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Calculating a determinant associated with multiplicative functions


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Calculating a Determinant Associated with Multiplicative Functions.

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Sunto. – Sia h una funzione moltiplicativa a valori complessi. Per ogni $N \in \mathbb{N}$, calcoliamo il determinante $D_N := \det_{i \mid N, j \mid N} \left( \frac{h(i, j)}{ij} \right)$, dove $(i, j)$ indica il massimo comune divisore di $i$ e $j$, che figurano in ordine crescente in righe e colonne. Precisamente dimostriamo che

$$D_N = \prod_{p \mid N} \left( \frac{1}{p^{k(i+1)}} \prod_{i=1}^{l} (h(p^i) - h(p^{i-1})) \right)^{\nu(N/p^i)}.$$ 

Dunque $D_N^{1/h(N)}$ è effettivamente una funzione moltiplicativa di $N$. L’apparato algebrico associato a questo risultato ci consente di dimostrarne altri due. Il primo è la caratterizzazione delle funzioni reali moltiplicative $f(n)$, con $0 \leq f(p) < 1$, come valori minimi di certe forme quadratiche sulla sfera unità $\tau(N)$ dimensionale. Il secondo è la determinazione esplicita dei valori minimi di certe altre forme quadratiche su detta sfera.

Summary. – Let $h$ be a complex valued multiplicative function. For any $N \in \mathbb{N}$, we compute the value of the determinant $D_N := \det_{i \mid N, j \mid N} \left( \frac{h(i, j)}{ij} \right)$, where $(i, j)$ denotes the greatest common divisor of $i$ and $j$, which appear in increasing order in rows and columns. Precisely we prove that

$$D_N = \prod_{p \mid N} \left( \frac{1}{p^{k(i+1)}} \prod_{i=1}^{l} (h(p^i) - h(p^{i-1})) \right)^{\nu(N/p^i)}.$$ 

This means that $D_N^{1/h(N)}$ is a multiplicative function of $N$. The algebraic apparatus associated with this result allows us to prove the following two results. The first one is the characterization of real multiplicative functions $f(n)$, with $0 \leq f(p) < 1$, as minimal values of certain quadratic forms on the $\tau(N)$ unit sphere. The second one is the explicit evaluation of the minimal values of certain others quadratic forms also on the unit sphere.
1. – Introduction.

Let \( h(n) \) be a complex-valued multiplicative function i.e. \( h(1) = 1 \) and for \( (m, n) = 1, m, n \in \mathbb{N} \), we have that \( h(mn) = h(m)h(n) \). Here \( (m, n) \) denotes the greatest common divisor of \( m \) and \( n \).

For any \( N \in \mathbb{N} \), denote by \( d_1, \ldots, d_{\tau(N)} \) the positive divisors of \( N \) in increasing order. We define the \( \tau(N) \times \tau(N) \) matrix \( M_N \) to be that with \( (i, j)^{th} \) element \( h((d_i, d_j))/d_id_j \). This we abbreviate to

\[
M_N = \left( \frac{h((i, j))}{ij} \right)_{i|N, j|N}
\]

and we write

\[
D_N = \det M_N.
\]

This matrix can arise in several natural ways some of which we shall now describe.

Let

\[
\Delta(x, N) = \sum_{n \leq xN} 1 - x\varphi(N)
\]

where \( \varphi \) denotes Euler’s function. This expression measures, via

\[
\Delta(N) = \sup_{x \in \mathbb{R}} |\Delta(x, N)|,
\]

the maximal deviation of the number of integers \( \leq xN \) which are coprime to \( N \) from the expected quantity.

Various results on \( \Delta(N) \) may be found in [2]. By writing \( \Delta(x, N) \) in the form

\[
\sum_{n \leq xN} \sum_{d|(n, N)} \mu(d) - x \sum_{n \leq xN} \sum_{d|(n, N)} \mu(d)
\]

and reversing summations, it readily follows that for square-free \( N \)

\[
\Delta(x, N) = -\mu(N) \sum_{d|N} \mu(d) \{xd\}
\]

where \( \mu \) is the Möbius function and \( \{t\} \) denotes the fractional part of \( t \). A simple computation then yields that

\[
\int_0^1 \Delta^2(x, N) \, dx = \frac{1}{12} \sum_{i|N, j|N} \mu(i) \mu(j) \frac{(i, j)^2}{ij}.
\]

This sum can be evaluated to be \( 2^{\omega(N)} \varphi(N)/N \), where \( \omega(N) \) is the number of distinct prime factors of \( N \) and \( \varphi \) is Euler’s function, and, as shown by Perelli-
Zannier [5], is $2^{v(N)}$ multiplied by the minimal value of the quadratic form
\[
\sum_{i|N, j|N} x_i x_j \frac{(i, j)^2}{ij}
\]
subject to the constraint $\sum_{i|N} x_i^2 = 1$. Observe that the associated matrix is of the form $M_N$ with $h(n) = n^2$.

This type of discussion can be presented in a more general context. For any fixed $N \in \mathbb{N}$, define a function $f$ by
\[
f(n) = f_N(n) = \sum_{d|(n, N)} \theta_d.
\]

Since $f$ is periodic with period $N$, denoting by $m(f)$ its mean value, it is easily seen that
\[
\sum_{n \leq x} f(n) = m(f) x - \sum_{d|N} \theta_d \left\{ \frac{x}{d} \right\}
\]
and hence
\[
\frac{1}{N} \int_0^N \left( \sum_{n \leq x} f(n) - m(f) x \right)^2 \, dx = \frac{1}{4} \left( \sum_{d|N} \theta_d \right)^2 + \frac{1}{12} \sum_{i|N, j|N} \alpha_i \alpha_j \frac{(i, j)^2}{ij}
\]
where $\alpha_i = \theta_{N/i}$. The matrix $M_N$ associated with $h(n) = n^2$ is once more present in the second term on the right.

Further, since $f^2$ is also periodic of period $N$, its mean value $m(f^2)$ can be expressed as
\[
m(f^2) := \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f^2(n) = \sum_{i|N, j|N} \theta_i \theta_j \frac{(i, j)^2}{ij}
\]
and hence we have here an occurrence of $M_N$ with $h(n) = n$.

Problems concerned with determining minimal values of certain quadratic forms subject to constraints on the variables also give rise to matrices of the form $M_N$. For example, let $g(n)$ be any positive real multiplicative function and define the multiplicative function $h$ by $h(n) = \sum_{d|n} g(d)$. Consider the quadratic form $A$ defined by
\[
A = \sum_{i|N, j|N} x_i x_j \frac{h((i, j))}{ij}
\]
or equivalently, in terms of $g$,
\[
A = \sum_{d|N} g(d) \left( \sum_{u|N \atop d|u} x_u^2 \right)^2,
\]
which transparently shows that $A$ is positive definite. To determine the mini-
mal value of $A$ on the $\tau(N)$-dimensional unit sphere $\sum_{d | N} x_d^2 = 1$ reduces to calculating the smallest eigenvalue of $M_N$. Such problems are often difficult to resolve completely since the eigenvalues can be zeros of polynomials of high degree. In our Theorem 3, we determine the required minimal value of $A$ for square-free $N$. This includes the special case of $h(n) = n^2$ corresponding to $g(n) = n^2 \prod_{p | n} \left(1 - \frac{1}{p^2}\right)$ which yields the minimal value $\varphi(N)/N$ for the Perelli-Zannier quadratic form cited above. In Theorem 2, we show that this property of the function $\varphi(n)/n$ as a minimal value of a constrained quadratic form is, in fact, one shared by any real multiplicative function $f(n)$ with $0 \leq f(p) < 1$.

Our first result is a complete evaluation of $\det M_N$ for any multiplicative function $h$.

**Theorem 1.** Let $h$ be a complex-valued multiplicative function. Then

$$D_N := \det_{i \mid N, j \mid N} \left( \frac{h((i, j))}{ij} \right) = \prod_{p \mid N} \left( \prod_{i=1}^{\varphi(p)} \frac{1}{p^{i+1}} \prod_{i=1}^{1} (h(p^i) - h(p^{i-1})) \right)^{\tau(p^i)}$$

for any $N \in \mathbb{N}$.

The interest here is in the fact that this can be achieved without calculating the individual eigenvalues of the matrix $M_N$. We also prove the following curious result regarding the characterization of certain multiplicative functions as minimal values of certain quadratic forms on the unit sphere.

**Theorem 2.** Any real-valued multiplicative function $f(n)$ with $0 \leq f(p) < 1$ for prime $p$ can be expressed as the minimal value, for square-free $n$, of a quadratic form

$$\sum_{i | n, j | n} x_i x_j \frac{h((i, j))}{ij},$$

with $h$ multiplicative, on the $\tau(n)$-dimensional sphere $\sum_{i | n} x_i^2 = 1$.

For example, as may be noted from the proof, the ubiquitous $M_N$ with $h(n) = n^2$ is associated with $f(n) = \frac{\varphi(n)}{n}$. We also note at the end of the proof that the condition $f(p) < 1$ cannot be relaxed to $f(p) \leq 1$.

Finally, as discussed earlier, we prove the following result.

**Theorem 3.** Let $N$ be a square-free integer. For any positive real multiplicative function $g(n)$, write $h(n) = \sum_{d | n} g(d)$. Then the minimal value of the
quadratic form

\[ A = \sum_{i \neq j, \in \mathbb{N}} x_i x_j \frac{h(i, j)}{ij} \]

subject to \( \sum_{\mathbb{N}} x_i^2 = 1 \) is given by

\[
\prod_{p \in \mathbb{N}} \left( 1 - \frac{(p^2 - h(p)) + \sqrt{(p^2 - h(p))^2 + 4p^2}}{2p^2} \right).
\]

It is obvious that substituting \( h(p) = p^2 \) in this expression immediately yields the minimal value \( q(N)/N \) as mentioned earlier.

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Preliminaries

We shall need three lemmas. Useful references for the facts we require on tensor products of matrices are \([1],[3],[4]\).

Lemma 1 is well known, Lemma 2 describes the eigenvalues of the matrix \( M_N \), whilst Lemma 3 is essentially a simple computation. We begin with the basic definitions.

Given an \( m \times n \) matrix \( A = (a_{ij}) \) and a \( p \times q \) matrix \( B = (b_{kl}) \), the tensor product \( A \otimes B \) is defined to be the \( mp \times nq \) matrix \( C = (c_{rs}) \) where \( c_{rs} = a_{ij} b_{kl} \) with \( r = p(i - 1) + k \) and \( s = q(j - 1) + l \). Equivalently, \( i = \left\lfloor \frac{r - 1}{p} \right\rfloor + 1 \), \( k \equiv r \pmod{p} \), \( j = \left\lfloor \frac{s - 1}{q} \right\rfloor + 1 \) and \( l \equiv s \pmod{q} \) where the square brackets indicate the integer part function and \( 1 \leq i \leq m \), \( 1 \leq j \leq n \), \( 1 \leq k \leq p \), \( 1 \leq l \leq q \).

It is a trivial but cumbersome exercise to show that tensor products are associative and that

\[(A \otimes B)(C \otimes D) = AC \otimes BD\]

for any matrices \( A, B, C, D \) whenever \( AC \) and \( BD \) are defined (see \([4]\), Theorems 8.8.3 and 8.8.6). In general, \( A \otimes B \neq B \otimes A \) so that the order in which the factors occur in a tensor product needs to be explicitly specified.

For any \( N \in \mathbb{N} \), \( N > 1 \), we shall denote by \( M_N^p \) the tensor product

\[ M_N^p = \otimes_{p \nmid N} M_p \]

where each factor \( M_p \) is as described in the introduction (with \( N = p^i \)) and occurs in the product in order of increasing \( p \).

All matrices considered in Lemmas 1 and 2 will be over \( \mathbb{C} \).
Lemma 1. – For any two square matrices $A$ and $B$, the eigenvalues of $A \otimes B$ are precisely all the products of eigenvalues of $A$ with eigenvalues of $B$ (both with multiplicity).

Proof. – See [4], Theorem 8.8.13.

Corollary 1. – The eigenvalues of $M_N \otimes$ are precisely all the products of eigenvalues of the individual factors $M_p$, taken one for each $p_i \parallel N$.

Proof. – Immediate.

Lemma 2. – The eigenvalues of $M_N$ are precisely all the products of eigenvalues of $M_p$, taken one for each $p_i \parallel N$.

Remark. – We shall prove that $MN$ and $M_N \otimes$ are similar matrices and the result then follows from Lemma 1.

Proof. – For any $(n, m) = 1$, let $A = \left( \frac{h((d_i, d_j))}{d_i d_j} \right)$, $1 \leq i, j \leq \tau(n)$, where $d_1, \ldots, d_{\tau(n)}$ are the $\tau(n)$ divisors of $n$, listed in some fixed order (not necessarily increasing) and, similarly, let $B = \left( \frac{h((\delta_k, \delta_l))}{\delta_k \delta_l} \right)$, $1 \leq k, l \leq \tau(m)$, where $\delta_1, \ldots, \delta_{\tau(m)}$ are the $\tau(m)$ divisors of $m$, also listed in some fixed order.

Then $A \otimes B = C = (c_{rs})$ satisfies, by definition of $\otimes$,

$$c_{rs} = \frac{h((d_i, d_j)) \cdot h((\delta_k, \delta_l))}{d_i d_j \cdot \delta_k \delta_l} = \frac{h((d_i \delta_k, d_j \delta_l))}{d_i \delta_k d_j \delta_l}$$

due to the fact that $h$ is multiplicative and $(n, m) = 1$.

Here $i = \left\lfloor \frac{r - 1}{\tau(m)} \right\rfloor + 1$ and $k \equiv r \pmod{\tau(m)}$, $1 \leq k \leq \tau(m)$, and so clearly $r$ fixes $i$ and $k$ and hence $d_i \delta_k$. Similarly, $s$ fixes $j$ and $l$ and hence $d_j \delta_l$. Therefore, writing $a_r = d_i \delta_k$ (and so $a_s = d_j \delta_l$), we see that $c_{rs} = \frac{h((a_r, a_s))}{a_r a_s}$ where clearly, as with $A$ and $B$, the sequence $a_r, 1 \leq r \leq \tau(nm)$, runs through all $\tau(nm)$ divisors of $nm$ in some fixed order (depending on the ordering of the $d_i$ and $\delta_k$). Observe also that, as with $A$ and $B$, $C$ is symmetric. Iterating this, it follows that the $(u, v)^{th}$ element of $M_N \otimes$ is given by $\frac{h((\beta_u, \beta_v))}{\beta_u \beta_v}$ where the sequence $\beta_u$ runs precisely through all the divisors of $N$ in some fixed order.

Now suppose that in the first row of $M_N \otimes$, a denominator $\beta_1 \beta_{v_0}$ (in the $v_0^{th}$ column) satisfies $\beta_1 \beta_{v_0} < \beta_1 \beta_{v_1}$ for some $v_1 < v_0$. Then, clearly, $\beta_{v_0} < \beta_{v_1}$ and hence $\beta_u \beta_{v_0} < \beta_u \beta_{v_1}$ for all $u$ i.e. for all rows. Therefore this property holds for the $v_0^{th}$ and $v_1^{th}$ column of $M_N \otimes$. A suitable column exchange in $M_N \otimes$ would
therefore remove this particular problem. This means post-multiplying $M_N$ by a non-singular elementary matrix $P$ with $\det P = -1$. Also since $P$ corresponds to a column exchange, $P^2 = I$ or $P^{-1} = P$.

However, since $M_N$ is symmetric, an analogous problem also occurs in the $v_0^{th}$ and $v_1^{th}$ row of $M_N$ and this is similarly removed by pre-multiplying $M_N$ by the same matrix $P$. After a certain number of such twin operations, $M_N$ will have all the denominators of the elements in its rows in increasing order as well as the corresponding property for its columns i.e. $M_N$ reduces to $M_N$. Hence

$$M_N = P_1 P_2 \cdots P_k M_N P_k P_{k-1} \cdots P_1$$

with each $P_i$ non-singular and satisfying $P_i^{-1} = P_i$.

Writing $P = P_k \cdots P_1$, we deduce that $M_N$ is similar to $M_N$ and the result follows from Lemma 1.

**Corollary 2**

$$D_N = \prod_{p' \mid N} (D_{p'})^{(N/p')}. $$

**Proof.** – This follows immediately from Lemma 2 and the fact that the determinant of a matrix over $C$ is the products of its eigenvalues.

We now calculate explicitly the eigenvalues and eigenvectors of $M_N$ in the case $N = p$ and, in addition when $h(n) = n^2$, also for $N = p^2$.

**Lemma 3.** – (i) For each prime $p$ and any real multiplicative function $h$, the two eigenvalues of $M_p$ are, in the case $h(p) \neq 1$, given by

$$\lambda_p^{(i)} = 1 + \frac{a_i(p)}{p}, \quad i = 1, 2$$

with corresponding eigenvectors

$$c^{(i)} = (1, a_i(p))^T, \quad i = 1, 2$$

where

$$a_1(p) = \frac{h(p) - p^2 + \sqrt{(p^2 - h(p))^2 + 4p^2}}{2p},$$

$$a_2(p) = \frac{h(p) - p^2 - \sqrt{(p^2 - h(p))^2 + 4p^2}}{2p}.$$ 

If $h(p) = 1$, $M_p$ has eigenvalues 0 and $1 + 1/p^2$ with corresponding eigenvectors $(1, -p)^T$ and $(1, 1/p)^T$ respectively.
(ii) For each prime $p$ and the multiplicative function $h(n) = n^2$, the three eigenvalues of $M_p^2$ are given by

$$\lambda^{(1)} = 1 - \frac{1}{p^2}, \quad \lambda^{(2)} = 1 + \frac{\alpha_p}{p} + \frac{1}{p^2}, \quad \lambda^{(3)} = 1 + \frac{\beta_p}{p} + \frac{1}{p^2}$$

with corresponding eigenvectors

$$\mathbf{c}^{(1)} = (1, 0, -1)^T, \quad \mathbf{c}^{(2)} = (1, \alpha_p, 1)^T, \quad \mathbf{c}^{(3)} = (1, \beta_p, 1)^T$$

where

$$\alpha_p = -1 - \sqrt{1 + 8p^2} \quad \text{and} \quad \beta_p = -1 + \sqrt{1 + 8p^2} \quad \frac{2p}{2p}$$

Remark. – Since, for any multiplicative function $h(n)$, $\det(M_p^3 - tI_3) = 0$ is a cubic in $t$, all eigenvalues of $M_p^3$ can always, in principle, be calculated and indeed this can also be done for $M_p^3$. The details, however, become excessively tedious. We have merely included (ii) above as an example of such a calculation.

Proof. – (i) $M_p$ has characteristic equation

$$p^2 t^2 - t(h(p) + p^2) + h(p) - 1 = 0.$$ 

If $h(p) = 1$, the eigenvalues are clearly 0 and $1 + 1/p^2$ and a simple computation confirms the corresponding eigenvectors as announced.

If $h(p) \neq 1$, the eigenvalues obtained by solving the quadratic characteristic equation and their corresponding eigenvectors are easily computed to be as announced above.

(ii) A typical eigenvalue $\lambda$ and its corresponding eigenvectors $(1, x, y)^T$ are obtained by solving the equations

$$1 + \frac{x}{p} + \frac{y}{p^2} = \lambda$$

$$1 + px + y = \lambda px$$

$$1 + px + p^2 y = \lambda p^2 y.$$
Eliminating \( \lambda \) from the first two equations and also from the first and third yields

\[
p_x \left( 1 + \frac{x}{p} + \frac{y}{p^2} \right) = 1 + px + y
\]
\[
p^2 y \left( 1 + \frac{x}{p} + \frac{y}{p^2} \right) = 1 + px + p^2 y.
\]

An easy exercise now shows that either \( y = 1 \) and \( px^2 + x - 2p = 0 \) or \( x = 0 \) and \( y = -1 \) and this yields the announced results.

**Proofs of Theorems**

**Proof of Theorem 1.** – The fact that

\[
D_p^l = \frac{1}{p^{(l+1)}} \prod_{i=1}^{l} (h(p^i) - h(p^{i-1}))
\]

is easily confirmed by reducing the determinant to upper triangular form. This is achieved by multiplying the \( i \)th row by \( 1/p \) and subtracting it from the \( (i+1) \)th row, starting with \( i = l \) and ending with \( i = 1 \). Our determinant identity (1) then follows immediately from Corollary 2.

**Proof of Theorem 2.** – Let \( n \) be square-free. The minimal value of the quadratic form \( \sum_{i|n, j|n} x_i x_j \frac{h(i,j)}{ij} \) on the unit sphere \( \sum_{i|n} x_i^2 = 1 \) is given by the smallest eigenvalue of \( M_n \). Since the above quadratic form can be rewritten as

\[
\sum_{i|n} g(i) \left( \sum_{j|n, i|j} x_j \right)^2,
\]

where \( h(n) = \sum_{d|n} g(d) \), the imposition of the condition \( g(i) \geq 0 \) for all \( i|n \) ensures that the quadratic form only takes non-negative values and hence the smallest eigenvalue (and therefore all eigenvalues) of \( M_n \) are also non-negative. The condition \( g(i) \geq 0 \) \( \forall i|n \) is equivalent, by Möbius inversion, to the condition \( h(p) \geq 1 \) \( \forall p|n \).

If \( f(p) = 0 \) for any \( p|n \) then \( f(n) = 0 \). Choosing \( h(n) = 1 \) for all \( n \), the eigenvalues of \( M_p \) are easily calculated to be \( 0 \) and \( 1 - \frac{1}{p^2} \) and hence by Lemma 2 and the above discussion, \( M_p \) has a zero eigenvalue and all others strictly positive. Thus the smallest eigenvalue of \( M_n \) is \( f(n) \) in this case.
If, on the other hand, $f(p) > 0$ for all $p|n$, we define $h$ multiplicative with

$$h(p) = \frac{1}{1 - f(p)} + p^2 f(p)$$

and $h(p^i), i \geq 2$, arbitrary with $h(p^i) \neq h(p^{i-1})$ for $i \geq 2$. This choice of $h$ ensures that $M_p$ is nonsingular and hence, by our earlier discussion, all the eigenvalues of $M_p$ are strictly positive. Define for each $p|n, k = h(p) = 2/(p(1 - f(p)))$. A simple computation shows that

$$h(p) = p^2 + \frac{(k^2 - 4) p}{2k}$$

and that $h(p) - p^2 + \sqrt{(p^2 - h(p))^2 + 4p^2} = kp$.

Since $1 > f(p) > 0$, we have that $k > 2/p$ and Lemma 3(i) implies that the eigenvalues of $M_p$ are $1 + k/(2p)$ and $1 - 2/(kp)$. Thus by Lemma 2, the smallest eigenvalue of $M_n$ is the product of the smallest eigenvalues of the various $M_p$ i.e. will equal

$$\prod_{p|n} \left(1 - \frac{2}{kp}\right) = \prod_{p|n} f(p) = f(n).$$

Thus in both cases the minimal value is $f(n)$ as required.

Note: The condition $f(p) < 1$ in Theorem 2 cannot be relaxed to $f(p) \leq 1$. In fact, if $n = p$, the quadratic form $\sum_{i|n, j|n} x_i x_j \frac{h((i, j))}{ij}$ reduces to $x_1^2 + 2x_1 x_p/p + h(p)x_p^2/p^2$ and this can indeed take values strictly less than 1 on the sphere $x_1^2 + x_p^2 = 1$. For example, if $h(p) > 0$, take $x_1 = \cos \theta, x_p = -\sin \theta$ with $\theta$ sufficiently small and positive and, if $h(p) \leq 0$, we may take the same values of $x_1$ and $x_p$ but instead for any $\theta \in (0, \pi/2)$.

Proof of Theorem 3. – As mentioned earlier, we can write $A$ as follows.

$$A = \sum_{i|N, j|N} x_i x_j \frac{h((i, j))}{ij} = \sum_{i|N, j|N} x_i x_j \sum_{d|N} g(d) =$$

$$= \sum_{d|N} g(d) \left( \sum_{n|N} \frac{x_n}{n} \right)^2$$

where $h(n) = \sum_{d|n} g(d)$. Thus the problem of minimizing $A$ subject to $\sum_{d|N} x_d^2 = 1$ reduces to determining the smallest eigenvalue of $M_N$. Since $g(n) > 0, A$ is a positive definite quadratic form and hence all the eigenvalues of $M_N$ are positive. We also have that $h(p) = 1 + g(p) > 1$ and hence $D_p = (h(p) - 1)/p^2 > 0$. 
Lemma 3 (i) implies that, for each $p | N$, the eigenvalues of $M_p$ are positive and together with Lemma 2 implies that the required minimal value of $A$ is the product of the smallest eigenvalues of each $M_p$ i.e.

$$\prod_{p | N} \left(1 - \frac{(p^2 - h(p)) + \sqrt{(p^2 - h(p))^2 + 4p^2}}{2p^2}\right).$$

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