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Calculating a Determinant Associated with Multiplicative Functions.

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Sunto. – *Sia h una funzione moltiplicativa a valori complessi. Per ogni* $N \in \mathbf{N}$, *calcoliamo il determinante* $D_N := \det_{i|N,\, j|N}\left(\frac{h((i,\, j))}{ij}\right)$, *dove* $(i,\, j)$ *indica il massimo comun divisore di i e j, che figurano in ordine crescente in righe e colonne. Precisamente dimostriamo che*

$$D_N = \prod_{p^l \| N} \left(\frac{1}{p^{l(l+1)}} \prod_{i=1}^{l} (h(p^i) - h(p^{i-1})) \right)^{\tau(N/p^l)}.$$

Dunque $D_N^{1/\tau(N)}$ *è effettivamente una funzione moltiplicativa di* N. *L'apparato algebrico associato a questo risultato ci consente di dimostrarne altri due. Il primo è la caratterizzazione delle funzioni reali moltiplicative* $f(n)$, *con* $0 \leqslant f(p) < 1$, *come valori minimi di certe forme quadratiche sulla sfera unità* $\tau(N)$ *dimensionale. Il secondo è la determinazione esplicita dei valori minimi di certe altre forme quadratiche su detta sfera.*

Summary. – *Let h be a complex valued multiplicative function. For any* $N \in \mathbf{N}$, *we compute the value of the determinant* $D_N := \det_{i|N,\, j|N}\left(\frac{h((i,\, j))}{ij}\right)$, *where* $(i,\, j)$ *denotes the greatest common divisor of i and j, which appear in increasing order in rows and columns. Precisely we prove that*

$$D_N = \prod_{p^l \| N} \left(\frac{1}{p^{l(l+1)}} \prod_{i=1}^{l} (h(p^i) - h(p^{i-1})) \right)^{\tau(N/p^l)}.$$

This means that $D_N^{1/\tau(N)}$ *is a multiplicative function of* N. *The algebraic apparatus associated with this result allows us to prove the following two results. The first one is the characterization of real multiplicative functions* $f(n)$, *with* $0 \leqslant f(p) < 1$, *as minimal values of certain quadratic forms on the* $\tau(N)$ *unit sphere. The second one is the explicit evaluation of the minimal values of certain others quadratic forms also on the unit sphere.*

1. – Introduction.

Let $h(n)$ be a complex-valued multiplicative function i.e. $h(1) = 1$ and for $(m, n) = 1$, $m, n \in \boldsymbol{N}$, we have that $h(mn) = h(m)h(n)$. Here (m, n) denotes the greatest common divisor of m and n.

For any $N \in \boldsymbol{N}$, denote by $d_1, \ldots, d_{\tau(N)}$ the positive divisors of N in increasing order. We define the $\tau(N) \times \tau(N)$ matrix M_N to be that with $(i, j)^{\text{th}}$ element $h((d_i, d_j))/d_i d_j$. This we abbreviate to

$$M_N = \left(\frac{h((i, j))}{ij} \right)_{i|N,\, j|N}$$

and we write

$$D_N = \det M_N .$$

This matrix can arise in several natural ways some of which we shall now describe.

Let

$$\Delta(x, N) = \sum_{\substack{n \leq xN \\ (n, N) = 1}} 1 - x\phi(N)$$

where ϕ denotes Euler's function. This expression measures, via

$$\Delta(N) = \sup_{x \in \boldsymbol{R}} |\Delta(x, N)| ,$$

the maximal deviation of the number of integers $\leq xN$ which are coprime to N from the expected quantity.

Various results on $\Delta(N)$ may be found in [2]. By writing $\Delta(x, N)$ in the form

$$\sum_{n \leq xN} \sum_{d | (n, N)} \mu(d) - x \sum_{n \leq N} \sum_{d | (n, N)} \mu(d)$$

and reversing summations, it readily follows that for square-free N

$$\Delta(x, N) = -\mu(N) \sum_{d | N} \mu(d) \{xd\}$$

where μ is the Möbius function and $\{t\}$ denotes the fractional part of t. A simple computation then yields that

$$\int_0^1 \Delta^2(x, N)\, dx = \frac{1}{12} \sum_{i|N,\, j|N} \mu(i)\, \mu(j)\, \frac{(i, j)^2}{ij} .$$

This sum can be evaluated to be $2^{\omega(N)} \varphi(N)/N$, where $\omega(N)$ is the number of distinct prime factors of N and φ is Euler's function, and, as shown by Perelli-

Zannier [5], is $2^{\omega(N)}$ multiplied by the minimal value of the quadratic form $\sum_{i|N, j|N} x_i x_j \frac{(i, j)^2}{ij}$ subject to the constraint $\sum_{i|N} x_i^2 = 1$. Observe that the associated matrix is of the form M_N with $h(n) = n^2$.

This type of discussion can be presented in a more general context. For any fixed $N \in \boldsymbol{N}$, define a function f by

$$f(n) = f_N(n) = \sum_{d|(n, N)} \theta_d.$$

Since f is periodic with period N, denoting by $m(f)$ its mean value, it is easily seen that

$$\sum_{n \leq x} f(n) = m(f)\, x - \sum_{d|N} \theta_d \left\{ \frac{x}{d} \right\}$$

and hence

$$\frac{1}{N} \int_0^N \Big(\sum_{n \leq x} f(n) - m(f)\, x \Big)^2 dx = \frac{1}{4} \Big(\sum_{d|N} \theta_d \Big)^2 + \frac{1}{12} \sum_{i|N, j|N} \alpha_i \alpha_j \frac{(i, j)^2}{ij}$$

where $\alpha_i = \theta_{N/i}$. The matrix M_N associated with $h(n) = n^2$ is once more present in the second term on the right.

Further, since f^2 is also periodic of period N, its mean value $m(f^2)$ can be expressed as

$$m(f^2) := \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f^2(n) = \sum_{i|N, j|N} \theta_i \theta_j \frac{(i, j)}{ij}$$

and hence we have here an occurrence of M_N with $h(n) = n$.

Problems concerned with determining minimal values of certain quadratic forms subject to constraints on the variables also give rise to matrices of the form M_N. For example, let $g(n)$ be *any* positive real multiplicative function and define the multiplicative function h by $h(n) = \sum_{d|n} g(d)$. Consider the quadratic form Λ defined by

$$\Lambda = \sum_{i|N, j|N} x_i x_j \frac{h((i, j))}{ij}$$

or equivalently, in terms of g,

$$\Lambda = \sum_{d|N} g(d) \left(\sum_{\substack{n|N \\ d|n}} \frac{x_n}{n} \right)^2,$$

which transparently shows that Λ is positive definite. To determine the minimal value of Λ on the $\tau(N)$-dimensional unit sphere $\sum_{d|N} x_d^2 = 1$ reduces to calculating the smallest eigenvalue of M_N. Such problems are often difficult to resolve completely since the eigenvalues can be zeros of polynomials of high degree. In our Theorem 3, we determine the required minimal value of Λ for *square-free* N. This includes the special case of $h(n) = n^2$ corresponding to $g(n) = n^2 \prod_{p|n}\left(1 - \frac{1}{p^2}\right)$ which yields the minimal value $\varphi(N)/N$ for the Perelli-Zannier quadratic form cited above. In Theorem 2, we show that this property of the function $\varphi(n)/n$ as a minimal value of a constrained quadratic form is, in fact, one shared by any real multiplicative function $f(n)$ with $0 \leqslant f(p) < 1$.

Our first result is a complete evaluation of $\det M_N$ for any multiplicative function h.

THEOREM 1. – *Let h be a complex-valued multiplicative function. Then*

$$(1) \quad D_N := \det_{i|N,\, j|N}\left(\frac{h((i,\, j))}{ij}\right) = \prod_{p^l \| N}\left(\frac{1}{p^{l(l+1)}}\prod_{i=1}^{l}(h(p^i) - h(p^{i-1}))\right)^{\tau(N/p^l)}$$

for any $N \in \boldsymbol{N}$.

The interest here is in the fact that this can be achieved without calculating the individual eigenvalues of the matrix M_N. We also prove the following curious result regarding the characterization of certain multiplicative functions as minimal values of certain quadratic forms on the unit sphere.

THEOREM 2. – *Any real-valued multiplicative function $f(n)$ with $0 \leqslant f(p) < 1$ for prime p can be expressed as the minimal value, for square-free n, of a quadratic form*

$$\sum_{i|n,\, j|n} x_i x_j \frac{h((i,\, j))}{ij},$$

with h multiplicative, on the $\tau(n)$-dimensional sphere $\sum_{i|n} x_i^2 = 1$.

For example, as may be noted from the proof, the ubiquitous M_N with $h(n) = n^2$ is associated with $f(n) = \frac{\phi(n)}{n}$. We also note at the end of the proof that the condition $f(p) < 1$ cannot be relaxed to $f(p) \leqslant 1$.

Finally, as discussed earlier, we prove the following result.

THEOREM 3. – *Let N be a square-free integer. For any positive real multiplicative function $g(n)$, write $h(n) = \sum_{d|n} g(d)$. Then the minimal value of the* *quadratic form*

$$\Lambda = \sum_{i|N,\, j|N} x_i x_j \frac{h((i,\, j))}{ij}$$

subject to $\sum_{d|N} x_d^2 = 1$ *is given by*

$$\prod_{p|N}\left(1 - \frac{(p^2 - h(p)) + \sqrt{(p^2 - h(p))^2 + 4p^2}}{2p^2}\right).$$

It is obvious that substituting $h(p) = p^2$ in this expression immediately yields the minimal value $\varphi(N)/N$ as mentioned earlier.

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Preliminaries

We shall need three lemmas. Useful references for the facts we require on tensor products of matrices are [1], [3] and [4].

Lemma 1 is well known, Lemma 2 describes the eigenvalues of the matrix M_N, whilst Lemma 3 is essentially a simple computation. We begin with the basic definitions.

Given an $m \times n$ matrix $A = (a_{ij})$ and a $p \times q$ matrix $B = (b_{kl})$, the tensor product $A \otimes B$ is defined to be the $mp \times nq$ matrix $C = (c_{rs})$ where $c_{rs} = a_{ij} b_{kl}$ with $r = p(i-1) + k$ and $s = q(j-1) + l$. Equivalently, $i = \left[\frac{r-1}{p}\right] + 1$, $k \equiv r \pmod p$, $j = \left[\frac{s-1}{q}\right] + 1$ and $l \equiv s \pmod q$ where the square brackets indicate the integer part function and $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant p$, $1 \leqslant l \leqslant q$.

It is a trivial but cumbersome exercise to show that tensor products are associative and that

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

for any matrices A, B, C, D whenever AC and BD are defined (see [4], Theorems 8.8.3 and 8.8.6). In general, $A \otimes B \neq B \otimes A$ so that the order in which the factors occur in a tensor product needs to be explicitly specified.

For any $N \in \boldsymbol{N}$, $N > 1$, we shall denote by $M_N^{\otimes}$ the tensor product

$$M_N^{\otimes} = \otimes_{p^l \| N} M_{p^l}$$

where each factor M_{p^l} is as described in the introduction (with $N = p^l$) and occurs in the product in order of increasing p.

All matrices considered in Lemmas 1 and 2 will be over $\boldsymbol{C}$.

LEMMA 1. – *For any two square matrices A and B, the eigenvalues of $A \otimes B$ are precisely all the products of eigenvalues of A with eigenvalues of B (both with multiplicity).*

PROOF. – See [4], Theorem 8.8.13.

COROLLARY 1. – *The eigenvalues of $M_N^{\otimes}$ are precisely all the products of eigenvalues of the individual factors M_{p^l}, taken one for each $p^l \| N$.*

PROOF. – Immediate.

LEMMA 2. – *The eigenvalues of M_N are precisely all the products of eigenvalues of M_{p^l}, taken one for each $p^l \| N$.*

REMARK. – We shall prove that M_N and $M_N^{\otimes}$ are similar matrices and the result then follows from Lemma 1.

PROOF. – For any $(n, m) = 1$, let $A = \left(\dfrac{h((d_i, d_j))}{d_i d_j}\right)$, $1 \leqslant i,\ j \leqslant \tau(n)$, where $d_1, \ldots, d_{\tau(n)}$ are the $\tau(n)$ divisors of n, listed in some fixed order (not necessarily increasing) and, similarly, let $B = \left(\dfrac{h((\delta_k, \delta_l))}{\delta_k \delta_l}\right)$, $1 \leqslant k,\ l \leqslant \tau(m)$, where $\delta_1, \ldots, \delta_{\tau(m)}$ are the $\tau(m)$ divisors of m, also listed in some fixed order.

Then $A \otimes B = C = (c_{rs})$ satisfies, by definition of $\otimes$,

$$c_{rs} = \frac{h((d_i, d_j))}{d_i d_j} \frac{h((\delta_k, \delta_l))}{\delta_k \delta_l} = \frac{h((d_i \delta_k, d_j \delta_l))}{d_i \delta_k d_j \delta_l}$$

due to the fact that h is multiplicative and $(n, m) = 1$.

Here $i = \left[\dfrac{r-1}{\tau(m)}\right] + 1$ and $k \equiv r \pmod{\tau(m)}$, $1 \leqslant k \leqslant \tau(m)$, and so clearly r fixes i and k and hence $d_i \delta_k$. Similarly, s fixes j and l and hence $d_j \delta_l$. Therefore, writing $\alpha_r = d_i \delta_k$ (and so $\alpha_s = d_j \delta_l$), we see that $c_{rs} = \dfrac{h((\alpha_r, \alpha_s))}{\alpha_r \alpha_s}$ where clearly, as with A and B, the sequence α_r, $1 \leqslant r \leqslant \tau(nm)$, runs through all $\tau(nm)$ divisors of nm in some fixed order (depending on the ordering of the d_i and δ_k). Observe also that, as with A and B, C is symmetric. Iterating this, it follows that the $(u, v)^{\text{th}}$ element of $M_N^{\otimes}$ is given by $\dfrac{h((\beta_u, \beta_v))}{\beta_u \beta_v}$ where the sequence β_u runs precisely through all the divisors of N in some fixed order.

Now suppose that in the *first* row of $M_N^{\otimes}$, a denominator $\beta_1 \beta_{v_0}$ (in the v_0^{th} column) satisfies $\beta_1 \beta_{v_0} < \beta_1 \beta_{v_1}$ for some $v_1 < v_0$. Then, clearly, $\beta_{v_0} < \beta_{v_1}$ and hence $\beta_u \beta_{v_0} < \beta_u \beta_{v_1}$ for all u i.e. for *all* rows. Therefore this property holds for the v_0^{th} and v_1^{th} column of $M_N^{\otimes}$. A suitable column exchange in $M_N^{\otimes}$ would therefore remove this particular problem. This means post-multiplying $M_N^{\otimes}$ by a non-singular elementary matrix P with det $P = -1$. Also since P corresponds to a column exchange, $P^2 = I$ or $P^{-1} = P$.

However, since $M_N^{\otimes}$ is symmetric, an analogous problem also occurs in the v_0^{th} and v_1^{th} row of $M_N^{\otimes}$ and this is similarly removed by pre-multiplying $M_N^{\otimes}$ by the same matrix P. After a certain number of such twin operations, $M_N^{\otimes}$ will have all the denominators of the elements in its rows in increasing order as well as the corresponding property for its columns i.e. $M_N^{\otimes}$ reduces to M_N. Hence

$$M_N = P_1 P_2 \cdots P_k M_N^{\otimes} P_k P_{k-1} \cdots P_1$$

with each P_i non-singular and satisfying $P_i^{-1} = P_i$.

Writing $P = P_k \cdots P_1$, we deduce that M_N is similar to $M_N^{\otimes}$ and the result follows from Lemma 1.

COROLLARY 2

$$D_N = \prod_{p^l \| N} (D_{p^l})^{\tau(N/p^l)}.$$

PROOF. – This follows immediately from Lemma 2 and the fact that the determinant of a matrix over $\boldsymbol{C}$ is the products of its eigenvalues.

We now calculate explicitly the eigenvalues and eigenvectors of M_N in the case $N = p$ and, in addition when $h(n) = n^2$, also for $N = p^2$.

LEMMA 3. – (i) *For each prime p and any real multiplicative function h, the two eigenvalues of M_p are, in the case $h(p) \neq 1$, given by*

$$\lambda_p^{(i)} = 1 + \frac{a_i(p)}{p}, \qquad i = 1, 2$$

with corresponding eigenvectors

$$c^{(i)} = (1, a_i(p))^T, \qquad i = 1, 2$$

where

$$a_1(p) = \frac{h(p) - p^2 + \sqrt{(p^2 - h(p))^2 + 4p^2}}{2p},$$

$$a_2(p) = \frac{h(p) - p^2 - \sqrt{(p^2 - h(p))^2 + 4p^2}}{2p}.$$

If $h(p) = 1$, M_p has eigenvalues 0 and $1 + 1/p^2$ with corresponding eigenvectors $(1, -p)^T$ and $(1, 1/p)^T$ respectively.

(ii) *For each prime p and the multiplicative function $h(n) = n^2$, the three eigenvalues of M_{p^2} are given by*

$$\lambda^{(1)} = 1 - \frac{1}{p^2}, \qquad \lambda^{(2)} = 1 + \frac{\alpha_p}{p} + \frac{1}{p^2}, \qquad \lambda^{(3)} = 1 + \frac{\beta_p}{p} + \frac{1}{p^2}$$

with corresponding eigenvectors

$$\boldsymbol{c}^{(1)} = (1,\, 0,\, -1)^T, \qquad \boldsymbol{c}^{(2)} = (1,\, \alpha_p,\, 1)^T, \qquad \boldsymbol{c}^{(3)} = (1,\, \beta_p,\, 1)^T$$

where

$$\alpha_p = \frac{-1 - \sqrt{1 + 8p^2}}{2p} \qquad \text{and} \qquad \beta_p = \frac{-1 + \sqrt{1 + 8p^2}}{2p}$$

REMARK. – Since, for *any* multiplicative function $h(n)$, $\det(M_{p^2} - tI_3) = 0$ is a cubic in t, all eigenvalues of M_{p^2} can always, in principle, be calculated and indeed this can also be done for M_{p^3}. The details, however, become excessively tedious. We have merely included (ii) above as an example of such a calculation.

PROOF. – (i) M_p has characteristic equation

$$p^2 t^2 - t(h(p) + p^2) + h(p) - 1 = 0\,.$$

If $h(p) = 1$, the eigenvalues are clearly 0 and $1 + 1/p^2$ and a simple computation confirms the corresponding eigenvectors as announced.

If $h(p) \neq 1$, the eigenvalues obtained by solving the quadratic characteristic equation and their corresponding eigenvectors are easily computed to be as announced above.

(ii) A typical eigenvalue λ and its corresponding eigenvectors $(1,\, x,\, y)^T$ are obtained by solving the equations

$$\begin{aligned} 1 + \frac{x}{p} + \frac{y}{p^2} &= \lambda \\ 1 + px + y &= \lambda p x \\ 1 + px + p^2 y &= \lambda p^2 y\,. \end{aligned}$$

Eliminating λ from the first two equations and also from the first and third yields

$$px\left(1+\frac{x}{p}+\frac{y}{p^2}\right) = 1+px+y$$

$$p^2y\left(1+\frac{x}{p}+\frac{y}{p^2}\right) = 1+px+p^2y\,.$$

An easy exercise now shows that either $y=1$ and $px^2+x-2p=0$ or $x=0$ and $y=-1$ and this yields the announced results.

Proofs of Theorems

PROOF OF THEOREM 1. – The fact that

$$D_{p^l}=\frac{1}{p^{l(l+1)}}\prod_{i=1}^{l}\left(h(p^i)-h(p^{i-1})\right)$$

is easily confirmed by reducing the determinant to upper triangular form. This is achieved by multiplying the i^{th} row by $1/p$ and subtracting it from the $(i+1)^{\text{th}}$ row, starting with $i=l$ and ending with $i=1$. Our determinant identity (1) then follows immediately from Corollary 2.

PROOF OF THEOREM 2. – Let n be square-free. The minimal value of the quadratic form $\sum\limits_{i|n,\, j|n} x_i x_j \frac{h((i,\, j))}{ij}$ on the unit sphere $\sum\limits_{i|n} x_i^2=1$ is given by the smallest eigenvalue of M_n. Since the above quadratic form can be rewritten as

$$\sum_{i|n} g(i)\left(\sum_{j|n,\, i|j}\frac{x_j}{j}\right)^2,$$

where $h(n)=\sum\limits_{d|n} g(d)$, the imposition of the condition $g(i)\geqslant 0$ for all $i|n$ ensures that the quadratic form only takes non-negative values and hence the smallest eigenvalue (and therefore all eigenvalues) of M_n are also non-negative. The condition $g(i)\geqslant 0\ \forall i|n$ is equivalent, by Möbius inversion, to the condition $h(p)\geqslant 1\ \forall p|n$.

If $f(p)=0$ for any $p|n$ then $f(n)=0$. Choosing $h(n)=1$ for all n, the eigenvalues of M_p are easily calculated to be 0 and $1+\frac{1}{p^2}$ and hence by Lemma 2 and the above discussion, M_n has a zero eigenvalue and all others strictly positive. Thus the smallest eigenvalue of M_n is $f(n)$ in this case.

If, on the other hand, $f(p) > 0$ for all $p|n$, we define h multiplicative with

$$h(p) = \frac{1}{1 - f(p)} + p^2 f(p)$$

and $h(p^i)$, $i \geqslant 2$, arbitrary with $h(p^i) \neq h(p^{i-1})$ for $i \geqslant 2$. This choice of h ensures that M_p is nonsingular and hence, by our earlier discussion, all the eigenvalues of M_p are strictly positive. Define for each $p|n$, $k = k(p) = 2/(p(1 - f(p)))$. A simple computation shows that

$$h(p) = p^2 + \frac{(k^2 - 4)\, p}{2k} \quad \text{and that} \quad h(p) - p^2 + \sqrt{(p^2 - h(p))^2 + 4p^2} = kp\,.$$

Since $1 > f(p) > 0$, we have that $k > 2/p$ and Lemma 3(i) implies that the eigenvalues of M_p are $1 + k/(2p)$ and $1 - 2/(kp)$. Thus by Lemma 2, the smallest eigenvalue of M_n is the product of the smallest eigenvalues of the various M_p i.e. will equal

$$\prod_{p|n} \left(1 - \frac{2}{kp}\right) = \prod_{p|n} f(p) = f(n)\,.$$

Thus in both cases the minimal value is $f(n)$ as required.

Note: The condition $f(p) < 1$ in Theorem 2 cannot be relaxed to $f(p) \leqslant 1$. In fact, if $n = p$, the quadratic form $\sum_{i|n,\, j|n} x_i x_j \frac{h((i,\, j))}{ij}$ reduces to $x_1^2 + 2x_1 x_p/p + h(p) x_p^2/p^2$ and this can indeed take values strictly less than 1 on the sphere $x_1^2 + x_p^2 = 1$. For example, if $h(p) > 0$, take $x_1 = \cos\theta$, $x_p = -\sin\theta$ with θ sufficiently small and positive and, if $h(p) \leqslant 0$, we may take the same values of x_1 and x_p but instead for any $\theta \in (0, \pi/2)$.

Proof of Theorem 3. – As mentioned earlier, we can write Λ as follows.

$$\Lambda = \sum_{i|N,\, j|N} x_i x_j \frac{h((i,\, j))}{ij} = \sum_{i|N,\, j|N} \frac{x_i x_j}{ij} \sum_{d|(i,\, j)} g(d) =$$

$$= \sum_{d|N} g(d) \left(\sum_{\substack{n|N \\ d|n}} \frac{x_n}{n} \right)^2$$

where $h(n) = \sum_{d|n} g(d)$. Thus the problem of minimizing Λ subject to $\sum_{d|N} x_d^2 = 1$ reduces to determining the smallest eigenvalue of M_N. Since $g(n) > 0$, Λ is a positive definite quadratic form and hence all the eigenvalues of M_N are positive. We also have that $h(p) = 1 + g(p) > 1$ and hence $D_p = (h(p) - 1)/p^2 > 0$.

Lemma 3 (i) implies that, for each $p|N$, the eigenvalues of M_p are positive and together with Lemma 2 implies that the required minimal value of Λ is the product of the smallest eigenvalues of each M_p i.e.

$$\prod_{p|N}\left(1-\frac{(p^2-h(p))+\sqrt{(p^2-h(p))^2+4p^2}}{2p^2}\right).$$

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