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Classification of Initial Data for the Riccati Equation.

N. CHERNYAVSKAYA - L. SHUSTER

Sunto. – *Consideriamo un problema di Cauchy*

$$y'(x) + y^2(x) = q(x), \qquad y(x)\,|_{x=x_0} = y_0$$

dove $x_0, y_0 \in R$ *e* $q(x) \in L_1^{\mathrm{loc}}(R)$ *è una funzione non negativa che soddisfa la condizione:*

$$\int_{-\infty}^{x} q(t)\,dt > 0, \qquad \int_{x}^{\infty} q(t)\,dt > 0 \qquad \text{for } x \in R\,.$$

Otteniamo le condizioni nelle quali $y(x)$ *può essere continuata in tutto* R. *Questo dipende da* x_0, y_0 *e dalle proprietà di* $q(x)$.

Summary. – *We consider a Cauchy problem*

$$y'(x) + y^2(x) = q(x), \qquad y(x)\,|_{x=x_0} = y_0$$

where $x_0, y_0 \in R$ *e* $q(x) \in L_1^{\mathrm{loc}}(R)$ *is a non-negative function satisfying the condition:*

$$\int_{-\infty}^{x} q(t)\,dt > 0, \qquad \int_{x}^{\infty} q(t)\,dt > 0 \qquad \text{for } x \in R\,.$$

We obtain the conditions under which $y(x)$ *can be continued to all of* R. *This depends on* x_0, y_0 *and the properties of* $q(x)$.

1. – Introduction.

In this paper we study a Cauchy problem for a Riccati equation

$$y'(x) + y^2(x) = q(x) \tag{1.1}$$

$$y(x)\,|_{x=x_0} = y_0 \tag{1.2}$$

where $x_0, y_0 \in R$ and $q(x) \in L_1^{\mathrm{loc}}(R)$ is a non-negative function satisfying the condition

$$\int_{-\infty}^{x} q(t)\,dt > 0\,, \qquad \int_{x}^{\infty} q(t)\,dt > 0 \qquad \text{for } x \in R\,. \tag{1.3}$$

Throughout the sequel we assume requirement for $q(x)$ to be satisfied. The following assertions are well known (see the remark at the ends of § 2):

I) For any point (x_0, y_0) in the XOY-plane, the Cauchy problem (1.1)-(1.2) has a unique solution in some neighborhood of x_0.

II) The solution of (1.1)-(1.2) cannot be continued from the neighborhood where it exists to the whole axis R for all initial data $(x_0, y_0) \in XOY$.

Here the requirements of the initial data x_0, y_0 for which the solution of (1.1)-(1.2) can be continued to the whole number axis are unknown. Therefore, in many papers which are concerned with equation (1.1), the assertions are of a conditional nature. For example, in [1, Ch. I, § 8, § 11], the main results are given under the assumption that a solution of the Riccati equation exists on some segment $[a, b]$; it is emphasized that such an assumption is essential. In this paper, we study conditions under which a solution of (1.1)-(1.2) exists on the whole number axis.

To be more precise, our goal is as follows: For problem (1.1)-(1.2), to distinguish between Cauchy data for which the solution can be continued to R and those for which such a continuation is impossible. Note that such a classification may be useful as a priori information for solving (1.1)-(1.2) by numeric methods. Here the a priori nature of the information is guaranteed by the fact that the main results of the paper are formulated in terms of the initial data x_0, y_0 and some auxiliary functions in $q(x)$. See § 3 for a more detailed analysis of our statements.

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2. – Preliminaries.

In this section, we give some assertions which will be used in the proofs. Throughout the sequel we denote by c absolute positive constants which are not essential for exposition and may differ within a single chain of calculations.

THEOREM 2.1. – [2] *Consider an equation*

$$z''(x) = q(x)\, z(x), \qquad x \in R \tag{2.1}$$

Equation (2.1) *has a fundamental system of solutions* (FSS) $\{u(x), v(x)\}$ *such that*

$$\begin{gathered} v(x) > 0, \quad u(x) > 0, \quad v'(x) > 0, \quad u'(x) < 0, \quad x \in R, \\ v'(x)\, u(x) - u'(x) v(x) = 1, \\ \lim_{x \to -\infty} \frac{v(x)}{u(x)} = \lim_{x \to \infty} \frac{u(x)}{v(x)} = 0. \end{gathered} \tag{2.2}$$

A FSS of (2.1) with properties (2.2) is called a principal FSS (PFSS) because $v(x)$ and $u(x)$ are principal solutions of (2.1) on $(-\infty, 0)$ and $(0, \infty)$, respectively [9, Ch. 11, § 6].

THEOREM 2.2. – [7] *For* $x \in R$ *the* PFSS *of* (2.1) *admits a representation*

$$v(x) = \sqrt{\varrho(x)}\exp\left(\frac{1}{2}\int_{x_0}^{x}\frac{dt}{\varrho(t)}\right), \qquad u(x) = \sqrt{\varrho(x)}\exp\left(-\frac{1}{2}\int_{x_0}^{x}\frac{dt}{\varrho(t)}\right). \tag{2.3}$$

Here $\varrho(x) \overset{\text{def}}{=} u(x)\,v(x)$, x_0 *is the unique root of the equation* $u(x) = v(x)$.

In the above form, representation (2.3) was given in [2, 5]. Note the following inequality [5]:

$$|\varrho'(x)| < 1, \qquad x \in R. \tag{2.4}$$

For a fixed $x \in R$, consider an equation in $d \geq 0$:

$$2 = d\int_{x-d}^{x+d} q(\xi)\,d\xi. \tag{2.5}$$

For every $x \in R$, equation (2.5) has a unique positive continuous solution [2, 5]. Let $d(x)$ denote this solution, $q^*(x) \overset{\text{def}}{=} d^{-2}(x)$. The functions $d(x)$, $q^*(x)$ were introduced by M. Otelbaev [10]. Note that the function $q^*(x)$ is a Steklov-type averaging of $q(t)$ on the segment $[x - d(x), x + d(x)]$ ([11, 5]).

DEFINITION 2.1. – [3] We say that $q(x)$ belongs to the class $\mathcal{H}$ (and write $q(x) \in \mathcal{H}$) if there exists a continuous function $k(x) \geq 2$, $x \in R$ such that $k(x) \to \infty$ as $|x| \to \infty$, and for $x \in R$, the following conditions hold:

$$1)\quad c^{-1}k(x) \leq k(t) \leq ck(x) \qquad \text{for } |t - x| \leq k(x)\,q^*(x)^{-1/2} \tag{2.6}$$

$$2)\quad \Phi(x) \overset{\text{def}}{=} \frac{k(x)}{\sqrt{q^*(x)}}\sup_{|z| \leq k(x)\,q^*(x)^{-1/2}}\left|\int_0^z [q(x+t) - q(x-t)]\,dt\right| \leq c. \tag{2.7}$$

THEOREM 2.3. – [3] *If* $q(x) \in \mathcal{H}$, *one has*

$$|\varrho'(x)| \leq ck(x)^{-1/2}, \qquad x \in R \tag{2.8}$$

$$\varrho(x) = \frac{1 + \varepsilon(x)}{2\sqrt{q^*(x)}}, \qquad |\varepsilon(x)| \leq \frac{c}{\sqrt{k(x)}}, \qquad |x| \gg 1. \tag{2.9}$$

REMARK. – One can obtain stronger estimates for $|\varrho'(x)|$ and $|\varepsilon(x)|$ [3].

THEOREM 2.4. – [3] *Let* $q(x) = q_1(x) + q_2(x)$ *where* $q_1(x)$ *is positive and continuous for* $x \in R$, $q_2(x) \in L_1^{\mathrm{loc}}(R)$, $A(x) = [0, 2q_1(x)^{-1/2}]$ *and*

$$h_1(x) = \frac{1}{\sqrt{q_1(x)}} \sup_{t \in A(x)} \left| \int_0^t [q_1(x+s) - 2q_1(x) + q_1(x-s)]\, ds \right|, \qquad x \in R \tag{2.10}$$

$$h_2(x) = \frac{1}{\sqrt{q_1(x)}} \sup_{t \in A(x)} \left| \int_{x-t}^{x+t} q_2(s)\, ds \right|, \qquad x \in R . \tag{2.11}$$

If $h_1(x) \to 0$, $h_2(x) \to 0$ *as* $|x| \to \infty$, *then* (*see* (2.5))

$$d(x) = \frac{1+\delta(x)}{\sqrt{q_1(x)}}, \qquad |\delta(x)| \leq c(h_1(x) + h_2(x)), \qquad |x| \gg 1 \tag{2.12}$$

$$c^{-1} q_1(x)^{-1/2} \leq d(x) \leq c q_1(x)^{-1/2}, \qquad x \in R . \tag{2.13}$$

REMARK. – Theorems 2.3 and 2.4 were obtained in [3] under the assumption

$$1 \leq q(x) \in L_1^{\mathrm{loc}}(R). \tag{2.14}$$

By minor modification of the proofs, we can keep their statement with condition (2.14) replaced by the requirement

$$q_0^* \overset{\mathrm{def}}{=} \inf_{x \in R} q^*(x) > 0 . \tag{2.15}$$

THEOREM 2.5. – [8, Ch. III, § 40]. *The general solution of equation* (1.1) *is of the following form:*

$$y(x) = \frac{\theta v'(x) + u'(x)}{\theta v(x) + u(x)} . \tag{2.16}$$

Here $\{u(x), v(x)\}$ *is a PFSS of* (2.1), θ *is an arbitrary constant.*

REMARK. – Assertioins 1)-2) follow easily from Theorems 2.5 and 2.1.

3. – Statement of results, analysis and examples.

In this section we present the results of the paper. Their proofs are given in § 4.

LEMMA 3.1. – *For a fixed* $x \in R$ *consider an equation in* $d \geqslant 0$

$$(3.1) \qquad 1 = \int_0^{\sqrt{2}d} \int_{x-t}^{x} q(\xi)\, d\xi\, dt\,, \quad 1 = \int_0^{\sqrt{2}d} \int_{x}^{x+t} q(\xi)\, d\xi\, dt\,, \quad 2 = \int_0^{\sqrt{2}d} \int_{x-t}^{x+t} q(\xi)\, d\xi\, dt\,.$$

Each of equations (3.1) *has a unique positive solution.*

Denote by $d_1(x)$, $d_2(x)$, $\widehat{d}(x)$ the solutions of (3.1). These functions were introduced in [4] and were used in [3, 6].

THEOREM 3.1. – *For* $x \in R$ *one has*

$$(3.2) \qquad \frac{1}{\sqrt{2}}\frac{1}{d_1(x)} < \frac{v'(x)}{v(x)} < \frac{\sqrt{2}}{d_1(x)}\,, \qquad \frac{1}{\sqrt{2}}\frac{1}{d_2(x)} < \frac{|u'(x)|}{u(x)} < \frac{\sqrt{2}}{d_2(x)}$$

$$(3.3) \qquad \frac{1}{\sqrt{2}}\frac{d_1(x)\, d_2(x)}{d_1(x) + d_2(x)} < \varrho(x) < \sqrt{2}\,\frac{d_1(x)\, d_2(x)}{d_1(x) + d_2(x)}$$

$$(3.4) \qquad \frac{\widehat{d}(x)}{2\sqrt{2}} < \varrho(x) < \sqrt{2}\,\widehat{d}(x)\,.$$

REMARK. – Under condition (2.14), estimates (3.2)-(3.3) were obtained in [5], and (3.4) was obtained in [6]. In the present paper, (3.2)-(3.4) are proved under minimal requirements to $q(x)$.

LEMMA 3.2. – *For* $x \in R$ *one has* (*see* (2.5) *and* (3.1)

$$(3.5) \qquad \frac{\widehat{d}(x)}{\sqrt{2}} \leqslant d(x) \leqslant \sqrt{2}\,\widehat{d}(x)\,.$$

REMARK. – Estimates (3.5) show that inequality (3.4) and formula (2.9) agree. Moreover, one can show [6, preprint] that under the hypotheses of Theorem 2.4 the functions $d(x)$ and $\widehat{d}(x)$ are asymptotically equivalent.

Let T and P be the following subsets of the plane XOY:

$$(3.6) \qquad T = \{(x, y) : y d_1(x) \geqslant \sqrt{2}\} \cup \{(x, y) : y d_2(x) \leqslant -\sqrt{2}\}$$

$$(3.7) \qquad P = \left\{(x, y) : y d_1(x) \leqslant \frac{1}{\sqrt{2}}\right\} \cap \left\{(x, y) : y d_2(x) \geqslant -\frac{1}{\sqrt{2}}\right\}.$$

THEOREM 3.2. – *Let* $y(x)$ *be the solution of* (1.1)-(1.2). *If* $(x_0, y_0) \in P$, *one can continue* $y(x)$ *to* R; *if* $(x_0, y_0) \in T$, *one cannot continue* $y(x)$ *to* R.

COROLLARY 3.2.1. – *Suppose that the solution of* (1.1)-(1.2) *can be continued to* R, *and let* $\tilde{y}(x)$ *denote this continuation. Then*

$$-2\sqrt{2} \leqslant \tilde{y}(x)\, \widehat{d}(x) \leqslant 2\sqrt{2}, \qquad x \in R\,. \tag{3.8}$$

Under additional requirements to $q(x)$, one can sharpen Theorem 3.2 for $|x| \gg 1$. Let $\varepsilon \in (0, 1]$, $a > 0$ be given. We introduce the following sets (see (2.5)):

$$T(\varepsilon, a) = \{(x, y) : yd(x) \geqslant 1 + \varepsilon,\ x \geqslant a\} \cup \{(x, y) : yd(x) \leqslant -(1+\varepsilon),\ x \leqslant -a\} \tag{3.9}$$

$$P(\varepsilon, a) = \{(x, y) :\ -1 + \varepsilon \leqslant yd(x) \leqslant 1 - \varepsilon,\ |x| \geqslant a\}\,. \tag{3.10}$$

THEOREM 3.3. – *Let* $q(x) \in \mathcal{H}$, $q_0^* > 0$ *(see* (2.15)*), and let* $y(x)$ *be the solution of* (1.1)-(1.2). *Then for any* $\varepsilon \in (0, 1]$ *there is* $a = a(\varepsilon) \gg 1$ *such that if* $(x_0, y_0) \in T(\varepsilon, a)$, $y(x)$ *cannot be continued to* R, *and if* $(x_0, y_0) \in P$, $y(x)$ *can be continued to* R. *Moreover, any solutions* $y_-(x)$, $y_+(x)$ *of* (1.1) *defined on* $(-\infty, a]$, $[a, \infty)$, *respectively, satisfy*

$$\lim_{x \to -\infty} y_-(x)\, d(x) = -1\,, \qquad \lim_{x \to \infty} y_+(x)\, d(x) = 1\,. \tag{3.11}$$

Analysis of results.

We emphasize that the classification of initial data (x_0, y_0) given in Theorem 3.2 applies to all equations (1.1) with non-negative coefficients $q(x) \in L_1^{\text{loc}}(R)$ satisfying condition (1.3). We impose no restriction to $q(x)$ such as smoothness, oscillation, etc., and in this sense the requirements to $q(x)$ in Theorem 3.2 are minimal. Moreover, the classification of initial data (x_0, y_0) is asymptotically exact (as $|x| \gg 1$) in the class of equations (1.1) with $q(x) \in \mathcal{H}$. The class $\mathcal{H}$ is large enough since it contains not only «ordinary» functions $q(x)$ but also non-differentiable, rapidly increasing and rapidly oscillating functions (see [3]). Note that it is usually difficult to study equations (1.1) and (2.1) with such coefficients. We bring reader's attention to the fact that the classification of initial data given in Theorem 3.2 is not full since it does not include the points $(x_0, y_0) \in S = XOY \backslash (PUT)$. Such points are defined by one of the following inequalities (see (3.6)-(3.7)):

$$\frac{1}{\sqrt{2}}\frac{1}{d_1(x_0)} < y_0 < \frac{\sqrt{2}}{d_1(x_0)}\,, \qquad -\frac{\sqrt{2}}{d_2(x_0)} < y_0 < -\frac{1}{\sqrt{2}d_2(x_0)}\,. \tag{3.12}$$

This gap in the classification is explained by the fact that Theorem 3.2 follows from Theorems 3.1 and 2.5, and the following main lemma.

LEMMA 3.3. – *Let* $y(x)$ *be the solution of* (1.1)-(1.2). *One can continue* $y(x)$ *to* R *if and only if*

$$\frac{u'(x_0)}{u(x_0)} \leqslant y_0 \leqslant \frac{v'(x_0)}{v(x_0)}. \tag{3.13}$$

Thus points (3.12) arise from the following reason: instead of the exact values of logarithmic derivatives $\frac{u'(x)}{u(x)}$, $\frac{v'(x)}{v(x)}$ we use their estimates (3.2). On the other hand, by Liouville's well-known theorem, it is impossible to give exact formulas for $\frac{v'(x)}{v(x)}$, $\frac{u'(x)}{u(x)}$ in the case of general equation (1.1). Taking this into account, one can pose an interesting problem of constructing an effective numerical method for computing $\frac{v'(x)}{v(x)}$, $\frac{u'(x)}{u(x)}$ in an arbitrary point $x_0 \in R$ with prescribed error. We consider this problem in a forthcoming paper. Finally, note that it is usually impossible to find exact values of $d_1(x)$ and $d_2(x)$ in an analytic form. However, for a given x_0 one can find the values $d_1(x_0)$, $d_2(x_0)$, $d(x_0)$ with prescribed error by using standard numerical methods since the functions

$$\begin{cases} F_1(d) = \int\limits_0^{\sqrt{2}d} \int\limits_{x_0-t}^{x_0} q(\xi)\, d\xi\, dt, \quad F_2(d) = \int\limits_0^{\sqrt{2}d} \int\limits_{x_0}^{x_0+t} q(\xi)\, d\xi\, dt, \\ F_3(d) = \int\limits_0^{\sqrt{2}d} \int\limits_{x_0-t}^{x_0+t} q(\xi)\, d\xi\, dt. \end{cases} \tag{3.14}$$

are non-negative, non-decreasing for $d \in [0, \infty)$ and $F_i(0) = 0$, $F_i(\infty) = \infty$, $i = \overline{1, 3}$. Therefore, the root of equation (3.1) can be localized in intervals with arbitrarily close ends. See Example 1 for a realization of this scheme in its simplest form (without computing all integrals).

EXAMPLE 1. – Consider a Cauchy problem

$$y'(x) + y^2(x) = (1 + x^2) + (1 + x^2)\cos\left(x + \frac{x^3}{3}\right) \tag{3.15}$$

$$y(0) = y_0. \tag{3.16}$$

To apply Theorem 3.2, we need two-sided estimates for $d_1(0)$ and $d_2(0)$. From (3.14) for $x = 0$ we obtain

$$F_2(d) = \int\limits_0^{\sqrt{2}d} \int\limits_0^t (1+\xi^2)\, d\xi\, dt + \int\limits_0^{\sqrt{2}d} \int\limits_0^t (1+\xi^2)\cos\left(\xi + \frac{\xi^3}{3}\right) d\xi\, dt$$

$$= d^2 + \frac{d^4}{3} + \int\limits_0^{\sqrt{2}d} \sin\left(t + \frac{t^3}{3}\right) dt \leqslant d^2 + \frac{d^4}{3} + \sqrt{2}d \overset{\text{def}}{=} f^{(+)}(d).$$

Since $f^{(+)}\left(\frac{1}{2}\right) < 1$, one has $F_2\left(\frac{1}{2}\right) \leqslant f^{(+)}\left(\frac{1}{2}\right) < 1 \Rightarrow d_2(0) > \frac{1}{2}$. Similarly,

$$F_2(d) = d^2 + \frac{d^4}{3} + \int_0^{\sqrt{2}d} \sin\left(t + \frac{t^3}{3}\right) dt \geqslant d^2 + \frac{d^4}{3} - \sqrt{2}d \overset{\text{def}}{=} f^{(-)}(d).$$

Since $F_2\left(\frac{27}{20}\right) \geqslant f^{(-)}\left(\frac{27}{20}\right) > 1$, one has $d_2(0) < \frac{27}{20}$. Since in this case $F_1(d) = F_2(d)$ for $d \geqslant 0$, we have $d_1(0) = d_2(0)$, and hence

$$2^{-1} < d_2(0), \qquad d_1(0) < 27\cdot 20^{-1}. \tag{3.17}$$

From (3.17) and Theorem 3.2, it follows that if $y_0 \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, one can continue the solution of (3.15)-(3.16), and if $y_0 \geqslant 3$ or $y_0 \leqslant -3$, one cannot continue it. In the cases $y_0 \in \left(-3, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 3\right)$, Theorem 3.2 gives not answer to the question on the existence of continuation of the solution of (3.15)-(3.16) to R. (Here we, in fact, slightly weakened the result in order to obtain «round» numbers.)

EXAMPLE 2. – Below we present a possible approach to the problem of finding asymptotics of the solution of (1.1)-(1.2) continued to R, and to applications of Theorem 3.3. To express the asymptotics in terms of $q(x)$, we assume below, in addition to the requirements from § 1, that the following conditions hold:

$$q(x) > 0 \quad \text{for} \quad x \in R, \qquad q(x) \to \infty \ \ as \ \ |x| \to \infty \tag{3.18}$$

$$|q(s) - q(x)| \leqslant c q(x)^\alpha |s - x|^\beta \quad \text{for} \quad |s - x| \leqslant \frac{k(x)}{\sqrt{q(x)}}, \qquad x \in R. \tag{3.19}$$

Here $\alpha > 0, \beta > 0, 2^{-1}\beta + 1 - \alpha \overset{\text{def}}{=} \delta > 0, k(x) \leqslant q(x)^\gamma, \gamma = \frac{\delta}{2(\beta+2)}$, $k(x) \geqslant 2$ for $x \in R$, $k(x)$ is a continuous function and $k(x) \to \infty$ as $|x| \to \infty$. To apply Theorem 3.3, we verify that $q(x) \in \mathcal{H}$. Let us find $q^*(x)$ for $|x| \gg 1$. According to Theorem 2.4, one has $q_1(x) := q(x)$, $q_2(x) := 0$. Then

$$h_1(x) = \frac{1}{\sqrt{q(x)}} \sup_{t \in A(x)} \left| \int_0^t [(q(x+s) - q(x)) + (q(x-s) - q(x))]\, ds \right|$$

$$\leqslant \frac{2cq(x)^\alpha}{\sqrt{q(x)}} \sup_{t \in A(x)} \left| \int_0^t s^\beta ds \right| = \frac{c}{q(x)^\delta} \to 0 \quad \text{for} \ \ |x| \to \infty \ \Rightarrow$$

$$q^*(x)^{-1/2} = d(x) = \frac{1 + \alpha(x)}{\sqrt{q(x)}}, \qquad |\alpha(x)| \leqslant c q(x)^{-\delta}, \ |x| \gg 1. \tag{3.20}$$

Let us check (2.6). From (3.18)-(3.19) for $|x| \gg 1$ and $s \in \left[x - \frac{k(x)}{\sqrt{q(x)}}, x + \frac{k(x)}{\sqrt{q(x)}}\right]$, it follows that

$$\frac{q(s)}{q(x)} \gtrless 1 \pm cq(x)^{\alpha-1}\left(\frac{k(x)}{\sqrt{q(x)}}\right)^{\beta} \gtrless 1 \pm \frac{c}{q(x)^{\delta/2}} \leqslant 1 \pm c^{\pm 1}. \tag{3.21}$$

Since $k(x)$, $q(x)$ are continuous, (3.21) remains true for all $x \in R$ (possibly, with a bigger constant c). Using (3.20) for $|x| \gg 1$ we check (2.7) in a similar way:

$$\Phi(x) = \frac{k(x)}{\sqrt{q^*(x)}} \sup_{|z| \leqslant k(x)q^*(x)^{-1/2}} \left| \int_0^z [(q(x+s)-q(x))-(q(x-s)-q(x))]\,ds \right| \tag{3.22}$$

$$\leqslant \frac{c}{\sqrt{q(x)}} \sup_{|z| \leqslant 2k(x)q(x)^{-1/2}} \left[\int_0^z 2cq(x)^{\alpha}|s|^{\beta} ds \right] \leqslant \frac{ck(x)^{\beta+2}}{q(x)^{\delta}} \to 0 \text{ for } |x| \to \infty.$$

Hence $\Phi(x) \leqslant c$ for $|x| \gg 1$. Since $d(x)$ is a continuous positive function (see [3]), from (2.13) it follows that $\Phi(x)$ is absolutely bounded for $x \in R$. Thus $q(x) \in \mathcal{H}$. Then by Theorem 3.3, for any $\varepsilon \in (0, 1]$, problem (1.1)-(1.2) has a continuation to R for all y_0 and $|x_0| \gg 1$ such that $y_0 q(x_0)^{-1/2} \in [-1+\varepsilon, 1-\varepsilon]$. Conversely, it has no continuation to R for all y_0 and $|x_0| \gg 1$ such that either $y_0 q(x_0)^{-1/2} \geqslant 1+\varepsilon$, or $y_0 q(x_0)^{-1/2} \leqslant -(1+\varepsilon)$. In addition (see (3.10)), one has

$$\lim_{x \to -\infty} \frac{y_-(x)}{\sqrt{q(x)}} = -1, \qquad \lim_{x \to \infty} y_+(x) \frac{1}{\sqrt{q(x)}} = 1\,.$$

REMARK. – The scheme suggested in Example 2 is also convenient for studying equations (1.1) with oscillating functions $q(x)$. Consider, for example, (3.15). Set $q_1(x) = 1 + x^2$, $q_2(x) = (1+x^2)\cos\left(x + \frac{x^3}{3}\right)$. It is easy to verify that $q_1(x)$ satisfies (3.18) and (3.19) for $\alpha = \frac{1}{2}$, $\beta = 1$, $k(x) = q_1(x)^{1/6}$. Repeating the above computation, we obtain that in this case, by Theorem 2.4 and Definition 2.1, one has

$$d(x) = \frac{1+\delta(x)}{\sqrt{1+x^2}}, \qquad |\delta(x)| \leqslant \frac{c}{\sqrt{1+x^2}}, \qquad |x| \to \infty$$

and $q(x) \in H$. Thus $q_2(x)$ does not have any influence on the asumptotic behavior $y_-(x)$ and $y_+(x)$, and by Theorem 3.3, for any $\varepsilon > 0$, problem (3.15)-(3.16) has a continuation to R for all y_0 and $|x_0|\gg 1$ such that $\frac{y_0}{\sqrt{1+x_0^2}}\in[-1+\varepsilon,$ $1-\varepsilon]$. Conversely, it has not continuation to R for all y_0 and $|x_0|\gg 1$ such that either $\frac{y_0}{\sqrt{1+x_0^2}}\geqslant 1+\varepsilon$, or $\frac{y_0}{\sqrt{1+x_0^2}}\leqslant -(1+\varepsilon)$.

4. – Proofs.

In this section, we prove the assertions from § 3.

Proof of Lemma 3.1. – All the equations from (3.1) are considered in the same way. Let us verify, for example, the assertion for the second equation. From (3.14) it follows that

$$F_2(d)=\int_0^{\sqrt{2}d}\int_x^{x+t}q(\xi)\,d\xi\,dt\geqslant\int_{d/\sqrt{2}}^{\sqrt{2}d}\int_x^{x+t}q(\xi)\,d\xi\,dt\geqslant\frac{d}{\sqrt{2}}\int_x^{x+d/\sqrt{2}}q(\xi)\,d\xi\to\infty$$

as $d\to\infty$. Hence $F_2(\infty)=\infty$. Since $F_2(0)=0$, $F_2(d)$ does not decrease and is continuous on $[0,\infty)$, we conclude that the equation $F_2(d)=1$ has at least one positive root. Suppose that there are two roots α and β, $\alpha>\beta$. Then

$$1=\int_0^{\sqrt{2}\beta}\int_x^{x+t}q(\xi)\,d\xi\,dt\leqslant\sqrt{2}\beta\int_x^{x+\sqrt{2}\beta}q(\xi)\,d\xi\Rightarrow\int_x^{x+\sqrt{2}\beta}q(\xi)\,d\xi\neq 0\Rightarrow$$

$$0=\int_{\sqrt{2}\beta}^{\sqrt{2}\alpha}\int_x^{x+t}q(\xi)\,dt\geqslant\sqrt{2}(\alpha-\beta)\int_x^{x+\sqrt{2}\beta}q(\xi)\,d\xi\Rightarrow\alpha=\beta\,,$$

a contradiction. ■

Proof of Theorem 3.1. – We integrate the equations $v''(\xi)=q(\xi)\,v(\xi)$ and $u''(\xi)=q(\xi)\,u(\xi)$ along $[x-t,x]$ and $[x,x+t]$, $t\geqslant 0$, respectively. We get

$$v'(x)-v'(x-t)=\int_{x-t}^{x}q(\xi)\,v(\xi)\,d\xi\,,\qquad |u'(x)|+u'(x+t)=\int_x^{x+t}q(\xi)\,d\xi\,. \tag{4.1}$$

Let us integrate equalities (4.1) by $t\in[0,\mu]$, $\mu\geqslant 0$. We get

$$v'(x)\mu=v(x)-v(x-\mu)+\int_0^{\mu}\int_{x-t}^{x}q(\xi)\,v(\xi)\,d\xi\,dt\,,\qquad \mu\geqslant 0 \tag{4.2}$$

$$|u'(x)|\mu=u(x)-u(x+\mu)+\int_0^{\mu}\int_x^{x+t}q(\xi)\,u(\xi)\,d\xi\,dt,\qquad \mu\geqslant 0\,. \tag{4.3}$$

In (4.2) and (4.3), set $\mu = \sqrt{2}d_1(x)$ and $\mu = \sqrt{2}d_2(x)$, respectively. In the estimates presented below, we use (2.2) and the definition of $d_1(x)$:

$$v'(x)\sqrt{2}d_1(x) < v(x) + v(x)\int_0^{\sqrt{2}d_1(x)}\int_{x-t}^{x} q(\xi)\,d\xi\,dt = 2v(x)$$

$$v'(x)\sqrt{2}d_1(x) > v(x) - v(x-\sqrt{2}d_1(x)) + v(x-\sqrt{2}d_1(x))\int_0^{\sqrt{2}d_1(x)}\int_{x-t}^{x} q(x)\,d\xi\,dt = v(x)\,.$$

Thus inequalities (3.2) for $v(x)$ are proved. The estimates for $u(x)$ are proved in a similar way. Furthermore, (3.2) and (2.2) imply (3.3):

$$\frac{d_1(x)\,d_2(x)}{d_1(x)+d_2(x)}\,\frac{1}{\varrho(x)} = \frac{\dfrac{v'(x)}{v(x)} + \dfrac{|u'(x)|}{u(x)}}{d_1(x)^{-1}+d_2(x)^{-1}} \leqslant \max\left\{\frac{v'(x)}{v(x)}d_1(x), \frac{|u'(x)|}{u(x)}d_2(x)\right\} < \sqrt{2}$$

$$\frac{d_1(x)\,d_2(x)}{d_1(x)+d_2(x)}\,\frac{1}{\varrho(x)} = \frac{\dfrac{v'(x)}{v(x)} + \dfrac{|u'(x)|}{u(x)}}{d_1(x)^{-1}+d_2(x)^{-1}} \geqslant \min\left\{\frac{v'(x)}{v(x)}d_1(x), \frac{|u'(x)|}{u(x)}d_2(x)\right\} > \frac{1}{\sqrt{2}}\,.$$

Let us check (3.4). Let $\eta(x) = d_1(x)\,d_2(x)(d_1(x)+d_2(x))^{-1}$. Then $\eta(x) < d_1(x)$, $\eta(x) < d_2(x)$, and therefore from the definition of $d_1(x)$, $d_2(x)$ it follows that

$$\begin{aligned} 2 &= \int_0^{\sqrt{2}d_1(x)}\int_{x-t}^{x} q(\xi)\,d\xi\,dt + \int_0^{\sqrt{2}d_2(x)}\int_{x}^{x+t} q(\xi)\,d\xi\,dt \\ &\geqslant \int_0^{\sqrt{2}\eta(x)}\int_{x-t}^{x+t} q(\xi)\,d\xi\,dt\,. \end{aligned} \tag{4.4}$$

From (4.4) it follows that $\widehat{d}(x) > \eta$, and we thus obtain $\varrho(x) < \sqrt{2}\eta(x) < \sqrt{2}\widehat{d}(x)$ by (3.3). From (2.2) and the definition of $\widehat{d}(x)$ we then obtain

$$\begin{aligned} \frac{1}{\varrho(x)} &= \frac{v'(x)}{v(x)} + \frac{|u'(x)|}{u(x)} = \frac{v(x)-v(x-\sqrt{2}\widehat{d}(x))}{v(x)\sqrt{2}\widehat{d}(x)} + \frac{1}{\sqrt{2}\widehat{d}(x)}\int_0^{\sqrt{2}\widehat{d}(x)}\int_{x-t}^{x} q(\xi)\frac{v(\xi)}{v(x)}\,d\xi\,dt \\ &\quad + \frac{u(x)-u(x+\sqrt{2}\widehat{d}(x))}{\sqrt{2}\widehat{d}(x)\,u(x)} + \frac{1}{\sqrt{2}\widehat{d}(x)}\int_0^{\sqrt{2}\widehat{d}(x)}\int_{x}^{x+t} q(\xi)\frac{u(\xi)}{u(x)}\,d\xi\,dt \\ &< \frac{\sqrt{2}}{\widehat{d}(x)} + \frac{1}{\sqrt{2}\widehat{d}(x)}\int_0^{\sqrt{2}\widehat{d}(x)}\int_{x-t}^{x+t} q(\xi)\,d\xi\,dt = \frac{\sqrt{2}}{\widehat{d}(x)} + \frac{\sqrt{2}}{\widehat{d}(x)} = \frac{2\sqrt{2}}{\widehat{d}(x)}\,. \quad \blacksquare \end{aligned}$$

PROOF OF LEMMA 3.2. – From the definitions of $d(x)$ and $\widehat{d}(x)$, we deduce the following relations which prove the assertion:

$$2=\int_0^{\sqrt{2}\widehat{d}(x)}\int_{x-t}^{x+t}q(\xi)\,d\xi\,dt\leqslant\sqrt{2}\widehat{d}(x)\int_{x-\sqrt{2}\widehat{d}(x)}^{x+\sqrt{2}\widehat{d}(x)}q(\xi)\,d\xi\Rightarrow\sqrt{2}\widehat{d}(x)\geqslant d(x)$$

$$2=\int_0^{\sqrt{2}\widehat{d}(x)}\int_{x-t}^{x+t}q(\xi)\,d\xi\,dt\geqslant\int_{\widehat{d}(x)/\sqrt{2}}^{\sqrt{2}\widehat{d}(x)}\int_{x-t}^{x+t}q(\xi)\,d\xi\,dt\geqslant\frac{\widehat{d}(x)}{\sqrt{2}}\int_{x-\widehat{d}(x)/\sqrt{2}}^{x+\widehat{d}(x)/\sqrt{2}}q(\xi)\,d\xi$$

$$\Rightarrow d(x)\geqslant\frac{\widehat{d}(x)}{\sqrt{2}}. \qquad \blacksquare$$

PROOF OF LEMMA 3.3. – By assertion I) from § 1, it is enough to prove the statement of the lemma with signs $\leqslant$, $\geqslant$ in (3.13) replaced by $<$, $>$, respectively.

NECESSITY. – Suppose that the solution of (1.1)-(1.2) can be continued to R. By Theorem 2.5, this solution is of the form (2.18) for some $\theta=\theta_0$, and $\theta_0\neq0$, $\theta_0\neq\pm\infty$ because of the above assumption on inequality signs in (3.13). Then only one of the following can hold:

$$1)\quad y_0>\frac{v'(x_0)}{v(x_0)},\qquad 2)\quad\frac{u'(x_0)}{u(x_0)}<y_0<\frac{v'(x_0)}{v(x_0)},\qquad 3)\quad y_0<\frac{u'(x_0)}{u(x_0)}.$$

Let us show that if 1) or 3) holds then $\theta_0<0$. Indeed, in the case 1) we obtain, using (2.2) and (2.18):

$$0<\frac{\theta_0v'(x_0)+u'(x_0)}{\theta_0v(x_0)+u(x_0)}-\frac{v'(x_0)}{v(x_0)}=-\frac{1}{v(x_0)[\theta_0v(x_0)+u(x_0)]}\Rightarrow\theta_0<-\frac{u(x_0)}{v(x_0)}<0.$$

Similarly, in the case 3), one has

$$0<\frac{u'(x_0)}{u(x_0)}-\frac{\theta_0v'(x_0)+u'(x_0)}{\theta_0v(x_0)+u(x_0)}=-\frac{\theta_0}{u(x_0)[\theta_0v(x_0)+u(x_0)]}. \tag{4.5}$$

The assumption $\theta_0>0$ contradicts (4.5) because of (2.2). Thus in the cases 1) and 3) we have $\theta_0<0$. But then there is $x_1\in R$ such that $\theta_0v(x_1)+u(x_1)=0$. Indeed, from Theorem 2.1 it follows that the function $\varphi(x)=-\dfrac{u(x)}{v(x)}$, $x\in R$ is continuous, $\varphi(x) < 0$ for all $x \in R$, and in addition

$$\varphi'(x) = \frac{1}{v^2(x)} > 0,\ x \in R\,; \qquad \lim_{x \to -\infty} \varphi(x) = -\infty\,, \quad \lim_{x \to \infty} \varphi(x) = 0\,.$$

Therefore, the equation $\theta_0 v(x) + u(x) = 0$ has a unique finite root x_1. Then the solution of (1.1)-(1.2) has vertical asymptotics at the point $x = x_1$, and thus the solution cannot be continued to R. This implies that the case 2) holds.

SUFFICIENCY. – Suppose that (3.13) holds. By assertion I) from §1 and Theorem 2.5, in some neighborhood of the point $x = x_0$ there exists a unique soluition of (1.1)-(1.2), and it is of the form (2.18) with some $\theta = \theta_0$, $\theta_0 \neq 0$, $\theta_0 \neq \pm\infty$. Then we deduce from (3.13) and (2.2):

$$0 < \frac{\theta_0 v'(x_0) + u'(x_0)}{\theta_0 v(x_0) + u(x_0)} - \frac{u'(x_0)}{u(x_0)} = \frac{\theta_0}{u(x_0)[\theta_0 v(x_0) + u(x_0)]}\,. \tag{4.6}$$

$$0 < \frac{v'(x_0)}{v(x_0)} - \frac{\theta_0 v'(x_0) + u'(x_0)}{\theta_0 v(x_0) + u(x_0)} = \frac{1}{v(x_0)[\theta_0 v(x_0) + u(x_0)]}\,. \tag{4.7}$$

From (4.6)-(4.7) and (2.2) it follows that $\theta_0 > 0$. But then in view of (2.2) the solution (2.18) is defined for all $x \in R$ and satisfies (1.1) almost everywhere. ■

PROOF OF THEOREM 3.2. – Follows from Theorems 3.1, 2.5 and Lemma 3.3. ■

PROOF OF COROLLARY 3.2.1. – From (2.3) one can easily deduce the equalities

$$\frac{v'(x)}{v(x)} = \frac{1 + \varrho'(x)}{2\varrho(x)}\,, \qquad \frac{u'(x)}{u(x)} = -\frac{1 - \varrho'(x)}{2\varrho(x)}\,, \qquad x \in R\,. \tag{4.8}$$

Since the solution (1.1) of equation (1.1) exists for all $x \in R$, it is of the form (2.18) with $\theta > 0$ (see the above proof of Theorem 3.2). Then from (2.18), (3.13), (2.4) and (3.4), it follows that

$$y(x) \leqslant \frac{v'(x)}{v(x)} = \frac{1 + \varrho'(x)}{2\varrho(x)} < \frac{1}{\varrho(x)} < \frac{2\sqrt{2}}{\widehat{d}(x)}\,, \qquad x \in R$$

$$y(x) \geqslant \frac{u'(x)}{u(x)} = -\frac{1 - \varrho'(x)}{2\varrho(x)} > -\frac{1}{\varrho(x)} > -\frac{2\sqrt{2}}{\widehat{d}(x)}\,, \qquad x \in R\,. \quad ■$$

PROOF OF THEOREM 3.3. – Since $q(x) \in \mathcal{H}$, by Theorem 2.3 one has $\varrho'(x) \to 0$ as $|x| \to \infty$, and (2.8)-(2.9) hold. Let $\varepsilon \in (0, 1]$, $(x_0, y_0) \in \{(x, y): yd(x) \geq 1 + \varepsilon, x \geq a\} \subset T(\varepsilon, a)$. Then for $a \gg 1$ we obtain using (4.8), (2.8) and (2.9):

$$\frac{v'(x_0)}{v(x_0)} = \frac{1+\varrho'(x_0)}{2\varrho(x_0)} = \frac{1+\varrho'(x_0)}{d(x_0)(1+\varepsilon(x_0))} < \frac{1+\varepsilon}{d(x_0)} \leq y_0 .$$

By Lemma 3.3, this implies that the solution of problem (1.1)-(1.2) cannot be continued to R. The cases $(x_0, y_0) \in \{(x, y): \ yd(x) \leq -(1+\varepsilon)\} \in T(\varepsilon, a)$, $a \gg 1$ and $(x_0, y_0) \in P(\varepsilon, a)$, $\gg 1$ are considered in a similar way. Let us check (3.11). From (2.2), (2.18) and Theorem 2.3 for $\theta \neq 0$, $\theta \neq \pm\infty$, we get

$$\lim_{x \to \infty} y_+(x)\, d(x) = \lim_{x\to\infty} \frac{\theta v'(x) + u'(x)}{\theta v(x) + u(x)} d(x) = \lim_{x\to\infty} \frac{v'(x)}{v(x)} \frac{1 + \dfrac{1}{\theta}\dfrac{u'(x)}{v'(x)}}{1+\dfrac{1}{\theta}\dfrac{u(x)}{v(x)}} d(x)$$

$$= \lim_{x\to\infty} (1+\varrho'(x)) \frac{d(x)}{2\varrho(x)} \frac{1 - \dfrac{1}{\theta}\dfrac{1-\varrho'(x)}{1+\varrho'(x)}\dfrac{u(x)}{v(x)}}{1+\dfrac{1}{\theta}\dfrac{u(x)}{v(x)}} = 1 .$$

The second equality of (3.11) can be verified in a similar way. ■

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