+m}{2}, & \frac{1+x+m}{2} \\ & \delta+\frac{1}{2}, & 1-\delta+x\end{matrix};1\right]=\frac{m!}{(\delta+2m)!}\,\frac{(x-\delta)_m(2\delta-x)_m}{(x)_m}\,. \tag{2.4}$$

This is in fact the reversal of (2.2). According to (1.2a), we may write it as

$$\sum_{k\geqslant 0}(-1)^k\binom{m}{k}(x+2k)_m\,\frac{k!\,(x)_{2k}}{(\delta+2k)!\,(1-\delta+x)_k}=\frac{m!}{(\delta+2m)!}(x-\delta)_m(2\delta-x)_m$$

which enables us, in view of the converse (1.2b), to derive the dual relation

$$\sum_{k\geqslant 0}(-1)^k\binom{m}{k}\frac{x+3k}{(x+2m)_{1+k}}\,\frac{k!}{(\delta+2k)!}(x-\delta)_k(2\delta-x)_k=\frac{m!\,(x)_{2m}}{(\delta+2m)!\,(1-\delta+x)_m}\,.$$

The last relation reads, in terms of hypergeometric series, as

$$ {}_4F_3\left[\begin{matrix}-m, & 1+\frac{x}{3}, & x-\delta, & 2\delta-x \\ & \frac{x}{3}, & \frac{1}{2}+\delta, & 1+x+2m\end{matrix};\frac{1}{4}\right]=\frac{m!}{(\delta+2m)!}\,\frac{(1+x)_{2m}}{(1-\delta+x)_{2m}}$$

whose reversal may be reformulated, under parameter replacements $x\to-$

$v-3m$ and $\delta+2m\to n$, as

$$(2.5)\qquad {}_4F_3\left[\begin{matrix} v, & 1+\frac{v}{3}, & -\frac{n}{2}, & \frac{1-n}{2} \\ & \frac{v}{3}, & 1+v+n, & 1-v-2n \end{matrix};4\right]=\frac{(v)_n(1+v)_n}{(v)_{2n}}$$

which will be the starting point for us to attack the problem in this paper.

3. – Hypergeometric evaluations with $z=1/4$.

For a complex indeterminate v and three complex numbers $\{a, b, c\}$, putting

$$(3.1a)\qquad \phi(x;m)=\ (b+v+x)_m$$

$$(3.1b)\qquad \psi(y;n)\ =(1-c-v-2y)_n$$

in the Theorem, then we may resolve the system of equations

$$(3.2a)\qquad \frac{(a+v)_n(b+v)_n}{(c+v)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{(b+v+3k)\,f(k)}{(b+v+n)_{1+k}(1-c-v-2n)_k}$$

$$(3.2b)\qquad +\sum_{k\geqslant 0}\binom{n}{1+2k}\frac{(1+c+v+3k)\,f'(k)}{(b+v+n)_{1+k}(1-c-v-2n)_{1+k}}$$

$$(3.2c)\qquad \frac{(a+v)_n(b+v)_n}{(c+v)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{(1-c-v-3k)\,f(k)}{(b+v+n)_k(1-c-v-2n)_{1+k}}$$

$$(3.2d)\qquad -\sum_{k\geqslant 0}\binom{n}{1+2k}\frac{(1+b+v+3k)\,f''(k)}{(b+v+n)_{1+k}(1-c-v-2n)_{1+k}}$$

as follows

$$f(n)=\sum_{k=0}(-1)^k\binom{2n}{k}\frac{(a+v)_k(b+v)_k}{(c+v)_{2k}}(b+v+k)_n(1-c-v-2k)_n$$

$$f'(n)=\sum_{k=0}(-1)^k\binom{1+2n}{k}\frac{(a+v)_k(b+v)_k}{(c+v)_{2k}}(b+v+k)_{1+n}(1-c-v-2k)_n$$

$$f''(n)=\sum_{k\geqslant 0}(-1)^k\binom{1+2n}{k}\frac{(a+v)_k(b+v)_k}{(c+v)_{2k}}(b+v+k)_n(1-c-v-2k)_{1+n}.$$

They may be reformulated in terms of hypergeometric series

(3.3a) $${}_3F_2\left[\begin{matrix}-2n, & a+v, & b+v+n \\ & \frac{c+v-n}{2}, & \frac{1+c+v-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{f(n)}{(b+v)_n(1-c-v)_n}$$

(3.3b) $${}_3F_2\left[\begin{matrix}-1-2n, & a+v, & 1+b+v+n \\ & \frac{c+v-n}{2}, & \frac{1+c+v-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{f'(n)}{(b+v)_{1+n}(1-c-v)_n}$$

(3.3c) $${}_3F_2\left[\begin{matrix}-1-2n, & a+v, & b+v+n \\ & \frac{c+v-n}{2}, & \frac{c-1+v-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{f''(n)}{(b+v)_n(1-c-v)_{1+n}}$$

which satisfy the following relations

(3.4a) $$f(n)=\frac{f'(n)\{1+c+v+3n\}+f''(n)\{1+b+v+3n\}}{(1+2n)\{1+2b-c+v+3n\}}$$

(3.4b) $$f'(n)=f(n)\,\frac{(1+2n)\{1+2b-c+v+3n\}}{1+c+v+3n}-f''(n)\,\frac{1+b+v+3n}{1+c+v+3n}$$

(3.4c) $$f''(n)=f(n)\,\frac{(1+2n)\{1+2b-c+v+3n\}}{1+b+v+3n}-f'(n)\,\frac{1+c+v+3n}{1+b+v+3n}.$$

Following this scheme, we are ready to demonstrate several terminating hypergeometric evaluations with variable $z=1/4$. For this reason, we need to rewrite (2.5) in terms of binomial summation

(3.5) $$\frac{(v)_n(v)_n}{(1+v)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{-v(v+3k)}{(v+n)_{1+k}(-v-2n)_{1+k}}\left\{4^k(v)_k(1/2)_k\right\}$$

whose incidence structure resembles to those of (3.2*a*)-(3.2*b*) and (3.2*c*)-(3.2*d*). By adjusting the factor $v(v+3k)$ slightly, we will match them completely. In such cases, the «so-called» inversion techniques will work out without obstacle.

3.1. It is obvious that (3.5) is identical with

$$\frac{(v)_n(v)_n}{(v)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{v+3k}{(v+n)_{1+k}(1-v-2n)_k}\left\{4^k(v)_k(1/2)_k\right\}$$

which is the case $a=b=c=0$ of (3.2*a*) and (3.2*b*) with

$$f(n)=4^n(v)_n(1/2)_n$$

$$f'(n)\equiv 0$$

$$f''(n)=4^n(v)_n(1/2)_n(1+2n)$$

where the last evaluation is obtained from the first two via (3.4c). In view of (3.3a), (3.3b) and (3.3c), we can write down directly the following hypergeometric identities

$$(3.6a)\qquad {}_3F_2\left[\begin{matrix}-2n, & v, & v+n \\ & \frac{v-n}{2}, & \frac{1+v-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{(2n)!}{n!\,(1-v)_n}$$

$$(3.6b)\qquad {}_3F_2\left[\begin{matrix}-1-2n, & v, & 1+v+n \\ & \frac{v-n}{2}, & \frac{1+v-n}{2}\end{matrix};\frac{1}{4}\right]=0$$

$$(3.6c)\qquad {}_3F_2\left[\begin{matrix}-1-2n, & v, & v+n \\ & \frac{v-n}{2}, & \frac{v-1-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{(1+2n)!}{n!\,(1-v)_{1+n}}$$

which confirm the binomial convolutions (0.2) and (0.4), and the corresponding hypergeometric evaluations (0.3) and (0.5) appeared at the begining of this article.

3.2. According to the factor separation

$$-v-3k=(k-v-2n)+2(n-2k)$$

the relation (3.5) may be splitted as

$$\frac{(v)_n(v)_n}{(1+v)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{(v+3k)\times f(k)}{(v+n)_{1+k}(-v-2n)_k}$$
$$+\sum_{k\geqslant 0}\binom{n}{1+2k}\frac{(2+v+3k)\times f'(k)}{(v+n)_{1+k}(-v-2n)_{1+k}}$$

which correspond to the case $a=b=0$ and $c=1$ of (3.2a) and (3.2b) with

$$f(n)=\frac{v}{v+3n}\,4^n(v)_n(1/2)_n$$

$$f'(n)=\frac{2v(1+2n)}{2+v+3n}\,4^n(v)_n(1/2)_n$$

$$f''(n)=\frac{-v(1+2n)}{1+v+3n}\,4^n(v)_n(1/2)_n$$

where the last evaluation is obtained from the first two via (3.4c). In view of (3.3a), (3.3b) and (3.3c), the dual relations may be stated as the following hypergeometric evaluations

$$(3.7a)\quad {}_3F_2\left[\begin{matrix}-2n, & v, & v+n \\ & \frac{1+v-n}{2}, & \frac{2+v-n}{2}\end{matrix} ; \frac{1}{4}\right] = \frac{(2n)!}{n!\,(-v)_n}\,\frac{v}{v+3n}$$

$$(3.7b)\quad {}_3F_2\left[\begin{matrix}-1-2n, & v, & 1+v+n \\ & \frac{1+v-n}{2}, & \frac{2+v-n}{2}\end{matrix} ; \frac{1}{4}\right] = \frac{(2n)!}{n!\,(-v)_n}\,\frac{2v(1+2n)}{(v+n)(2+v+3n)}$$

$$(3.7c)\quad {}_3F_2\left[\begin{matrix}-1-2n, & v, & v+n \\ & \frac{v-n}{2}, & \frac{1+v-n}{2}\end{matrix} ; \frac{1}{4}\right] = \frac{(2n)!}{n!\,(1-v)_n}\,\frac{1+2n}{1+v+3n}$$

which may also be obtained in the same way from (3.2*c*) and (3.2*d*) through the the factor separation $-v-3k=(n-2k)-(v+n+k)$.

3.3. On account of the factor separation

$$-v-3k=(k-v-2n)\,\frac{1+v+3k}{1+v+n+k}+(n-2k)\,\frac{2+v+3k}{1+v+n+k}$$

the relation (3.5) may be expressed accordingly as

$$\frac{(v)_n(1+v)_n}{(1+v)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{(1+v+3k)\times f(k)}{(1+v+n)_{1+k}(-v-2n)_k}$$
$$+\sum_{k\geqslant 0}\binom{n}{1+2k}\frac{(2+v+3k)\times f'(k)}{(1+v+n)_{1+k}(-v-2n)_{1+k}}$$

which is a reduced form of (3.2*a*) and (3.2*b*) with $a=0$, $b=c=1$ and

$$f(n)=4^n(v)_n(1/2)_n$$
$$f'(n)=4^n(v)_n(1/2)_n(1+2n)$$
$$f''(n)=0$$

where the last evaluation is obtained from the first two via (3.4*c*). By means of (3.3*a*) and (3.3*b*), the dual relations are obtained as the following hypergeometric formulas

$$(3.8a)\quad {}_3F_2\left[\begin{matrix}-2n, & v, & 1+v+n \\ & \frac{1+v-n}{2}, & \frac{2+v-n}{2}\end{matrix} ; \frac{1}{4}\right] = \frac{(2n)!}{n!\,(-v)_n}\,\frac{v}{v+n}$$

$$(3.8b)\quad {}_3F_2\left[\begin{matrix}-1-2n, & v, & 2+v+n \\ & \frac{1+v-n}{2}, & \frac{2+v-n}{2}\end{matrix} ; \frac{1}{4}\right] = \frac{(2n)!}{n!\,(-v)_n}\,\frac{v(1+2n)}{(v+n)(1+v+n)}$$

where the evaluation derived from $f''(n)$ via (3.3c) is identical with (3.6b) and so has been omitted. In addition, these formulas may also be obtained from (3.2c) and (3.2d) through the dual separation

$$-v-3k=(n-2k)\,\frac{v+1+3k}{1-v-2n+k}-(v+n+k)\,\frac{v-1+3k}{1-v-2n+k}\,.$$

3.4. Separating the factor

$$-v-3k=(k-v-2n)\,\frac{v+2k}{v+n}+(n-2k)\,\frac{v+k}{v+n}$$

we may reformulate the relation (3.5) as

$$\frac{(v)_n(1+v)_n}{(1+v)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{(v+3k)\times f(k)}{(v+n)_{1+k}(-v-2n)_k}$$

$$+\sum_{k\geqslant 0}\binom{n}{1+2k}\frac{(2+v+3k)\times f'(k)}{(v+n)_{1+k}(-v-2n)_{1+k}}$$

which correspond to (3.2a) and (3.2b) specified with $a=c=1,\ b=0$ and

$$f(n)=4^n(v)_n(1/2)_n\,\frac{v+2n}{v+3n}$$

$$f'(n)=4^n(v)_n(1/2)_n\,\frac{(v+n)(1+2n)}{2+v+3n}$$

$$f''(n)=4^n(v)_n(1/2)_n\,\frac{n(1+2n)}{1+v+3n}$$

where the last evaluation is obtained from the first two via (3.4c). In view of (3.3a) and (3.3c), we can write down directly the following hypergeometric identities

$$(3.9a)\quad {}_3F_2\left[\begin{matrix}-2n, & 1+v, & v+n \\ & \frac{1+v-n}{2}, & \frac{2+v-n}{2}\end{matrix};\ \frac{1}{4}\right]=\frac{(2n)!}{n!\,(-v)_n}\,\frac{v+2n}{v+3n}$$

$$(3.9b)\quad {}_3F_2\left[\begin{matrix}-1-2n, & 1+v, & v+n \\ & \frac{v-n}{2}, & \frac{1+v-n}{2}\end{matrix};\ \frac{1}{4}\right]=\frac{(1+2n)!}{n!\,(-v)_{1+n}}\,\frac{n}{1+v+3n}$$

where the evaluation derived from $f'(n)$ via (3.3b) has been omitted for its equivalence to (3.7c). These formulas may also be obtained from (3.2c) and

(3.2d) through the dual separation

$$-v-3k=-(n-2k)\,\frac{k}{v+n}-(v+n+k)\,\frac{v+2k}{v+n}.$$

3.5. Now we consider a more general case. Given two complex numbers $\{\alpha,\gamma\}$, the factor separation

$$-v-3k=(k-v-2n)\,\frac{\alpha+2\gamma k}{\alpha+\gamma n}+(n-2k)\,\frac{2\alpha-\gamma v+\gamma k}{\alpha+\gamma n}$$

leads the relation (3.5) to

$$\text{(3.10}a\text{)}\qquad \frac{\alpha+\gamma n}{\alpha}\,\frac{(v)_n(v)_n}{(1+v)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{(v+3k)\times f(k)}{(v+n)_{1+k}(-v-2n)_k}$$

$$\text{(3.10}b\text{)}\qquad +\sum_{k\geqslant 0}\binom{n}{1+2k}\frac{(2+v+3k)\times f'(k)}{(v+n)_{1+k}(-v-2n)_{1+k}}$$

where

$$f(k)=4^k(v)_k(1/2)_k\,\frac{v(\alpha+2\gamma k)}{\alpha(v+3k)}$$

$$f'(k)=4^k(v)_k(1/2)_k\,\frac{(1+2k)\,v(2\alpha-\gamma v+\gamma k)}{\alpha(2+v+3k)}$$

$$f''(k)=4^k(v)_k(1/2)_k\,\frac{(1+2k)\,v(-\alpha+\gamma v+\gamma k)}{\alpha(1+v+3k)}$$

with the last evaluation derived from the first two via (1.5c). Then (3.10a) and (3.10b) are special cases of (1.3a) and (1.3b) with

$$\phi(x;m)=(v+x)_m$$

$$\psi(y;n)=(-v-2y)_n$$

$$g(k)=\frac{\alpha+k\gamma}{\alpha}\,\frac{(v)_k(v)_k}{(1+v)_{2k}}.$$

Their dual relations read through (1.4a), (1.4b) and (1.4c), as the following hypergeometric identities

$$\text{(3.11}a\text{)}\qquad {}_4F_3\left[\begin{matrix}-2n, & 1+\frac{\alpha}{\gamma}, & v & v+n\\ & \frac{\alpha}{\gamma}, & \frac{1+v-n}{2} & \frac{2+v-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{(2n)!}{n!\,(-v)_n}\,\frac{v(\alpha+2\gamma n)}{\alpha(v+3n)}$$

$$(3.11b)\quad {}_4F_3\left[\begin{matrix}-1-2n,\ 1+\frac{\alpha}{\gamma}, & v & 1+v+n \\ \frac{\alpha}{\gamma}, & \frac{1+v-n}{2} & \frac{2+v-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{v(2\alpha-\gamma v+\gamma n)}{\alpha(v+n)(2+v+3n)}$$

$$(3.11c)\quad {}_4F_3\left[\begin{matrix}-1-2n,\ 1+\frac{\alpha}{\gamma}, & v & v+n \\ \frac{\alpha}{\gamma}, & \frac{v-n}{2} & \frac{1+v-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{(1+2n)!}{n!\,(1-v)_n}\,\frac{\alpha-\gamma v-\gamma n}{\alpha(1+v+3n)}$$

which may also be obtained in the same way from (3.2c) and (3.2d) through the dual separation

$$-v-3k=(n-2k)\,\frac{\alpha-\gamma v-\gamma k}{\alpha+\gamma n}-(v+n+k)\,\frac{\alpha+2\gamma k}{\alpha+\gamma n}\,.$$

It is not hard to check that these three hypergeometric evaluations are the common generalizations of the previous established hypergeometric identities in this section.

4. – Recurrence relations and more evaluations.

In order to simplify the notation, we introduce the following hypergeometric series:

$$(4.1a)\qquad F\left[\begin{matrix}a,\,b,\,c\\ n,\,v\end{matrix}\right]={}_3F_2\left[\begin{matrix}-2n, & a+v, & b+v+n \\ & \frac{c+v-n}{2}, & \frac{1+c+v-n}{2}\end{matrix};\frac{1}{4}\right]$$

$$(4.1b)\qquad \mathcal{F}\left[\begin{matrix}a,\,b,\,c\\ n,\,v\end{matrix}\right]={}_3F_2\left[\begin{matrix}-1-2n, & a+v, & b+v+n \\ & \frac{c+v-n}{2}, & \frac{1+c+v-n}{2}\end{matrix};\frac{1}{4}\right]$$

where the former is (3.3a) and the latter the unified form of (3.3b) and (3.3c).

Then the relations (3.4a), (3.4b) and (3.4c) may be translated into a single one between F and $\mathcal{F}$:

$$(4.2a)\qquad F\left[\begin{matrix}a,\,b,\,c\\ n,\,v\end{matrix}\right]=\mathcal{F}\left[\begin{matrix}a,\,1+b,\,c\\ n,\,v\end{matrix}\right]\frac{(b+v+n)(1+c+v+3n)}{(1+2n)(1+2b-c+v+3n)}$$

$$(4.2b)\qquad +\,\mathcal{F}\left[\begin{matrix}a,\,b,\,c-1\\ n,\,v\end{matrix}\right]\frac{(1+b+v+3n)(1-c-v+n)}{(1+2n)(1+2b-c+v+3n)}\,.$$

By means of the factor separations

$$a+v+k-1=k+(a+v-1)$$

$$b+v+n+k-1=k+(b+v+n-1)$$

we may derive the following «vertical» recurrence relations

$$(4.3a)\quad F\begin{bmatrix} a,b,c \\ n,v \end{bmatrix} = F\begin{bmatrix} a-1,b,c \\ n,v \end{bmatrix} - \mathscr{F}\begin{bmatrix} a-1,1+b,c \\ n-1,1+v \end{bmatrix} \frac{2n(b+v+n)}{(c+v-n)(1+c+v-n)}$$

$$(4.3b)\qquad = F\begin{bmatrix} a,\, b-1,\, c \\ n,\, v \end{bmatrix} - \mathscr{F}\begin{bmatrix} a,\, b,\, c \\ n-1,\, 1+v \end{bmatrix} \frac{2n(a+v)}{(c+v-n)(1+c+v-n)}$$

$$(4.3c)\quad \mathscr{F}\begin{bmatrix} a,\, b,\, c \\ n,\, v \end{bmatrix} = \mathscr{F}\begin{bmatrix} a-1,\, b,\, c \\ n,\, v \end{bmatrix} - F\begin{bmatrix} a-1,\, b,\, 1+c \\ n,\, 1+v \end{bmatrix} \frac{(1+2n)(b+v+n)}{(c+v-n)(1+c+v-n)}$$

$$(4.3d)\qquad = \mathscr{F}\begin{bmatrix} a,\, b-1,\, c \\ n,\, v \end{bmatrix} - F\begin{bmatrix} a,\, b-1,\, 1+c \\ n,\, 1+v \end{bmatrix} \frac{(1+2n)(a+v)}{(c+v-n)(1+c+v-n)}.$$

Instead, the factor separations

$$1 = \frac{2(a+v+k)}{1+2a-c+v+n} - \frac{c+v-1-n+2k}{1+2a-c+v+n}$$

$$1 = \frac{2(b+v+n+k)}{1+2b-c+v+3n} - \frac{c+v-1-n+2k}{1+2b-c+v+3n}$$

lead us to the following recurrence relations of contiguous hypergeometric series

$$(4.4a)\quad F\begin{bmatrix} a,b,c \\ n,v \end{bmatrix} = F\begin{bmatrix} 1+a,b,c \\ n,v \end{bmatrix} \frac{2(a+v)}{1+2a-c+v+n} + F\begin{bmatrix} a,b,c-1 \\ n,v \end{bmatrix} \frac{1-c-v+n}{1+2a-c+v+n}$$

$$(4.4b)\qquad = F\begin{bmatrix} a,1+b,c \\ n,v \end{bmatrix} \frac{2(b+v+n)}{1+2b-c+v+3n} + F\begin{bmatrix} a,b,c-1 \\ n,v \end{bmatrix} \frac{1-c-v+n}{1+2b-c+v+3n}$$

$$(4.4c)\quad \mathscr{F}\begin{bmatrix} a,\, b,\, c \\ n,\, v \end{bmatrix} = \mathscr{F}\begin{bmatrix} 1+a,\, b,\, c \\ n,\, v \end{bmatrix} \frac{2(a+v)}{1+2a-c+v+n} + \mathscr{F}\begin{bmatrix} a,\, b,\, c-1 \\ n,\, v \end{bmatrix} \frac{1-c-v+n}{1+2a-c+v+n}$$

$$(4.4d)\qquad = \mathscr{F}\begin{bmatrix} a,\, 1+b,\, c \\ n,\, v \end{bmatrix} \frac{2(b+v+n)}{1+2b-c+v+3n} + \mathscr{F}\begin{bmatrix} a,\, b,\, c-1 \\ n,\, v \end{bmatrix} \frac{1-c-v+n}{1+2b-c+v+3n}.$$

Based on these recurrences, further 55 evaluations of hypergeometric ${}_3F_2[1/4]$-series have been derived. Among them we display now only these with closed forms in terms of shifted factorial fractions, possibly adjusted by an additional linear factor. Other formulas of terminating and non-terminating hypergeometric series with the same variable may be found in [10, 11] and

[13], respectively.

$$F\begin{bmatrix}1,0,0\\n,v\end{bmatrix}=\frac{(2n)!}{n!\,(1-v)_n} \tag{4.5}$$

$$F\begin{bmatrix}0,1,0\\n,v\end{bmatrix}=\frac{(2n)!}{n!\,(1-v)_n} \tag{4.6}$$

$$F\begin{bmatrix}1,1,0\\n,v\end{bmatrix}=\frac{(2n)!}{n!\,(-v)_n} \tag{4.7}$$

$$\mathcal{F}\begin{bmatrix}1,2,0\\n,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{2}{n-v} \tag{4.8}$$

$$\mathcal{F}\begin{bmatrix}1,3,0\\n,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{3}{n-v} \tag{4.9}$$

$$F\begin{bmatrix}1,2,0\\n,v\end{bmatrix}=\frac{(2n)!}{n!\,(-v)_n}\,\frac{v-3n}{v-n} \tag{4.10}$$

$$F\begin{bmatrix}1,3,0\\n,v\end{bmatrix}=\frac{(2n)!}{n!\,(-v)_n}\,\frac{v-6n}{v-n} \tag{4.11}$$

$$\mathcal{F}\begin{bmatrix}0,3,1\\n,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{v(2+n)}{(v+n)_3} \tag{4.12}$$

$$F\begin{bmatrix}0,2,0\\n,v\end{bmatrix}=\frac{(2n)!}{n!\,(1-v)_n}\,\frac{1+v}{1+v+n} \tag{4.13}$$

$$\mathcal{F}\begin{bmatrix}0,2,0\\n,v\end{bmatrix}=\frac{(2n)!}{n!\,(1-v)_n}\,\frac{-(1+2n)}{1+v+n} \tag{4.14}$$

$$\mathcal{F}\begin{bmatrix}2,4,0\\n,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(1-v)_n}\,\frac{3n-5v-8}{v(1+v)} \tag{4.15}$$

$$\mathcal{F}\begin{bmatrix}2,1,0\\n,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{2+2v+n}{(1+v)(n-v)} \tag{4.16}$$

$$\mathcal{F}\begin{bmatrix}2,2,0\\n,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{4+3v+n}{(1+v)(n-v)} \tag{4.17}$$

$$\mathcal{F}\begin{bmatrix}2,3,0\\n,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{2(3+2v)}{(1+v)(n-v)} \tag{4.18}$$

$$\mathcal{F}\begin{bmatrix}2,\,5,\,0\\ n,\,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(1-v)_n}\,\frac{9n-6v-10}{v(1+v)} \tag{4.19}$$

$$\mathcal{F}\begin{bmatrix}3,\,1,\,0\\ n,\,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{3+2n+3v}{(1+v)(n-v)} \tag{4.20}$$

$$\mathcal{F}\begin{bmatrix}0,\,0,\,1\\ n,\,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{v(2+3v+9n)}{(v+3n)_3} \tag{4.21}$$

$$F\begin{bmatrix}0,\,2,\,1\\ n,\,v\end{bmatrix}=\frac{(2n)!}{n!\,(-v)_n}\,\frac{v(1+v+2n)}{(v+n)(1+v+n)} \tag{4.22}$$

$$\mathcal{F}\begin{bmatrix}1,\,4,\,0\\ n,\,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{4v+3n+12}{(n-v)(v+n+3)} \tag{4.23}$$

$$\mathcal{F}\begin{bmatrix}0,\,4,\,1\\ n,\,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{v(2-v)}{(v+n)_4}\{v+2n+3\} \tag{4.24}$$

$$\mathcal{F}\begin{bmatrix}0,\,3,\,0\\ n,\,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(1-v)_n}\,\frac{-(4+2v+3n)}{(1+v+n)(2+v+n)} \tag{4.25}$$

$$F\begin{bmatrix}0,\,1,\,2\\ n,\,v\end{bmatrix}=\frac{(2n)!}{n!\,(-1-v)_n}\,\frac{v(1+v)(v-1-n)}{(1+v+3n)(v-1+n)_2} \tag{4.26}$$

$$\mathcal{F}\begin{bmatrix}0,\,2,\,2\\ n,\,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-v)_n}\,\frac{v(1+v-n)(3v-3+n)}{(3+v+3n)(v-1+n)_3} \tag{4.27}$$

$$F\begin{bmatrix}0,\,2,\,2\\ n,\,v\end{bmatrix}=\frac{(2n)!}{n!\,(-1-v)_n}\,\frac{v(v+1)(v-1)}{(v+n)(v+1+n)(v-1+n)} \tag{4.28}$$

$$\mathcal{F}\begin{bmatrix}0,\,3,\,2\\ n,\,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-1-v)_n}\,\frac{v(1+v)}{(v+n-1)_4}\{2v+n-2\} \tag{4.29}$$

$$\mathcal{F}\begin{bmatrix}0,\,4,\,3\\ n,\,v\end{bmatrix}=\frac{(1+2n)!}{n!\,(-2-v)_n}\,\frac{(v-1)_4}{(v+n-2)_6}\{3v+2n-6\} \tag{4.30}$$

5. – Additional comment.

Besides the Saalschütz theorem, there exists also a 2-balanced summation formulae (cf. [4, 7])

$$(5.1)\qquad {}_3F_2\left[\begin{matrix}-m, & \frac{x+m}{2}, & \frac{1+x+m}{2} \\ & \delta+\frac{1}{2}, & 2-\delta+x\end{matrix};1\right]=$$

$$=\frac{3\delta-2-x+3m}{3\delta-2-x}\,\frac{m!\,(x-\delta)_m(2\delta-2-x)_m}{(\delta+2m)!\,(x)_m}.$$

We can express it, according to (1.2a), as

$$\sum_{k\geqslant 0}(-1)^k\binom{m}{k}\frac{(x+2k)_m\,k!\,(x)_{2k}}{(\delta+2k)!\,(2-\delta+x)_k}=$$

$$=\frac{3\delta-2-x+3m}{3\delta-2-x}\,\frac{m!\,(x-\delta)_m(2\delta-2-x)_m}{(\delta+2m)!}.$$

which leads us, in view of (1.2b), to the dual relation

$$\sum_{k\geqslant 0}(-1)^k\binom{m}{k}\frac{(x+3k)(3\delta-2-x+3k)}{(x+2m)_{1+k}(3\delta-2-x)}\,\frac{k!\,(2\delta-2-x)_k}{(\delta+2k)!/(x-\delta)_k}=$$

$$=\frac{m!\,(x)_{2m}}{(\delta+2m)!\,(2-\delta+x)_m}$$

whose reversal reads, under parameter replacements $x\to -v-3m$ and $\delta+2m\to n$, as the following hypergeometric evaluation

$$(5.2)\qquad {}_5F_4\left[\begin{matrix}v,\ 1+\frac{v}{3}, & \frac{5-v}{3}-n, & -\frac{n}{2}, & \frac{1-n}{2} \\ \frac{v}{3}, & \frac{2-v}{3}-n, & 1+v+n, & 3-v-2n\end{matrix};4\right]=$$

$$=\frac{v-2}{v-2+3n}\,\frac{(v-1)_n(1+v)_n}{(v-2)_{2n}}.$$

Similar to the applications of (2.5) in Section 3, we now use this result to show an alternative approach to the hypergeometric identities (4.6), (4.10) and (4.14).

Notice that (5.2) may be expressed in terms of binomial summation as

$$(5.3)\qquad \frac{(v)_n(v-1)_n}{(v-1)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{(v+3k)(2-v-3n+3k)}{(v+n)_{1+k}(2-v-2n)_{1+k}}\left\{4^k(v)_k(1/2)_k\right\}.$$

By means of the factor separation

$$2-v-3n+3k=(2-v-2n+k)-(n-2k)$$

the relation (5.3) may be divided into two parts

$$\frac{(v)_n(v-1)_n}{(v-1)_{2n}}=\sum_{k\geqslant 0}\binom{n}{2k}\frac{(v+3k)\times f(k)}{(v+n)_{1+k}(2-v-2n)_k}$$
$$+\sum_{k\geqslant 0}\binom{n}{1+2k}\frac{(v+3k)\times f'(k)}{(v+n)_{1+k}(2-v-2n)_{1+k}}$$

which correspond to the case $b=0$ and $a=c=-1$ of (3.2a) and (3.2b) with

$$\begin{aligned} f(n) &= 2^{0+2n}(v)_n(1/2)_n \\ f'(n) &= -2^{1+2n}(v)_n(1/2)_{1+n} \\ f''(n) &= 2^{2+2n}(v)_n(1/2)_{1+n} \end{aligned}$$

where the last evaluation is obtained from the first two via (3.4c). In view of (3.3a), (3.3b) and (3.3c), the dual relations may be stated as the following hypergeometric formulas

$${}_3F_2\left[\begin{matrix}-2n, & v-1, & v+n \\ & \frac{v-n}{2}, & \frac{v-1-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{(2n)!}{n!\,(2-v)_n}$$

$${}_3F_2\left[\begin{matrix}-1-2n, & v-1, & 1+v+n \\ & \frac{v-n}{2}, & \frac{v-1-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{(1+2n)!}{n!\,(2-v)_n}\,\frac{-1}{v+n}$$

$${}_3F_2\left[\begin{matrix}-1-2n, & v-1, & v+n \\ & \frac{v-1-n}{2}, & \frac{v-2-n}{2}\end{matrix};\frac{1}{4}\right]=\frac{(1+2n)!}{n!\,(3-v)_n}\,\frac{2}{2-v}$$

which are exactly (4.6), (4.14) and (4.10) with variable v being shifted.

The same results may also be derived from (3.2c) and (3.2d) through the dual separation

$$(v+3k)\{2-v-3n+3k\}=(2-v-3k)\{v+n+k\}$$
$$-2(1+v+3k)\{n-2k\}.$$

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