Bollettino Unione Matematica Italiana

P. Auscher, M. Qafsaoui

Observations on $W^{1,p}$ estimates for divergence elliptic equations with VMO coefficients

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **5-B** (2002), n.2, p. 487–509.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2002_8_5B_2_487_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2002.

Bollettino U. M. I.
(8) 5-B (2002), 487-509

Observations on $W^{1,p}$ Estimates for Divergence Elliptic Equations with VMO Coefficients.

P. Auscher - M. Qafsaoui

Sunto. – *In questo lavoro esponiamo alcune osservazioni circa il lavoro di Di Fazio riguardante le stime $W^{1,p}$ per $1 < p < \infty$ per soluzioni di equazioni ellittiche del tipo* $\mathrm{div}\, A \nabla u = \mathrm{div}\, f$ *su un dominio Ω con dati di Dirichlet nulli, A nella classe VMO ed f in L^p. Si considera il caso in cui i coefficienti della parte principale sono complessi e la frontiera di Ω è di classe C^1. Si considera inoltre il caso del problema di Neumann non omogeneo e si dimostrano risultati analoghi. Il principale strumento utilizzato è una conveniente formula di rappresentazione per la funzione di Green e di Neumann.*

Summary. – *In this paper, we make some observations on the work of Di Fazio concerning $W^{1,p}$ estimates, $1 < p < \infty$, for solutions of elliptic equations* $\mathrm{div}\, A \nabla u = \mathrm{div}\, f$, *on a domain Ω with Dirichlet data* 0 *whenever $A \in VMO(\Omega)$ and $f \in L^p(\Omega)$. We weaken the assumptions allowing real and complex non-symmetric operators and C^1 boundary. We also consider the corresponding inhomogeneous Neumann problem for which we prove the similar result. The main tool is an appropriate representation for the Green (and Neumann) function on the upper half space. We propose two such representations.*

Introduction.

In recent years, there has been a wide interest for elliptic equations with discontinuous coefficients that belong to *VMO*, [1], [4], [5], [6], [8], [7], [11], [12], [13], [15], [16], [18], [20], [21]. In particular, it is shown in [11] that for $1 < p < +\infty$ and $\boldsymbol{f} \in L^p(\Omega)$, the inhomogeneous Dirichlet problem

$$(1) \qquad \begin{cases} \mathrm{div}\, A \nabla u = \mathrm{div}\, \boldsymbol{f} \ \text{ in } \ \Omega \\ u \in W_0^{1,p}(\Omega) \end{cases}$$

has a unique solution, and $\|\nabla u\|_p \leq C \|\boldsymbol{f}\|_p$ with C independent of $\boldsymbol{f}$, provided Ω is a smooth (e.g. $C^{1,1}$) bounded open set in $\mathbb{R}^n$, $n \geq 2$, and A is a real, symmetric, uniformly elliptic matrix with coefficients in $VMO(\Omega) \cap L^\infty(\Omega)$. We recall that a locally integrable function g on Ω is in the space $BMO(\Omega)$ if

$$\sup_B \frac{1}{|B|} \int_B |g(x) - g_B|\, dx \equiv \|g\|_* < +\infty,$$

where B ranges over balls $B \subset \Omega$ and g_B denotes the mean of g on B. For $g \in BMO(\Omega)$ and $r > 0$, we set

$$\eta(r) = \sup_{\varrho \leqslant r} \frac{1}{|B_\varrho|} \int_{B_\varrho} |g(x) - g_B| \, dx .$$

Here ϱ denotes the radius of B_ϱ. A function $g \in BMO(\Omega)$ is in $VMO(\Omega)$ if $\lim_{r \to 0} \eta(r) = 0$ and we call η the *vmo* modulus of continuity of g. Our observations will be the followings.

1) We remove the hypothesis that A be real and symmetric and allow complex coefficients: Our assumptions on $A(x) = (a_{jk}(x))_{1 \leqslant j, k \leqslant n}$ are

$$\text{(2)} \qquad \begin{cases} a_{jk}(x) \in \mathbb{C} \text{ and } \|a_{jk}\|_\infty \leqslant \delta^{-1}, \\ A(x) + A^*(x) \geqslant 2\delta Id \;\; a.e. \\ a_{jk} \in vmo(\Omega) \end{cases}$$

for some $\delta > 0$ called the ellipticity constant of A.

What is really at stake is the representation of the Green function $G(x, y)$ for constant coefficients operators on the upper half space as the one chosen in [11] cannot extend to all matrices A satisfying (2).

Here, we use two different representations: one is taken from [14], [22]; L^p boundedness of the operator with kernel $\nabla_x \nabla_y G(x, y)$ follows from classical theory for singular integrals of convolution type. The other originates from [2]; it uses the reflection principle and L^p boundedness is a consequence of the T(1) theorem [10].

2) There is a technical step in [11] which can be avoided. We establish a representation of solutions that is simpler to use for obtaining interior and boundary a priori estimates via commutator results between Calderón-Zygmund operators and VMO functions [9].

3) We make clear that the proof works for C^1 domains by controlling the constants. Note that in [22], the similar problem with continuous coefficients was treated on C^1 domains. Our assumption on Ω is:

$$\text{(3)} \qquad \Omega \text{ is a bounded, open, connected set with } C^1 \text{ boundary} .$$

4) We can treat in the same way the corresponding inhomogeneous Neumann problem:

$$\text{(4)} \qquad \begin{cases} \operatorname{div} A \nabla u = \operatorname{div} \boldsymbol{f} \text{ in } \Omega \\ u \in W^{1,p}(\Omega) \text{ and } \nu . A \nabla u = \nu . f \text{ on } \partial\Omega \end{cases}$$

where ν is the outward unit normal. To avoid defining $\nu . A\nabla u$ as a distribution on the boundary of Ω we mean (4) variationally:

$$u \in W^{1,p}(\Omega) \text{ and } \forall \varphi \in Lip(\overline{\Omega}) \quad \int_\Omega A\nabla u \nabla \varphi = \int_\Omega \boldsymbol{f} \nabla \varphi . \tag{5}$$

Note that, similarly, (1) is equivalent to

$$u \in W_0^{1,p}(\Omega) \text{ and } \forall \varphi \in \mathrm{Lip}(\Omega), \operatorname{Supp}\varphi \subset \Omega \quad \int_\Omega A\nabla u \nabla \varphi = \int_\Omega \boldsymbol{f} \nabla \varphi . \tag{6}$$

The main result of this paper is

THEOREM 1. – *Let A satisfy* (2) *and Ω satisfy* (3), $1 < p < +\infty$ and $\boldsymbol{f} \in L^p(\Omega)$. *Then there exist a unique solution u_D of* (1) *and a unique solution u_N of* (4) *modulo constant. Moreover, the operators $\boldsymbol{f} \mapsto \nabla u_D$ and $\boldsymbol{f} \mapsto \nabla u_N$ are bounded on $L^p(\Omega)$, with norm bounded by a constant depending on n, δ, p, the VMO modulus of continuity of A, $\partial\Omega$ and $|\Omega|$.*

1. – Estimates for constant coefficients operators.

In this section A denotes a constant complex matrix satisfying (2).

1.1. *Main results.*

The fundamental solution of $-\operatorname{div} A\nabla = -\sum_{j,k} \partial_j(a_{jk}\partial_k)$ is given by, for $x \neq 0$,

$$\Gamma_A(x) = \frac{1}{(n-2)\,\omega_n(\det S)^{1/2}}(S^{-1}x, x)^{(2-n)/2} \qquad \text{if } n \geq 3 , \tag{7}$$

$$\Gamma_A(x) = -\frac{1}{4\pi(\det S)^{1/2}}\log(S^{-1}x, x) \qquad \text{if } n = 2 . \tag{8}$$

Here, $S = \frac{A + {}^tA}{2}$ is the symmetric part of A, $\log z$ denotes the principal branch of the logarithm and $z^\alpha = e^{\alpha \log z}$ for $z \in \mathbb{C} \setminus \mathbb{R}^-$ and $\alpha \in \mathbb{R}$. This is well known when A is real and symmetric, and can be easily checked in general. From this we obtain

$$|D^\alpha \Gamma_A(x)| \leq \frac{C}{|x|^{n-2+|\alpha|}}, \qquad x \neq 0 , \tag{9}$$

for $|\alpha| = 1, 2, 3$ and C depends only on dimension and ellipticity, and D^α is the usual symbol for the partial derivative associated with α.

Consider now the Green function $G_A(x, y)$ and the Neumann function $N_A(x, y)$ for $L = -\operatorname{div} A\nabla$ on $\mathbb{R}^n_+$, that is for all $y \in \mathbb{R}^n_+$,

$$\begin{cases} L_x G_A(x, y) = \delta_x(y) & \text{in } \mathcal{D}'(\mathbb{R}^n_+) \\ G_A(x, y) = 0 & \text{if } x \in \partial\mathbb{R}^n_+ \end{cases} \tag{10}$$

and

$$(11) \qquad \begin{cases} L_x N_A(x, y) = \delta_x(y) & \text{in } \mathcal{D}'(\mathbb{R}^n_+) \\ \nu . A\nabla_x N_A(x, y) = 0 & \text{if } x \in \partial\mathbb{R}^n_+ \end{cases}$$

and $\nu = (0, \dots, 0, -1)$.

Assume for a moment that A is real and symmetric. Then $G_A(x, y)$ can be computed as

$$(12) \qquad G_A(x, y) = \Gamma_A(x - y) - \Gamma_A(T(x) - y), \qquad x, y \in \mathbb{R}^n_+,$$

where T is the orthogonal symmetry with respect to $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ in the inner product induced by the inverse of A. It is characterized by

$$(13) \qquad T_{|\mathbb{R}^{n-1}} = Id, \qquad T^2 = Id, \qquad {}^tTA^{-1}T = A^{-1},$$

(${}^tT$ is the transpose of T) and calculations yield

$$(14) \qquad T(x) = x^* - x_n v,$$

where $x^* = (x_1, \dots, x_{n-1}, -x_n)$ if $x = (x_1, \dots, x_{n-1}, x_n)$ and $v \in \mathbb{R}^{n-1}$ with

$$v_k = \frac{2a_{kn}}{a_{nn}}, \qquad 1 \leqslant k \leqslant n-1.$$

Formula (12) extends to matrices A for which one can find T of the form (14) such that $\Gamma_A \circ T = \Gamma_A$. In (13), ${}^tTA^{-1}T = A^{-1}$ must be replaced by $TS\,{}^tT = S$, where $S = \frac{A + {}^tA}{2}$. One finds

$$(15) \qquad v_k = \frac{a_{nk} + a_{kn}}{a_{nn}}, \qquad 1 \leqslant k \leqslant n-1,$$

so that A must satisfy $a_{nk} + a_{kn} \in \mathbb{R}a_{nn}$ for T to be a map on $\mathbb{R}^n$.

The Neumann function when A is real symmetric is given by

$$(16) \qquad N_A(x, y) = \Gamma_A(x - y) + \Gamma_A(T(x) - y), \qquad x, y \in \mathbb{R}^n_+.$$

One checks that this formula is valid for any A provided $a_{kn} = a_{nk} \in \mathbb{R}a_{nn}$, $1 \leqslant k \leqslant n-1$.

One must therefore find another representation for G_A and N_A to remove these algebraic constraints on A. We achieve this goal by presenting two different approaches (see section 1.2, section 1.3). This enables us to prove the following results.

PROPOSITION 2. – *There is a constant C depending only on dimension and ellipticity of A such that*

$$|D^\alpha G_A(x, y)| + |D^\alpha N_A(x, y)| \leqslant \frac{C}{|x-y|^{n-2+|\alpha|}} \tag{17}$$

whenever $x, y \in \mathbb{R}^n_+$, $x \neq y$ and $\alpha \in \mathbb{N}^{2n}$, $|\alpha| = 1, 2, 3$ and D is any partial of order 1 in x and y.

Of course, one can differentiate indefinitely but then the constant C depends also of $|\alpha|$. Note also that these estimates hold up to the boundary of $\mathbb{R}^n_+$.

Define $\mathcal{G}_A$ and $\mathcal{N}_A$ as follows: for all $\boldsymbol{g}, \boldsymbol{h} \in \mathcal{D}(\mathbb{R}^n_+)$, valued in $\mathbb{C}^n$,

$$(\mathcal{G}_A \boldsymbol{g}, \boldsymbol{h}) = -\iint_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} \nabla_y G_A(x, y).\, \boldsymbol{g}(y) \operatorname{div} \boldsymbol{h}(x)\, dx\, dy\,, \tag{18}$$

$$(\mathcal{N}_A \boldsymbol{g}, \boldsymbol{h}) = -\iint_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} \nabla_y N_A(x, y).\, \boldsymbol{g}(y) \operatorname{div} \boldsymbol{h}(x)\, dx\, dy\,. \tag{19}$$

Note that the above integrals exist in the Lebesgue sense by (17). The main result is

THEOREM 3. – *For all $p \in (1, +\infty)$, there exists C depending on dimension, ellipticity and p such that for all $\boldsymbol{g} \in \mathcal{D}(\mathbb{R}^n_+)$, valued in $\mathbb{C}^n$,*

$$\|\mathcal{G}_A \boldsymbol{g}\|_p + \|\mathcal{N}_A \boldsymbol{g}\|_p \leqslant C \|\boldsymbol{g}\|_p. \tag{20}$$

1.2. *Fourier transform method.*

In this section we prove Proposition 2 and Theorem 3 using an algorithm based on the Fourier transform. We follow [14] to compute the Green function (and correct a mistake in that paper). Actually, such computations already appeared in [22]. We use similar ideas to compute th Neumann function. To this end, it is convenient to write $x \in \mathbb{R}^n_+$ as $x = (x', t)$ with $x' \in \mathbb{R}^{n-1}$ and $t = x_n \geqslant 0$ is the nth coordinate of x. Functions $f(x)$ defined on $\mathbb{R}^n$ will be denoted as $f(t)(x')$ and the variable x' will often be omitted.

Again, we assume A to be constant. We have

$$\forall\, \xi \in \mathbb{R}^n, \qquad \widehat{\Gamma_A}(\xi) = (A\xi.\, \xi)^{-1}$$

where $\widehat{f}$ is the Fourier transform of f in $\mathbb{R}^n$ (formally $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-ix.\,\xi}\, dx$). Write $\xi = (\xi', \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}$. For fixed $\xi' \in \mathbb{R}^{n-1}$, there exist $\tau_+(\xi')$ and $\tau_-(\xi')$ with $\Im m\, \tau_+(\xi') > 0$ and $\Im m\, \tau_-(\xi') < 0$ such that

$$A\xi.\, \xi = a_{nn}(\tau - \tau_+(\xi'))(\tau - \tau_-(\xi'))\,.$$

As functions of ξ', $\tau_+(\xi')$ and $\tau_-(\xi')$ are homogeneous of degree 1 on $\mathbb{R}^{n-1}$, $C^\infty(\mathbb{R}^{n-1}\setminus\{0\})$ and there exist $m, M > 0$ depending only on n and δ such that for all $\xi' \in \mathbb{R}^{n-1}$

$$\left\{\begin{array}{l} \Im m\, \tau_+(\xi') \geqslant m|\xi'|, \qquad \Im m\, \tau_-(\xi') \leqslant -m|\xi'|, \\ |\tau_+(\xi')| + |\tau_-(\xi')| \leqslant M|\xi'|. \end{array}\right. \tag{21}$$

The Fourier transform of $\Gamma_A(t)$ (in $\mathbb{R}^{n-1}$) is given by

$$\widehat{\Gamma_A(t)}(\xi') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{it\tau}}{a_{nn}(\tau-\tau_+(\xi'))(\tau-\tau_-(\xi'))}\,d\tau$$

and by the calculus of residues we find

$$\widehat{\Gamma_A(t)}(\xi') = \frac{ie^{it\tau_\pm(\xi')}}{a_{nn}(\tau_+(\xi')-\tau_-(\xi'))}, \tag{22}$$

where $\tau_+(\xi')$ occurs when $t \geqslant 0$ and $\tau_-(\xi')$ occurs when $t \leqslant 0$.

The solution of the homogeneous Dirichlet problem $\operatorname{div} A\nabla v = 0$ in $\mathbb{R}^n_+$, $v = g$ on $\mathbb{R}^{n-1}$ and $v = 0$ at ∞ is given by $v(t) = \mathcal{P}_t \star g$ where $\star$ is the convolution on $\mathbb{R}^{n-1}$ and $\mathcal{P}_t$ is the Poisson kernel. Using Fourier transform in $\mathbb{R}^{n-1}$ one has

$$\widehat{\mathcal{P}_t}(\xi') = e^{it\tau_+(\xi')}, \qquad t \geqslant 0, \qquad \xi' \in \mathbb{R}^{n-1}. \tag{23}$$

Then $G_A(x, y)$ can be computed as

$$\begin{aligned} G_A(x, y) &= (\Gamma_A(t-s) - \mathcal{P}_t\star\Gamma_A(-s))(x'-y') \\ &= \Gamma_A(x-y) - (\mathcal{P}_t\star\Gamma_A(-s))(x'-y'), \end{aligned} \tag{24}$$

where $x = (x', t)$, $y = (y', s) \in \mathbb{R}^n_+$.

To compute the Neumann function, we construct the Neumann to Dirichlet boundary operator as follows. We look for $N_A(x, y)$ in the form

$$N_A(x, y) = (\Gamma_A(t-s) - \mathcal{P}_t\star h(s))(x'-y')$$

for $x = (x', t)$, $y = (y', s) \in \mathbb{R}^n_+$. The condition on h is that

$$\forall s > 0,\ -\nu.\, A\nabla\Gamma(t-s)_{|t=0} = -\nu.\, A\nabla\, \mathcal{P}_t\star h(s)_{|t=0}, \tag{25}$$

where $\nabla = \nabla_x = \left(\nabla_{x'}, \dfrac{\partial}{\partial t}\right)$ and $\nu = (0, \dots, 0, -1)$. Set B_+, B_- the boundary operators with symbols b_+, b_- given by

$$b_\pm(\xi') = i\left(\sum_{k=1}^{n-1} a_{nk}\,\xi_k + a_{nn}\,\tau_\pm(\xi')\right)$$

with $\xi' = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. Then (25) is equivalent to

$$\forall s > 0, \qquad B_+ h(s) = B_- \Gamma(-s).$$

LEMMA 4. – *The symbols b_+, b_- are $C^\infty(\mathbb{R}^{n-1} \setminus \{0\})$ and homogeneous of degree 1 with*

$$\forall \xi' \in \mathbb{R}^{n-1}, \qquad m|\xi'| < |b_\pm(\xi')| \leq M|\xi'|,$$

where m, M depend only on n and δ in (2).

PROOF. – We only consider b_+. It is clear that $b_+ \in C^\infty(\mathbb{R}^{n-1} \setminus \{0\})$, and the upper bound for b_+ is obvious from (21). Next we study the lower bound. Fix $\xi' \in S^{n-2}$ and let $V: \mathbb{C}^2 \to \mathbb{C}^n$ be linear defined by $V\varepsilon_1 = \xi'$, $V\varepsilon_2 = e_n$, when $(\varepsilon_1, \varepsilon_2)$ is the canonical basis of $\mathbb{C}^2$ and $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{C}^n$. Since $\|\xi'\| = 1$, V is unitary so that

$$\forall v \in \mathbb{C}^2, \qquad \Re e\, {}^tVAVv \,.\, \bar{v} \geq \delta \|Vv\|^2 = \delta \|v\|^2,$$

where δ is the ellipticity constant of A. The matrix representing ${}^tVAV$ is given by

$$\begin{pmatrix} \sum\limits_{1 \leq j,\, k \leq n-1} a_{jk} \xi_k \xi_j & \sum\limits_{k=1}^{n-1} a_{kn} \xi_k \\ \sum\limits_{k=1}^{n-1} a_{nk} \xi_k & a_{nn} \end{pmatrix}$$

and by definition of τ_+ (we drop ξ') and b_+ we have

$${}^tVAV \begin{pmatrix} 1 \\ \tau_+ \end{pmatrix} = \begin{pmatrix} -\tau_+ b_+ \\ b_+ \end{pmatrix}.$$

Thus $\delta(1 + |\tau_+|^2) \leq b_+(-\tau_+ + \overline{\tau_+}) = -2ib_+(\Im m \tau_+)$ and the lower bound for b_+ follows from the properties of τ_+. ■

Set $m = b_+^{-1} b_-$. Using [23], p. 75, and the properties (9) on $\Gamma_A(-s)$, we see that $h(s) = P.V.K_m \star \Gamma_A(-s) + c_m \Gamma_A(-s)$ on $\mathbb{R}^{n-1}$, where K_m is a Calderón-Zygmund kernel on $\mathbb{R}^{n-1}$ and $c_m \in \mathbb{C}$. Hence,

(26) $N_A(x, y) = \Gamma_A(x - y) - (\mathcal{P}_t \star (P.V.K_m + c_m \delta_0) \star \Gamma_A(-s))(x' - y')$.

Note that $G_A(x, y)$ in (24) and $N_A(x, y)$ in (26) are of the same type.

We now turn to the proof of Proposition 2. Since the estimates on $D^\alpha \Gamma$ have been already observed, it remains to obtain estimates for functions of the form

$$f_\alpha(s, t, x') = D^\alpha(\mathcal{P}_t \star (P.V.K + c\delta_0) \star \Gamma(-s))(x')$$

where $c \in \mathbb{C}$, $\alpha \in \mathbb{N}^{n+1}$, $|\alpha| = 1, 2, 3$ and D is any of D_{x_j}, $1 \leqslant j \leqslant n-1$, D_t, D_s, and $K \in C^\infty(\mathbb{R}^{n-1} \backslash \{0\})$ is a Calderón-Zygmund kernel on $\mathbb{R}^{n-1}$. To see this we observe after straightforward computations using (21), (22), (23) and Lemma 4 that

$$f_\alpha(s, t, x') = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{it\tau_+(\xi') - is\tau_-(\xi')} \Lambda_\alpha(\xi')\, e^{ix' \cdot \xi'}\, d\xi',$$

where Λ_α is a homogeneous function of degree $|\alpha| - 1$ and $\Lambda_\alpha \in C^\infty(\mathbb{R}^{n-1} \backslash \{0\})$ with $|\Lambda_\alpha(\xi')| \leqslant M|\xi'|^{|\alpha|-1}$ and M depends only on n and δ. Using the fact that $\Im m \tau_+(\xi') \geqslant m|\xi'|$ and $\Im m \tau_-(\xi') \leqslant -m|\xi'|$, we routinely obtain

$$f_\alpha(s, t, x')| \leqslant \frac{M}{(s + t + |x'|)^{n-1+|\alpha|}} \tag{27}$$

for all $s > 0$, $t > 0$, $x' \in \mathbb{R}^{n-1}$ and M depends only on n and δ if $|\alpha|$ is restricted to 1, 2 or 3. It is then easy to deduce Proposition 2 from (27), and we skip further details.

It remains to prove Theorem 3. First we have

$$\left| \int_{\mathbb{R}^n_+} \left(\int_{\mathbb{R}^n_+} \nabla_y (\Gamma_A(x-y)).\, \boldsymbol{g}(y)\, dy \right) \operatorname{div} \boldsymbol{h}(x)\, dx \right| \leqslant C(n, \delta, p) \|\boldsymbol{g}\|_p \|\boldsymbol{h}\|_{p'} \tag{28}$$

for all $\boldsymbol{g}, \boldsymbol{h} \in \mathcal{D}(\mathbb{R}^n_+)$, valued in $\mathbb{C}^n$. Indeed, integrating by parts, the expression in the left hand side is equal to

$$\lim_{\varepsilon \to 0} \iint_{|x-y| \geqslant \varepsilon} ((\nabla\nabla\Gamma_A)(x-y)\, \boldsymbol{g}(y)).\, \boldsymbol{h}(x)\, dx\, dy - \int_{\mathbb{R}^n_+} \mathcal{C}\boldsymbol{g}(x).\, \boldsymbol{h}(x)\, dx\,, \tag{29}$$

where $\mathcal{C}$ is the constant matrix with entries

$$c_{jk} = \int_{S^{n-1}} (D_k \Gamma)(t)\, t_j\, d\sigma(t)$$

and σ is the surface measure on S^{n-1}. Classical Calderón-Zygmund theory yields (28).

Next, set $H(x, y) = (\mathcal{P}_t \star \Gamma(-s))(x' - y')$ for $x = (x', t)$ and $y = (y', s) \in \mathbb{R}^n_+$. Then for $\boldsymbol{g}, \boldsymbol{h} \in \mathcal{D}(\mathbb{R}^n_+)$ valued in $\mathbb{C}^n$,

$$-\int_{\mathbb{R}^n_+} \left(\int_{\mathbb{R}^n_+} \nabla_y H(x, y).\, \boldsymbol{g}(y)\, dy \right) \operatorname{div} \boldsymbol{h}(x)\, dx =$$

$$\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} (\nabla_x \nabla_y H(x, y)\, \boldsymbol{g}(y)).\, \boldsymbol{h}(x)\, dx\, dy\,,$$

the last integral being non singular since, by (27),

$$|\nabla_x \nabla_y H(x, y)| \leqslant \frac{M}{|x - y^*|^n}, \qquad M = C(n, \delta). \tag{30}$$

It is classical that (30) gives an upper bound of the form $C(n, \delta, p)\|\boldsymbol{g}\|_p \|\boldsymbol{h}\|_{p'}$ for the last integral by using Hardy inequality (see [5], or [3] for a short proof). This proof applies with $H(x, y)$ replaced by $\widetilde{H}(x, y) = (\mathcal{P}_t \star (P.\ V.\ K_m + c_m \delta_0) \star \Gamma(-s))(x' - y')$. This finishes the proof of Theorem 3 by this method.

1.3. *Reflection principle method.*

Here, we use the good old reflection principle across the boundary and then rely on estimates for fundamental solution of a specific class of elliptic operators studied in [2]. We will deduce Theorem 3 from the T(1) theorem. As for Proposition 2, we only obtain part of it. We miss some cases in (17), not because they cannot be obtained by this method, but because we feel that such calculations are out of the core of this article.

Define the orthogonal symmetry $\mathcal{S}$ of $\mathbb{R}^n$ across $\partial \mathbb{R}^n_+$ by

$$\mathcal{S}(x_1, \ldots, x_{n-1}, x_n) = \mathcal{S}(x', x_n) = (x', -x_n) \tag{31}$$

and let

$$A^\sharp(x) = \begin{cases} A & \text{if } x_n \geqslant 0 \\ \mathcal{S} A \mathcal{S} & \text{if } x_n < 0 . \end{cases} \tag{32}$$

Recall that A is constant, but $A^\sharp(x)$ may no longer be constant. Let $b_{jk}(x)$ be the coefficients of $A^\sharp(x)$. We have

$$\begin{cases} b_{jk}(x) = a_{jk} & \text{if } 1 \leqslant j, k \leqslant n-1 \ \text{ or } j = k = n , \\ b_{jk}(x) = a_{jk} \operatorname{sign}(x_n) & \text{otherwise} , \end{cases}$$

therefore, the coefficients of $A^\sharp$ depend only on x_n. Furthermore $A^\sharp$ is uniformly elliptic on $\mathbb{R}^n$ with the same ellipticity constant δ as A. The class of elliptic operators $\mathcal{L} = -\operatorname{div} \mathcal{A} \nabla$ on $\mathbb{R}^n$ associated with bounded uniformly elliptic matrices $\mathcal{A}$ depending on one coordinate variable is studied in [2] and in particular estimates are obtained for derivatives of the kernel of $e^{-t\mathcal{L}}$.

In our case, this gives us enough information to apply the T(1) theorem. Set

$L^\sharp = -\operatorname{div} A^\sharp \nabla$ and introduce the vectors fields

$$\begin{cases} X_k = \dfrac{\partial}{\partial x_k} \qquad \text{for } 1 \leqslant k \leqslant n-1\,, \\ X_n = b_{n1}\dfrac{\partial}{\partial x_1} + \ldots + b_{nn}\dfrac{\partial}{\partial x_n}\,, \end{cases}$$

and

$$\begin{cases} \widetilde{X}_l = \dfrac{\partial}{\partial x_l} \qquad \text{for } 1 \leqslant l \leqslant n-1\,, \\ \widetilde{X}_n = b_{1n}\dfrac{\partial}{\partial x_1} + \ldots + b_{nn}\dfrac{\partial}{\partial x_n}\,. \end{cases}$$

For $t > 0$, let $K_t(x, y)$, $M_t^k(x, y)$ and $\widetilde{M}_t^k(x, y)$ be respectively the kernels of $e^{-tL^\sharp}$, $t^{1/2} X_k e^{-tL^\sharp}$ and $t^{1/2} e^{-tL^\sharp}\, {}^t\widetilde{X}_k$, $1 \leqslant k \leqslant n$ (here, ${}^t\widetilde{X}_k$ is the transpose of $\widetilde{X}_k$).

LEMMA 5 ([2], Ch IV, Lemma 24 and Appendix B). – *We have for $\eta \in (0, 1)$, say $\eta = 1/2$,*

$$|K_t(x, y)| \leqslant \frac{C}{t^{n/2}} \exp\left(-\frac{a|x-y|^2}{t}\right)$$

$$|M_t^k(x, y)| + |\widetilde{M}_t^k(x, y)| \leqslant \frac{C}{t^{n/2}} \exp\left(-\frac{a|x-y|^2}{t}\right)$$

$$|M_t^k(x+h, y) - M_t^k(x, y)| + |\widetilde{M}_t^k(x, y+h) - \widetilde{M}_t^k(x, y)| \leqslant \frac{C}{t^{n/2}}\left(\frac{|h|}{t^{1/2}}\right)^\eta,$$

where $C \geqslant 0$ and $a > 0$ depend only on n and δ.

Define

$$\mathcal{C}_{kl} = X_k (L^\sharp)^{-1}\, {}^t\widetilde{X}_l = \int_0^{+\infty} X_k e^{-tL^\sharp}\, {}^t\widetilde{X}_l\, dt\,.$$

This is a formal definition. In fact, we have

$$\mathcal{C}_{kl} = \lim_{\varepsilon \to 0} \mathcal{C}_{kl}^\varepsilon$$

and

$$\mathcal{T}^{\varepsilon}_{kl} = \int_{\varepsilon}^{1/\varepsilon} X_k e^{-tL^{\sharp}} {}^t\widetilde{X}_l \, dt , \qquad \varepsilon > 0 .$$

The limit is taken in the strong topology of $\mathcal{B}(L^2(\mathbb{R}^n))$ once uniform estimates with respect to ε are obtained. Since this is a standard step in Calderón-Zygmund theory we ignore it and think $\mathcal{T}_{kl}$ as $\mathcal{T}^{\varepsilon}_{kl}$ in the sequel.

Denote by $K_{kl}(x, y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ the distribution kernel of $\mathcal{T}_{kl}$.

PROPOSITION 6. – *There is a constant C depending only on n and δ such that for all $x, y \in \mathbb{R}^n$, $x \neq y$,*

$$|K_{kl}(x, y)| \leq \frac{C}{|x-y|^n}, \tag{33}$$

$$|K_{kl}(x+h, y) - K_{kl}(x, y)| \leq \frac{C}{|x-y|^{n+1}} \left(\frac{|h|}{|x-y|} \right)^{\eta}, \tag{34}$$

$$|K_{kl}(x, y+h) - K_{kl}(x, y)| \leq \frac{C}{|x-y|^{n+1}} \left(\frac{|h|}{|x-y|} \right)^{\eta}, \tag{35}$$

when $|h| \leq \frac{1}{2}|x-y|$.

PROOF. – Using $e^{-tL^{\sharp}} = e^{-(t/2)L^{\sharp}} e^{-(t/2)L^{\sharp}}$, we write

$$K_{kl}(x, y) = 2 \int_0^{+\infty} \left(\int_{\mathbb{R}^n} M^k_{t/2}(x, z) \, \widetilde{M}^l_{t/2}(z, y) \, dz \right) dt .$$

The proof follows from the preceding lemma and routine computations. ■

PROPOSITION 7. – *$\mathcal{T}_{kl}$ is a Calderón-Zygmund operator on $\mathbb{R}^n$: for all $p \in (1, +\infty)$, there exists a constant C depending only on n, δ and p such that*

$$\|\mathcal{T}_{kl} f\|_p \leq C \|f\|_p$$

for $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

PROOF. – The argument is divided into four cases: we apply the T(1) theorem to obtain L^2 boundedness. Then the L^p estimate is a classical consequence of Proposition 6. In this argument, the spaces are defined on $\mathbb{R}^n$.

* *Case* 1: $1 \leqslant k,\, l \leqslant n-1$.

Since $X_k = \dfrac{\partial}{\partial x_k}$ and $\widetilde{X}_l = \dfrac{\partial}{\partial x_l}$, we have that $\mathcal{T}_{kl}(1) = \mathcal{T}^*_{kl}(1) = 0$. Thus $\mathcal{T}_{kl}$ is bounded by using the T(1) theorem (we recall the reader that the weak boundedness property follows from $\mathcal{T}_{kl}(1) = 0$ or $\mathcal{T}^*_{kl}(1) = 0$ by standard arguments [10]).

* *Case* 2: $k = n$ and $1 \leqslant l \leqslant n-1$.

We have

$$\mathcal{T}_{n,\,l} = X_n (L^\sharp)^{-1}\, {}^t\widetilde{X}_l = -\int_0^\infty \left(b_{n1} \frac{\partial}{\partial x_1} + \ldots + b_{nn} \frac{\partial}{\partial x_n} \right) e^{-tL^\sharp} \frac{\partial}{\partial x_l}\, dt$$

and thus $\mathcal{T}_{n,\,l}(1) = 0$ (and $\mathcal{T}_{n,\,l}$ has the weak boundedness property). On the other hand

$$\mathcal{T}^*_{n,\,l} = -\int_0^\infty \frac{\partial}{\partial x_l} e^{-t(L^\sharp)^*} \left(\frac{\partial}{\partial x_1} \overline{b_{n1}} + \ldots + \frac{\partial}{\partial x_n} \overline{b_{nn}} \right) dt\,.$$

Since b_{nn} is constant we have

$$\mathcal{T}^*_{n,\,l}(1) = \sum_{j=1}^{n-1} \mathcal{T}^*_{j,\,l}(\overline{b_{nj}})\,.$$

By the first case and Calderón-Zygmund theory we know that, for $1 \leqslant j \leqslant n-1$, the operators $\mathcal{T}^*_{j,\,l}$ extends boundedly from L^∞ to BMO. Hence $\mathcal{T}^*_{n,\,l}(1) \in BMO$. Thus $\mathcal{T}_{n,\,l}$ is bounded by invoking again the T(1) theorem.

* *Case* 3: $1 \leqslant k \leqslant n-1$ and $l = n$.

We have

$$\mathcal{T}_{k,\,n} = -\int_0^\infty \frac{\partial}{\partial x_k} e^{-tL^\sharp} \left(\frac{\partial}{\partial x_1} b_{1n} + \ldots + \frac{\partial}{\partial x_n} b_{nn} \right) dt\,.$$

Again $\mathcal{T}^*_{k,\,n}(1) = 0$ and since b_{nn} is constant

$$\mathcal{T}_{k,\,n}(1) = \sum_{j=1}^{n-1} \mathcal{T}_{k,\,j}(b_{jn})\,.$$

As for the case 2, we have that $\mathcal{T}_{k,\,n}(1) \in BMO$. Hence $\mathcal{T}_{k,\,n}$ is bounded by the T(1) theorem.

* *Case* 4: $k = l = n$.

We have

$$\mathcal{T}_{n,n} = -\int_0^\infty \left(b_{n1} \frac{\partial}{\partial x_1} + \ldots + b_{nn} \frac{\partial}{\partial x_n} \right) e^{-tL^\natural} \left(\frac{\partial}{\partial x_1} b_{1n} + \ldots + \frac{\partial}{\partial x_n} b_{nn} \right) dt$$

and

$$\mathcal{T}_{n,n}(1) = \sum_{j=1}^{n-1} \mathcal{T}_{n,j}(b_{jn}).$$

For the same reasons above, we have that $\mathcal{T}_{n,n}(1) \in BMO$. The same holds for $\mathcal{T}^*_{n,n}$:

$$\mathcal{T}^*_{n,n}(1) = \sum_{j=1}^{n-1} \mathcal{T}_{j,n}(\overline{b_{nj}}),$$

and $\mathcal{T}^*_{n,n}(1) \in BMO$.

It remains to prove the weak boundedness property for $\mathcal{T}_{n,n}$. We have

$$\mathcal{T}_{n,n} = -\int_0^\infty b_{nn} \frac{\partial}{\partial x_n} e^{-tL^\natural} \left(\frac{\partial}{\partial x_1} b_{1n} + \ldots + \frac{\partial}{\partial x_n} b_{nn} \right) dt + \mathcal{R}_n,$$

where

$$\mathcal{R}_n = \sum_{j=1}^{n-1} b_{nj} \mathcal{T}_{j,n}.$$

We have that $\mathcal{R}_n$ is bounded on L^2 since the $\mathcal{T}_{j,n}$ are bounded on L^2 and $b_{nj} \in L^\infty$. Let B be a ball (with center x_0 and radius r) and $\varphi, \psi \in \mathcal{C}^1$ compactly supported in B. We have

$$\begin{aligned} |(\mathcal{R}_n \varphi, \psi)| &\leq \|\mathcal{R}_n\|_{2,2} \|\varphi\|_2 \|\psi\|_2 \\ &\leq C_n r^n \|\mathcal{R}_n\|_{2,2} \|\varphi\|_\infty \|\psi\|_\infty . \end{aligned}$$

On the other hand, since

$$((\mathcal{T}_{n,n} - \mathcal{R}_n)\varphi, \psi) = b_{nn} \int_0^\infty \left(e^{-tL^\natural}\, {}^t\widetilde{X}_n \varphi, \frac{\partial}{\partial x_n} \psi \right) dt$$

and taking into consideration the estimates on the kernel of $t^{1/2} e^{-tL^\natural}\, {}^t\widetilde{X}_n$ and the fact that

$$\int_0^{+\infty} \frac{1}{t^{(n+1)/2}} e^{-(|x-y|^2/t)}\, dt \leqslant \frac{C}{|x-y|^{n-1}},$$

it then follows

$$\begin{aligned}\left|((\mathcal{T}_{n,n} - \mathcal{R}_n)\,\varphi, \psi)\right| &\leqslant C|B|\,\|\varphi\|_{L^\infty(B)} \left\| \frac{1}{|x-y|^{n-1}} \right\|_{L^1_y(B)} \left\| \frac{\partial \psi}{\partial x_n} \right\|_{L^\infty(B)} \\ &\leqslant C r^{n+1} \|\varphi\|_\infty \left\| \frac{\partial \psi}{\partial x_n} \right\|_\infty,\end{aligned}$$

Hence

$$|(\mathcal{T}_{n,n}\varphi, \psi)| \leqslant C r^n \|\varphi\|_\infty (\|\psi\|_\infty + r\|\nabla\psi\|_\infty).$$

This shows the weak boundedness property for $\mathcal{T}_{nn}$ and ends the proof of Proposition 7. ■

If $\Gamma_{A^\sharp}$ denotes the fundamental solution for the operator $L^\sharp = -\mathrm{div}\,(A^\sharp \nabla)$ then the Green and the Neumann functions are respectively given by

$$G_A(x, y) = \Gamma_{A^\sharp}(x, y) - \Gamma_{A^\sharp}(x, y^*), \tag{36}$$

and

$$N_A(x, y) = \Gamma_{A^\sharp}(x, y) + \Gamma_{A^\sharp}(x, y^*). \tag{37}$$

Let $X = (X_1, \dots, X_n)$, $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_n)$ and define for $x \in \mathbb{R}^n$

$$J(x) = J = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{n,n-1} & b_{nn} \end{bmatrix},$$

$$\tilde{J}(x) = \tilde{J} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ b_{1n} & b_{2n} & b_{3n} & \cdots & b_{n-1,n} & b_{nn} \end{bmatrix},$$

where J and $\tilde{J}$ have bounded coefficients. We have $X = J\nabla$, $\tilde{X} = \tilde{J}\nabla$. Here we understand X, $\tilde{X}$ and ∇ as column vectors. Thus

$$\nabla(L^{\sharp})^{-1}\,{}^t\nabla = \int_0^{+\infty} (J^{-1}X)\, e^{-tL^{\sharp}}({}^t\tilde{X}\,{}^t\tilde{J}^{-1})\, dt\,,$$

Let $\mathcal{K}(x, y)$ be the distribution kernel of $\mathcal{T} = \int_0^{+\infty} X e^{-tL^{\sharp}}\,{}^t\tilde{X}\, dt$. Note that $J(x)$ and $\tilde{J}(x)$ are constant on $\mathbb{R}^n_+$ and on $\mathbb{R}^n_- = \mathbb{R}^n \setminus \overline{\mathbb{R}^n_+}$. Thus

$$-\nabla_x \nabla_y \Gamma_{A^{\sharp}}(x, y) = J^{-1}(x)\, \mathcal{K}(x, y)\, {}^t\tilde{J}^{-1}(y)\,, \tag{38}$$

where the equality holds in $\mathcal{D}'(\mathbb{R}^n_{\pm} \times \mathbb{R}^n_{\pm})$. If $\mathcal{H}_A = \mathcal{G}_A$ or $\mathcal{N}_A$, we have for all $\boldsymbol{g}$, $\boldsymbol{h} \in \mathcal{D}(\mathbb{R}^n_+)$, valued in $\mathbb{C}^n$,

$$(\mathcal{H}_A \boldsymbol{g}, \boldsymbol{h}) = -\iint_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} \nabla_y(\Gamma_{A^{\sharp}}(x, y) \pm \Gamma_{A^{\sharp}}(x, y^*)).\, \boldsymbol{g}(y)\, \operatorname{div} \boldsymbol{h}(x)\, dx\, dy$$

where $-$ (resp. $+$) occurs when $\mathcal{H}_A = \mathcal{G}_A$ (resp. $\mathcal{H}_A = \mathcal{N}_A$). It follows from (38) that

$$(\mathcal{H}_A \boldsymbol{g}, \boldsymbol{h}) = (\mathcal{T}\,{}^t\tilde{J}^{-1}\boldsymbol{g}, {}^tJ^{-1}\boldsymbol{h}) \pm (\mathcal{T}\,{}^t\tilde{J}^{-1}\Theta\boldsymbol{g}, {}^tJ^{-1}\boldsymbol{h})$$

where $\Theta\boldsymbol{g}(x) = \mathcal{S}\boldsymbol{g}(\mathcal{S}x)$ and $\mathcal{S}$ is defined in (31). Hence, by Proposition 7 we obtain

$$|(\mathcal{H}_A \boldsymbol{g}, \boldsymbol{h})| \leqslant C \|\boldsymbol{g}\|_{L^p(\mathbb{R}^n_+)} \|\boldsymbol{h}\|_{L^{p'}(\mathbb{R}^n_+)}.$$

This proves Theorem 3.

REMARK. – Using the fact that $A^{\sharp}$, J and $\tilde{J}$ are constant on $\mathbb{R}^n_+$ and on $\mathbb{R}^n_-$, this formalism can be used to obtain $|D^{\alpha}\Gamma_{A^{\sharp}}(x, y)| \leqslant \dfrac{C}{|x-y|^{n-1+|\alpha|}}$, with $(x, y) \in \mathbb{R}^n_{\pm} \times \mathbb{R}^n_{\pm}$ and D^{α} is any derivative in x and y. This would give another proof of Proposition 2.

2. – Variations on commutator results.

Let (X, d, μ) be a space of homogeneous type with $\mu(X) = +\infty$. In our application X is the half-space $\mathbb{R}^n_+$.

Let T be a bounded operator on some $L^{\varrho}(X)$, $1 < \varrho < \infty$, with norm bounded by 1 and assume that T is associated with a kernel $k(x, y)$ in the sense that

$$Tf(x) = \int_X k(x, y)\, f(y)\, d\mu(y) \qquad \text{for a.e. } x \notin \operatorname{supp} f,\ f \in L^{\varrho}(X)\,, \tag{39}$$

and that k satisfies $|k(x, y)| \leqslant d(x, y)^{-n}$ and for some $\nu > 0$,

$$|k(x, y) - k(x', y)| \leqslant \frac{d(x, x')^{\nu}}{d(x, y)^{n+\nu}} \tag{40}$$

for every $x, x', y \in X$ such that $\frac{d(x, x')}{d(x, y)} \leqslant \frac{1}{2}$. Here n is the homogeneous dimension.

PROPOSITION 8. – *Let T and k be as above. Then, there exists C_0 such that for all $r > \varrho$ and all ball B in X if $g = T((a - a_{2B}) f)$ with $f \in L^r(X)$,* supp f *compact, and $x \in B$,*

$$\inf_{c \in \mathbb{C}} \frac{1}{|B|} \int_B |g - c| \, d\mu \leqslant C_0 \|a\|_* (M(|f|^r)(x))^{1/r}.$$

Here, M is the maximal operator

$$Mh(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |h(y)| \, d\mu(y).$$

This is classical and we include a proof adapted from [5], Theorem 3.19, for the reader's convenience.

PROOF. – Write $g = g_1 + g_2$ where $g_1 = T((a - a_{2B}) \chi_{2B} f)$ and $g_2 = T((a - a_{2B}) \chi_{^c(2B)} f)$. We have

$$\frac{1}{|B|} \int_B |g_1| \, d\mu \leqslant \left(\frac{1}{|B|} \int_B |g_1|^{\varrho} \, d\mu \right)^{1/\varrho} \leqslant \left(\frac{1}{|B|} \int_{2B} |a - a_{2B}|^{\varrho} \, |f|^{\varrho} \, d\mu \right)^{1/\varrho}$$

$$\leqslant C \|a\|_* \left(\frac{1}{|2B|} \int_{2B} |f|^r \, d\mu \right)^{1/r}$$

by the boundedness of T, John-Nirenberg inequality and Hölder inequality.

Next, set $c_B = \int_{y \notin 2B} k(x_B, y)(a(y) - a_{2B}) f(y) \, d\mu(y)$ where x_B is the center of B (c_B exists by using the pointwise estimate on k and the assumption on the support of f). Then

$$|g_2(x) - c_B| \leqslant \int_{y \notin 2B} \frac{d(x, x_B)^{\nu}}{d(x, y)^{n+\nu}} |a(y) - a_{2B}| \, |f(y)| \, d\mu(y),$$

so that using $d(x, x_B) \leqslant R$ (=radius of B) and $d(x, y) \geqslant c_1 d(x_B, y)$

$$\frac{1}{|B|} \int_B |g_2(x) - c_B| \, d\mu(x) \leqslant c r^{\nu} \int_{d(y, x_B) \geqslant 2R} \frac{1}{d(y, x_B)^{n+\nu}} |a(y) - a_{2B}| \, |f(y)| \, d\mu(y).$$

Breaking the integral in rings $2^k R \leqslant d(y, x_B) \leqslant 2^{k+1} R$, $k \geqslant 1$, and using

$$\frac{1}{\mu(2^k B)} \int_{2^k B} |a - a_{2B}| \, d\mu \leqslant c \ln(k+1) \|a\|_*$$

we obtain

$$\frac{1}{|B|} \int_B |g_2(x) - c_B| \, d\mu(x) \leqslant c \sum_{k=1}^{\infty} \ln(k+1) \|a\|_* 2^{-k\nu} \frac{1}{\mu(2^{k+1} B)} \int_{2^{k+1} B} |f(y)| \, d\mu(y)$$

so that

$$\frac{1}{\mu(B)} \int_B |g_2 - c_B| \, d\mu \leqslant c \|a\|_* Mf(x),$$

for all $x \in B$ and Proposition 8 follows.

3. – A priori estimates.

Let $B_\sigma^+ = B_\sigma \cap \mathbb{R}_+^n$, where B_σ is a ball of radius σ in $\mathbb{R}^n$. Let $p > 2$, u, $\boldsymbol{f}$, f_0 be compactly supported in $B_\sigma \cap \overline{\mathbb{R}_+^n}$ with $u \in W^{1,p}(B_\sigma^+)$, $\boldsymbol{f} \in [L^p(B_\sigma^+)]^n$ and $f_0 \in L^{p_*}(B_\sigma^+)$, $p_* = \dfrac{np}{n+p} > 1$. Assume A satisfy (2) in $\mathbb{R}_+^n$ with, in addition, $A \in \mathrm{BUC}(\mathbb{R}_+^n)$. Suppose that

$$\begin{cases} \operatorname{div} A \nabla u = -\operatorname{div} \boldsymbol{f} + f_0 & \text{in } \mathbb{R}_+^n \\ u = 0 & \text{on } \partial \mathbb{R}_+^n \end{cases}$$

in the sense that for all $\varphi \in \mathrm{Lip}(\mathbb{R}_+^n)$, $\operatorname{Supp} \varphi \subset \mathbb{R}_+^n$,

$$\int_{\mathbb{R}_+^n} A \nabla u \nabla \varphi = \int_{\mathbb{R}_+^n} \boldsymbol{f} \nabla \varphi + \int_{\mathbb{R}_+^n} f_0 \varphi .$$

We claim that

$$\|\nabla u\|_{L^p(B_\sigma^+)} \leqslant C \|A\|_* \|\nabla u\|_{L^p(B_\sigma^+)} + C \|\boldsymbol{f}\|_{L^p(B_\sigma^+)} + C \|f_0\|_{L^{p_*}(B_\sigma^+)}, \tag{41}$$

with C depending only on p, n and δ.

Note that if $B_\sigma \subset \mathbb{R}_+^n$ we obtain an interior estimate, while if $B_\sigma \cap \partial \mathbb{R}_+^n \neq \emptyset$ we obtain a boundary estimate. We do them in the same flow.

Let B be a ball in $\mathbb{R}^n$ with center in $\mathbb{R}_+^n$ and $B^+ = B \cap \mathbb{R}_+^n$ (those B^+ are the balls B used in section 2 on the space $X = \mathbb{R}_+^n$). If A_{2B^+} is the mean of A over

$2B^+$, we have

$$\begin{cases} -\operatorname{div} A_{2B^+} \nabla u = -\operatorname{div} \boldsymbol{f} - \operatorname{div}((A_{2B^+} - A)\nabla u) + f_0 & \text{in } \mathbb{R}^n_+ \\ u = 0 & \text{on } \partial\mathbb{R}^n_+ . \end{cases}$$

Thus for all $x \in \mathbb{R}^n_+$, if $G = G_{A_{2B^+}}$,

$$u(x) = \int_{\mathbb{R}^n_+} \nabla_y G(x, y)\, \boldsymbol{f}(y)\, dy + \int_{\mathbb{R}^n_+} \nabla_y G(x, y)(A_{2B^+} - A(y))\, \nabla u(y)\, dy + \int_{\mathbb{R}^n_+} G(x, y)\, f_0(y)\, dy .$$

For $\boldsymbol{h} \in L^p(\mathbb{R}^n_+)$ with compact support and since $|\nabla_y G(x, y)| \leqslant \frac{C}{|x-y|^{n-1}}$, we have that $x \mapsto \int_{\mathbb{R}^n_+} \nabla_y G(x, y)\, \boldsymbol{h}(y)\, dy \in L^1_{\text{loc}}(\mathbb{R}^n_+)$. Also, since $|G(x, y)| \leqslant \frac{C}{|x-y|^{n-2}}$ (or $C|\ln|x-y||$ if $n = 2$), we have $x \mapsto \int_{\mathbb{R}^n_+} G(x, y)\, f_0(y)\, dy \in L^1_{\text{loc}}(\mathbb{R}^n_+)$. Taking derivatives in $\mathcal{D}'(\mathbb{R}^n_+)$ and using the bounded extension of $\mathcal{G} = \mathcal{G}_{A_{2B^+}}$ (see Theorem 3) to $L^p(\mathbb{R}^n_+)$, we have

$$\nabla u(x) = \mathcal{G}\boldsymbol{f}(x) + \mathcal{G}((A_{2B^+} - A)\nabla u)(x) + I_D f_0(x) \qquad \text{in } \mathcal{D}'(\mathbb{R}^n_+).$$

This is where it is convenient to have $A \in \mathrm{BUC}(\mathbb{R}^n_+)$ since the term $\mathcal{G}((A_{2B^+} - A)\nabla u)$ is defined as $(A_{2B^+} - A)\nabla u \in L^p(\mathbb{R}^n_+)$. Here the integral $(I_D h)(x) = \int_{\mathbb{R}^n_+} \nabla_x G(x, y)\, h(y)\, dy$ defines a bounded operator from $L^{p_*}(\mathbb{R}^n_+)$ into $L^p(\mathbb{R}^n_+)$ since $1 < p_*$ and $p < +\infty$ using (17) with $|\alpha| = 1$. By Theorem 3

$$\|\nabla u\|_p \leqslant C\|\boldsymbol{f}\|_p + \|\mathcal{G}((A_{2B^+} - A)\nabla u)\|_p + C\|f_0\|_{p_*}, \tag{42}$$

where C depends only on p, n and δ (note that the ellipticity constant of A_{2B^+} is uniformly controlled by the one of A, hence the estimate does not depend on the choice of B). By Proposition 8, if $g = \mathcal{G}((A_{2B^+} - A)\nabla u)$ and $1 < r < p$, for all $x \in B^+$,

$$\inf_{c \in \mathbb{C}^n} \frac{1}{|B^+|} \int_{B^+} |g - c| \leqslant C\|A\|_* (M_+(|\nabla u|^r)(x))^{1/r}.$$

Here we used $\|A\|_* = \sup_{B^+} \frac{1}{|B^+|} \int_{B^+} |A - A_{B^+}|$ where the supremum is taken over all $B^+ = B \cap \mathbb{R}^n_+$, B being an Euclidean ball with center in $\mathbb{R}^n_+$. [It is easy to see that this norm is equivalent to the usual norm defined in the introduction, where the balls B are contained in $\mathbb{R}^n_+$]. Also $M_+(h)(x) = \sup_{B^+ \ni x} \frac{1}{|B^+|} \int_{B^+} |h|$.

Fixing x and taking the supremum over $B^+ \ni x$ leads to

$$g^\sharp(x) \leqslant C\|A\|_* (M_+(|\nabla u|^r)(x))^{1/r},$$

where $g^\sharp$ is the Fefferman-Stein sharp function on $\mathbb{R}^n_+$. Using $\|g\|_p \leqslant C_p \|g^\sharp\|_p$ and the Hardy-Littlewood theorem, since $r < p$, yields $\|g\|_p \leqslant C\|A\|_* \|\nabla u\|_p$. This proves (41).

Next, let us consider the Neumann problem. With the same hypotheses on u, $\boldsymbol{f}$, f_0 and A, we assume

$$\begin{cases} -\operatorname{div} A \nabla u = -\operatorname{div} \boldsymbol{f} + f_0 & \text{in } \mathbb{R}^n_+ \\ \nu . A \nabla u = \nu . f & \text{on } \partial\mathbb{R}^n_+ . \end{cases}$$

Again this is interpreted in the variational sense. Then we have

$$\|\nabla u\|_{L^p(B_\sigma^+)} \leqslant C\|A\|_* \|\nabla u\|_{L^p(B_\sigma^+)} + C\|\boldsymbol{f}\|_{L^p(B_\sigma^+)} + C\|f_0\|_{L^{p_*}(B_\sigma^+)}, \tag{43}$$

with C depending only on p, n and δ. The argument is entirely similar to the preceding one and is skipped.

4. – Proof of Theorem 1.

Now, we wish to prove Theorem 1 in its full generality. We only consider the existence issue when $p > 2$. Indeed, uniqueness (modulo constant) follows easily from the $p = 2$ case. Then a duality argument ends the proof when $p < 2$. We henceforth suppose that $p > 2$.

First assume that Ω has C^∞ boundary, that $A \in C^\infty(\Omega)$ and that $\boldsymbol{f} \in C_0^\infty(\Omega)$. Let us consider first the Neumann problem. By Lax-Milgram lemma, we have a solution $u \in W^{1,2}(\Omega)$ with $\int_\Omega u = 0$ such that

$$\int_\Omega A \nabla u \nabla v = \int_\Omega \boldsymbol{f} \nabla v , \qquad \forall\, v \in \operatorname{Lip}(\overline{\Omega}) .$$

Classical interior and boundary elliptic regularity tells us that $u \in C^\infty(\overline{\Omega})$. Hence $\nabla u \in L^p(\Omega)$ but we wish to prove

$$\|\nabla u\|_{L^p(\Omega)} \leqslant C\|\boldsymbol{f}\|_{L^p(\Omega)} \tag{44}$$

with C depending only on n, δ, p, the *VMO* modulus of continuity of A, $|\Omega|$ and the C^1 modulus of continuity of $\partial\Omega$ which we can define as the modulus of continuity of the outward unit normal on $\partial\Omega$.

Fix $\theta \in C_0^\infty(B_\sigma)$ where B_σ is a ball in $\mathbb{R}^n$. We get

$$\int_{\Omega \cap B_\sigma} \tilde{A} \nabla(u\theta) \nabla v = \int_{\Omega \cap B_\sigma} \tilde{\boldsymbol{f}} \nabla v + \int_{\Omega \cap B_\sigma} \tilde{f}_0 v , \qquad \forall v \in \operatorname{Lip}(\overline{\Omega}), \tag{45}$$

where $\tilde{\boldsymbol{f}} = \theta\boldsymbol{f} + A\nabla\theta u$ and $\tilde{f}_0 = \boldsymbol{f}\nabla\theta - A\nabla u\nabla\theta$ and $\tilde{A} = A\chi + I(1-\chi)$ where $\chi \in C_0^\infty(B_{2\sigma})$ and $\chi \equiv 1$ on supp θ. Note that $\tilde{A} \in C^\infty(\Omega)$ and if σ is chosen small enough we have that $\|\tilde{A}\|_*$ is small, as it is controlled by the *VMO* modulus of continuity of A.

If $B_\sigma \subset \Omega$ then we can consider B_σ in a half-space so that either a priori estimate (41) or (43) yields

$$\|\nabla(u\theta)\|_{L^p(B_\sigma)} \leqslant C(\|\tilde{\boldsymbol{f}}\|_{L^p(B_\sigma)} + \|\tilde{f}_0\|_{L^{p_*}(B_\sigma)}) \tag{46}$$

and C depends uniquely on p, n, δ and $\|\tilde{A}\|_*$.

If $B_\sigma \cap \partial\Omega \neq \emptyset$ with σ small enough, then we use a local C^∞ chart $\varphi : \mathbb{R}^n_+ \to \Omega$ to flatten the boundary. Then one can pull back the a priori estimate (43) to $(u\theta)\circ\varphi, \tilde{\boldsymbol{f}}\circ\varphi, f_0\circ\varphi$ and $\tilde{A}$ on Ω replaced by another C^∞ matrix $\tilde{\tilde{A}}$ on $\mathbb{R}^n_+$ and the important point is that $\|\tilde{\tilde{A}}\|_* \leqslant C\|\tilde{A}\|_*$ where C depends on dimension and on the C^1 modulus of continuity of $\partial\Omega$ (this is because $\mathrm{BUC}\times(VMO \cap L^\infty) \subset (VMO \cap L^\infty)$). Thus we also have (46) with C depending on p, n, δ, the *VMO* modulus of continuity of A and the C^1 modulus of continuity of $\partial\Omega$.

From here, it suffices to iterate as in [11], p. 416, to obtain

$$\|\nabla u\|_{L^p(\Omega)} \leqslant C(\|\nabla u\|_{L^2(\Omega)} + \|\boldsymbol{f}\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}).$$

Poincaré-Sobolev inequality (valid for connected lipschitz sets, if $\int_\Omega u = 0$) yields

$$\|u\|_{L^p(\Omega)} \leqslant C\|\nabla u\|_{L^{p_*}(\Omega)}$$

and a second iteration yields

$$\|\nabla u\|_{L^p(\Omega)} \leqslant C(\|\nabla u\|_{L^2(\Omega)} + \|\boldsymbol{f}\|_{L^p(\Omega)}).$$

Lastly the L^2 theory yields

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C\|\boldsymbol{f}\|_{L^2(\Omega)} \leqslant C|\Omega|^{1/2-1/p}\|\boldsymbol{f}\|_{L^p(\Omega)}$$

and (44) follows with the expected behavior of C.

The proof for the Dirichlet problem is entirely similar and skipped.

It remains to remove the a priori assumptions on Ω, A, f. Assume that A and Ω satisfy (2) and (3) and that $\boldsymbol{f} \in L^p(\Omega)$.

Dirichlet problem: Let $A_k \in C^\infty(\Omega)$ satisfying (2) uniformly such that $A_k \to A$ *a.e.* on Ω and $A_k \to A$ in $BMO(\Omega)$ and $\boldsymbol{f}_k \in C_0^\infty(\Omega)$ with $\boldsymbol{f}_k \to \boldsymbol{f}$ in $L^p(\Omega)$. Choose also an increasing sequence of C^∞ subdomains Ω_k converging to Ω, with C^1 modulus of continuity of $\partial\Omega_k$ uniform in k, that is $\sup_k \omega_{\Omega_k} \leqslant C\omega_\Omega$ (this can be done using a regularized distance function to $\partial\Omega$). One can arrange supp $\boldsymbol{f}_k \subset$

Ω_k. Let $u_k \in C^\infty(\Omega_k)$ be the solution of $u_k = 0$ on $\partial\Omega_k$

$$\int_{\Omega_k} A_k \nabla u_k \nabla\varphi = \int_{\Omega_k} \boldsymbol{f}_k \nabla\varphi$$

for all $\varphi \in C_0^\infty(\Omega_k)$. Extend u_k to be 0 outside of Ω_k. Then, using our a priori estimate on Ω_k,

$$\|\nabla u_k\|_{L^p(\Omega)} \leqslant \|\nabla u_k\|_{L^p(\Omega_k)} \leqslant C\|\boldsymbol{f}_k\|_{L^p(\Omega_k)} \leqslant C \sup_k \|\boldsymbol{f}_k\|_{L^p(\Omega)} \leqslant C\|\boldsymbol{f}\|_{L^p(\Omega)}.$$

Thus (u_k) has a weakly converging subsequence in $W_0^{1,p}(\Omega)$. In particular, there exists $u \in W_0^{1,p}(\Omega)$ such that, up to extraction, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$ and we obtain

$$\int_\Omega A\nabla u \nabla\varphi = \int_\Omega \boldsymbol{f}\nabla\varphi , \qquad \forall\varphi \in C_0^\infty(\Omega).$$

This shows the existence part of Theorem 1 in the case $p > 2$ for the Dirichlet problem.

Neumann problem: Pick $\boldsymbol{f}_k \in C_0^\infty(\Omega)$ with $\boldsymbol{f}_k \to \boldsymbol{f}$ in $L^p(\Omega)$ and $A_k \in C^\infty(\mathbb{R}^n)$ with $\sup_k \|A_k\|_{L^\infty(\mathbb{R}^n)} \leqslant 2\delta$, A_k uniformly elliptic on $\mathbb{R}^n$, with ellipticity constant $\frac{\delta}{2}$, $A_k \to A$ *a.e.* on Ω and $A_k \to A$ in $BMO(\Omega)$. Now let (Ω_k) be a decreasing sequence of C^∞ domains converging to Ω with C^1 modulus of continuity of $\partial\Omega_k$ uniform in k.

Let $u_k \in W^{1,2}(\Omega_k)$ with $\int_\Omega u_k = 0$ be the unique solution of

$$\int_{\Omega_k} A_k \nabla u_k \nabla\varphi = \int_{\Omega_k} \boldsymbol{f}_k \nabla\varphi \qquad \forall\varphi \in C^\infty(\overline{\Omega_k}).$$

Since $\Omega \subset \Omega_k$ and supp $\boldsymbol{f}_k \subset \Omega$,

$$\|\nabla u_k\|_{L^p(\Omega)} \leqslant \|\nabla u_k\|_{L^p(\Omega_k)} \leqslant C\|\boldsymbol{f}_k\|_{L^p(\Omega_k)} \leqslant C \sup_k \|\boldsymbol{f}_k\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}$$

with C uniform in k. Thus (u_k) has a weakly converging subsequence in $W^{1,p}(\Omega)$ and there exists $u \in W^{1,p}(\Omega)$ such that, up to extraction, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$. Now, for all $\varphi \in C^\infty(\overline{\Omega})$, write

$$\int_{\Omega_k} A_k \nabla u_k \nabla\varphi - \int_\Omega A\nabla u \nabla\varphi = \int_\Omega (A_k - A) \nabla u_k \nabla\varphi + \int_\Omega A(\nabla u_k - \nabla u) \nabla\varphi + \int_{\Omega_k\setminus\Omega} A_k \nabla u_k \nabla\varphi .$$

The first term tends to 0 by dominated convergence since $\left|\int_\Omega (A_k - A) \nabla u_k \nabla\varphi\right| \leqslant \|\nabla u_k\|_{L^p(\Omega)} \|(A_k - A) \nabla\varphi\|_{L^{p'}(\Omega)}$, the second by weak convergence and the last by $\|A_k \nabla u_k\|_{L^p(\Omega_k)} \leqslant 2\delta \sup_k \|\nabla u_k\|_{L^p(\Omega)}$ and $\|\nabla\varphi\|_{L^{p'}(\Omega_k\setminus\Omega)} \to 0$. Since

$\int_{\Omega_k} \boldsymbol{f}_k \nabla \varphi$ tends to $\int_{\Omega} \boldsymbol{f} \nabla \varphi$, the proof of Theorem 1 for the existence of a solution when $p > 2$ for the Neumann problem is finished. Taking into account the starting comments of this section Theorem 1 has been completely proved.

5. – Concluding remark.

One cannot take Ω with arbitrary lipschitz boundary. Indeed, already for $L = -\Delta$, $W^{1,p}$ estimates for both (1) and (4) are restricted to $p < p_0$ for some $p_0 < +\infty$ ([17], [19]). However, it is plausible that p_0 tends to $+\infty$ with the lipschitz constant of $\partial\Omega$ tending to 0 (*i.e.* $\partial\Omega$ tends to a C^1 boundary), whenever $A \in VMO(\Omega)$. Also for a given p, one can replace the hypothesis «$A \in VMO(\Omega)$» by «the distance of A to $VMO(\Omega)$ in $BMO(\Omega)$ being small enough depending on the value of p» (see [8] where this is observed for non divergence equations).

REFERENCES

[1] J. M. ANGELETTI - S. MAZET - PH. TCHAMITCHIAN, *Analysis of second order elliptic operators without boundary conditions and with VMO or Hölderian coefficients*. In *Multiscale wavelet methods for partial differential equations*, pages 495-539. Academic Press, San Diego, CA, 1997.

[2] P. AUSCHER - PH. TCHAMITCHIAN, *Square root problem for divergence operators and related topics*, volume 249 of Astérisque, Soc. Math. France, 1998.

[3] P. AUSCHER - PH. TCHAMITCHIAN, *On square roots of elliptic second order divergence operators on strongly lipschitz domains: L^p theory*, to appear in Math. Annalen.

[4] M. BRAMANTI - L. BRANDOLINI, *L^p estimates for nonvariational hypoelliptic operators with VMO coefficients*, Trans. Amer. Math. Soc., **352** (2) (2000), 781-822.

[5] F. CHIARENZA, *L^p regularity for systems of PDEs, with coefficients in VMO*. In *Nonlinear analysis, function spaces and applications*, Vol. **5** (Prague, 1994), pages 1-32, Prometheus, Prague, 1994.

[6] F. CHIARENZA - M. FRANCIOSI - M. FRASCA, *L^p-estimates for linear elliptic systems with discontinuous coefficients*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., **5** (1) (1994), 27-32.

[7] F. CHIARENZA - M. FRASCA - P. LONGO, *Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat., **40** (1) (1991), 149-168.

[8] F. CHIARENZA - M. FRASCA - P. LONGO, *$W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc., **336** (2) (1993), 841-853.
[9] R. R. COIFMAN - R. ROCHBERG - G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2), **103** (3) (1976), 611-635.
[10] G. DAVID - J. L. JOURNÉ, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. (2), **120** (2) (1984), 371-397.
[11] G. DI FAZIO, *L^p estimates for divergence form elliptic equations with discontinuous coefficients*, Boll. Un. Mat. Ital. A (7), **10** (2) (1996), 409-420.
[12] G. DI FAZIO - D. K. PALAGACHEV, *Oblique derivative problem for elliptic equations in non-divergence form with VMO coefficients*, Comment. Math. Univ. Carolin., **37** (3) (1996), 537-556.
[13] G. DI FAZIO - D. K. PALAGACHEV, *Oblique derivative problem for quasilinear elliptic equations with VMO coefficients*, Bull. Austral. Math. Soc., **53** (3) (1996), 501-513.
[14] E. B. FABES - D. S. JERISON - C. E. KENIG, *Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure*, Ann. of Math. (2), **119** (1) (1984), 121-141.
[15] D. FAN - SH. LU - D. YANG, *Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients*, Georgian Math. J., **5** (5) (1998), 425-440.
[16] T. IWANIEC - C. SBORDONE, *Riesz transforms and elliptic PDEs with VMO coefficients*, J. Anal. Math., **74** (1998), 183-212.
[17] D. JERISON - C. E. KENIG, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal., **130** (1) (1995), 161-219.
[18] A. MAUGERI - D. K. PALAGACHEV - C. VITANZA, *Oblique derivative problem for uniformly elliptic operators with VMO coefficients and applications*, C. R. Acad. Sci. Paris Sér. I Math., **327** (1) (1998), 53-58.
[19] O. MENDEZ - M. MITREA, *Complex powers of the Neumann laplacian in lipschitz domains*, Math. Nach., 1999, to appear.
[20] D. K. PALAGACHEV, *Quasilinear elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc., **347** (7) (1995), 2481-2493.
[21] M. A. RAGUSA, *Dirichlet problem in Morrey spaces for elliptic equations in nondivergence form with VMO coefficients*. In *Proceedings of the Eighth International Colloquium on Differential Equations (Plovdiv, 1997)*, pages 385-390, Utrecht, 1998, VSP.
[22] CH. G. SIMADER. *On Dirichlet's boundary value problem*, Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 268.
[23] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, N.J., 1970, Princeton Mathematical Series, No. 30.

P. Auscher and M. Qafsaoui: LAMFA, CNRS FRE 2270, 33, rue Saint-Leu
F-80039 Amiens Cedex 1
E-mail: pascal.auscher@u-picardie.fr and mahmoud.qafsaoui@u-picardie.fr

Pervenuta in Redazione
il 19 maggio 2000