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Remarks on the Quasiconvex Envelope of Some Functions Depending on Quadratic Forms.

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Sunto. – *In questo lavoro calcoliamo la chiusura quasi convessa di alcune funzioni definite sullo spazio M_{mn} delle matrici reali $m \times n$ attraverso forme quadratiche. I risultati sono applicati ad alcune funzioni relative alla densità di energia elastica di James e Ericksen.*

Summary. – *We compute the quasiconvex envelope of certain functions defined on the space M_{mn} of real $m \times n$ matrices. These functions are basically functions of a quadratic form on M_{mn}. The quasiconvex envelope computation is applied to densities that are related to the James-Ericksen elastic stored energy function.*

1. – Introduction.

We denote by M_{mn} the space of real $m \times n$ matrices. Let W be a function defined on M_{mn} with values in $\mathbb{R}$. Let D be a bounded domain in $\mathbb{R}^n$. We use the Einstein summation convention, unless otherwise specified.

In applications to problems of Continuum Mechanics, the fundamental issue of the Calculus of Variations consists in minimizing such energy functionals as

$$I(u) = \int_D W(\nabla u(x))\, dx\,, \tag{1.1}$$

where u is a mapping from D into $\mathbb{R}^m$ belonging to some subset of an appropriate Sobolev space. In this context, ∇u designates the gradient of u, *i.e.*, the $m \times n$ matrix

$$(\nabla u)_{ij} = \frac{\partial u_i}{\partial x_j}\,,$$

where $u_1, \ldots, u_m$ denote the Cartesian components of u. In applications to nonlinear elasticity, u is a deformation of a body occupying the domain D in its reference configuration, ∇u is the deformation gradient and W is the stored energy function of a hyperelastic material. Naturally, appropriate boundary conditions and loading terms must be added to give rise to a well-posed problem.

As a general rule, the functional I is not weakly lower semicontinuous on the above-mentioned Sobolev space. The direct method of the Calculus of Variations thus does not apply to minimize (1.1). One of the ways of getting around this difficulty is to consider the so-called relaxed problem, which in this case consists in minimizing the energy

$$\bar{I}(u) = \int_D QW(\nabla u(x))\, dx \tag{1.2}$$

where QW denotes the quasiconvex envelope of W, see [5]. Before going any further, let us recall the various convexity notions that are relevant in the vectorial case of the Calculus of Variations, see [5] again.

- Let $\tau(m, n)$ be the number of all minors of an $m \times n$ matrix F and $M(F)$ be the vector of all such minors. A function $W : M_{mn} \to \mathbb{R}$ is said to be *polyconvex* if there exists a convex function $\widehat{W} : \mathbb{R}^{\tau(m, n)} \to \mathbb{R}$ such that

$$\forall F \in M_{mn}, \qquad W(F) = \widehat{W}(M(F)). \tag{1.3}$$

- A function W is said to be *quasiconvex* if

$$\forall F \in M_{mn}, \qquad W(F) \leqslant \frac{1}{\text{meas } D} \int_D W(F + \nabla v(x))\, dx\,, \tag{1.4}$$

for all bounded domains $D \subset \mathbb{R}^n$ and all functions $v \in W_0^{1,\infty}(D; \mathbb{R}^m)$.

- A function W is said to be *rank-1-convex* if, for all couples of matrices (F, G) such that $\operatorname{rank}(F - G) \leqslant 1$ and all $\lambda \in [0, 1]$,

$$W(\lambda F + (1 - \lambda)\, G) \leqslant \lambda W(F) + (1 - \lambda)\, W(G). \tag{1.5}$$

Quasiconvexity was introduced by Morrey [10], [11] as a necessary and sufficient condition for the weak lower semicontinuity of I over Sobolev spaces, under appropriate assumptions of growth and bound below. It is clearly not easy to check in practice. Morrey also proved that rank-1-convexity is a necessary condition for such weak lower semicontinuity. In the case when W is of class C^2, condition (1.5) can be slightly strengthened to become the well-known Legendre-Hadamard, or strong ellipticity condition

$$\forall F \in M_{mn},\ \forall \xi \in \mathbb{R}^n,\ \forall \eta \in \mathbb{R}^m, \qquad \frac{\partial^2 W}{\partial F_{ij}\, \partial F_{kl}}(F)\, \xi_i \xi_k \eta_j \eta_l \geqslant c\, |\xi|^2\, |\eta|^2, \tag{1.6}$$

with $c > 0$. Polyconvexity was introduced by Ball [1] to deal with existence questions in nonlinear elasticity, for which the growth conditions required by Morrey's theorem are not satisfied. In particular, using polyconvexity, certain energy densities W that take the value $+\infty$ become amenable.

It is by now well-known that, in the finite-valued case,

$$W \text{ convex} \Rightarrow W \text{ polyconvex} \Rightarrow W \text{ quasiconvex} \Rightarrow W \text{ rank-1-convex}, \tag{1.7}$$

and that the reverse implications are false in general. The last one,

$$W \text{ rank-1-convex} \not\Rightarrow W \text{ quasiconvex},$$

which had been left standing for a long time, was recently established by Šverák, see [12] for dimensions $m \geqslant 3$ and $n \geqslant 2$. When $m = 1$ or $n = 1$, which is to say in the scalar case, all the above notions are equivalent.

Associated with the above convexity notions are the corresponding convex, polyconvex, quasiconvex and rank-1-convex envelopes defined by

$$\begin{aligned}
CW &= \sup\{Z;\, Z \text{ convex and } Z \leqslant W\},\\
PW &= \sup\{Z;\, Z \text{ polyconvex and } Z \leqslant W\},\\
QW &= \sup\{Z;\, Z \text{ quasiconvex and } Z \leqslant W\},\\
RW &= \sup\{Z;\, Z \text{ rank-1-convex and } Z \leqslant W\}.
\end{aligned}$$

By (1.7), we clearly have

$$CW \leqslant PW \leqslant QW \leqslant RW. \tag{1.8}$$

The four envelopes coincide when RW is convex.

The relationship between the quasiconvex envelope and the relaxed energy functional alluded to above, is that minimizing sequences for the original energy (1.1) weakly converge to minimizers of the relaxed functional (1.2), under appropriate technical assumptions, see [5]. The converse is also true in the sense that all minimizers of (1.2) are weak limits of a minimizing sequence for (1.1). Thus, the computation of the quasiconvex envelope of an energy density W provides information on the asymptotic behavior of minimizing sequences for the corresponding functional.

The goal of this article is to compute the quasiconvex envelope of certain functions W which depend on the gradient through a quadratic form. The explicit computation of the quasiconvex envelope of a given function W is in general a hopeless task, see [6], [7], [8], [9] for some examples in which it is possible to carry it out completely. Indeed, the results we obtain here are rather limited in scope. The main motivation for them is the study of an elastic stored energy function proposed by James and Ericksen to model phase transitions in elastic crystals in dimension two, see for example [4]. The James-Ericksen density is given by

$$W(F) = \widetilde{W}(C) = \kappa_1(\mathrm{tr}\,(C) - 2)^2 + \kappa_2 c_{12}^2 + \kappa_3\left(\left(\frac{c_{11} - c_{22}}{2}\right)^2 - \varepsilon^2\right)^2 \tag{1.9}$$

where

$$C = F^T F = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

is the Cauchy-Green or strain tensor, the nonnegative constants κ_1, κ_2 and κ_3 are elastic moduli and ε is a small parameter. The quasiconvex envelope of the James-Ericksen density was computed in the cases when $\kappa_1 = 0$ or $\kappa_3 = 0$ by Bousselsal and Brighi [3]. We recover their result in the case $\kappa_1 = 0$ as a consequence of more general relaxation results.

The general case $\kappa_1 \kappa_2 \kappa_3 > 0$ seems unfortunately to be out of reach of current methods. We nonetheless succeed in identifying a rather large set of matrices for which the quasiconvex envelope of W vanishes. This set is a region in the deformation gradient space where the relaxed energy is degenerate. If the deformation gradient of a solution of the relaxed minimization problem takes such values in a subset of the domain Ω, this indicates that relaxation is occuring in this subset and that minimizing sequences for the initial minimization problem will develop oscillations and exhibit microstructure in the same subset. After this work was completed, it was brought to our attention that the set we had found was not optimal. It actually is a strict subset of the actual set of matrices for which the quasiconvex envelope vanishes, see [2]. Our method nonetheless provides estimates from above for the quasiconvex envelope of the James-Ericksen density, and is not a priori limited to the 2D or 2D-3D cases.

We conclude the article by giving an example of how the case of functions depending on certain homogeneous functions of degree $p \neq 2$ can be handled along similar lines.

2. – Rank one decompositions and quadratic forms.

Let us first recall the following result, due to Bousselsal and Brighi [3]. The proof is given here for completeness.

THEOREM 2.1. – *Let $F \in M_{mn}$, $\alpha \in \mathbb{R}$ and q a quadratic form on M_{mn} such that $q(F) \leq \alpha$. We assume that there exists two vectors $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ such* *that* $q(a \otimes b) > 0$. *Then there exists* $\lambda \in [0, 1]$ *and* $t \in \mathbb{R}$ *such that if* $E = ta \otimes b$,

$$q(F + \lambda E) = q(F - (1 - \lambda)\, E) = \alpha\,. \tag{2.1}$$

PROOF. – For all $F, E \in M_{mn}$ and $\lambda \in [0, 1]$, we have

$$q(F + \lambda E) = q(F) + 2\lambda\beta(F, E) + \lambda^2 q(E) \tag{2.2}$$

$$q(F - (1 - \lambda)\, E) = q(F) - 2(1 - \lambda)\, \beta(F, E) + (1 - \lambda)^2 q(E) \tag{2.3}$$

where $\beta(\cdot, \cdot)$ is the symmetric bilinear form associated with q.

If $q(F) = \alpha$, we just take $t = 0$. Assume now that $q(F) < \alpha$. We want to solve the system

$$\begin{cases} q(F) + 2\lambda\beta(F, E) + \lambda^2 q(E) = \alpha\,, \\ q(F) - 2(1 - \lambda)\, \beta(F, E) + (1 - \lambda)^2 q(E) = \alpha\,, \end{cases} \tag{2.4}$$

for $t \in \mathbb{R}_-$ and $\lambda \in [0, 1]$.

It is clear that if we let $X_1 = \lambda t$ and $X_2 = (\lambda - 1)t$, then system (2.4) is equivalent to requiring that X_1 and X_2 be roots of the polynomial

$$P(X) = q(a \otimes b)\, X^2 + 2\beta(F, a \otimes b)\, X + q(F) - \alpha\,.$$

Now, since $q(a \otimes b) > 0$ and $q(F) - \alpha < 0$, this polynomial has two real simple roots so that we are left with solving the system

$$\begin{cases} \lambda t = \dfrac{-\beta(F, a \otimes b) - \sqrt{\Delta}}{q(a \otimes b)}\,, \\[2ex] (\lambda - 1)\, t = \dfrac{-\beta(F, a \otimes b) + \sqrt{\Delta}}{q(a \otimes b)}\,, \end{cases}$$

where

$$\Delta = \beta(F, a \otimes b)^2 + (\alpha - q(F))\, q(a \otimes b)\,.$$

It follows readily that

$$\begin{cases} t = -\dfrac{2\sqrt{\Delta}}{q(a \otimes b)} \\[2ex] \lambda = \dfrac{1}{2}\left(1 + \dfrac{\beta(F, a \otimes b)}{\sqrt{\Delta}}\right). \end{cases}$$

It is clear that $\lambda \in [0, 1]$. ■

Let us apply Theorem 2.1 in some examples that will be useful for the study of the James-Ericksen energy. We denote by F^k the k-th column-vector of a $m \times n$ matrix F and by $F^k \cdot F^l$ their scalar product in $\mathbb{R}^m$

$$F^k \cdot F^l = F_{ik} F_{il}.$$

We consider a quadratic form q on M_{mn} that can be expressed as

$$q(F) = s_{kl} F^k \cdot F^l,$$

where $S = (s_{kl})$ is a symmetric $n \times n$ matrix.

COROLLARY 2.2. – *Let $\alpha \in \mathbb{R}$ and $F \in M_{mn}$ be such that $q(F) \neq \alpha$.*

i) If $q(F) < \alpha$ and at least one diagonal coefficient of S is strictly positive, then there exists $\lambda \in [0, 1]$, $A, B \in M_{mn}$ such that $\operatorname{rank}(A - B) \leqslant 1$ *with*

$$F = \lambda A + (1 - \lambda) B \text{ and } q(A) = q(B) = \alpha . \tag{2.5}$$

ii) If $q(F) \neq \alpha$ and either there are two diagonal coefficients of S, s_{kk} and s_{ll} with $k \neq l$, such that $s_{kk} s_{ll} < 0$, or at least one off-diagonal coefficient is nonzero, then there exists $\lambda \in [0, 1]$, $A, B \in M_{mn}$ such that $\operatorname{rank}(A - B) \leqslant 1$ *with*

$$F = \lambda A + (1 - \lambda) B \text{ and } q(A) = q(B) = \alpha . \tag{2.6}$$

In both cases, we have in addition, $A_{ik} = B_{ik} = F_{ik}$ for all $1 \leqslant i \leqslant m - 1$ and $1 \leqslant k \leqslant n$.

PROOF. – i) Let $F \in M_{mn}$ be such that $q(F) < \alpha$ and let s_{kk} be a strictly positive diagonal coefficient of S. We take $a = (a_i) \in \mathbb{R}^m$ and $b = (b_j) \in \mathbb{R}^n$ with $a_i = \delta_{im}$ and $b_j = \delta_{jk}$. Then, $q(a \otimes b) = s_{kk} > 0$ and we can apply Theorem 2 to find $\lambda \in [0, 1]$ and t such that $A = F + \lambda t E$ and $B = F - (1 - \lambda) t E$ meet our requirements, since $E_{ik} = 0$ for all $1 \leqslant i \leqslant m - 1$ and $1 \leqslant k \leqslant n$.

ii) Assume now that $q(F) \neq \alpha$ and either there are two diagonal coefficients of S, s_{kk} and s_{ll} with $k \neq l$, such that $s_{kk} s_{ll} < 0$, or at least one off-diagonal coefficient is nonzero.

If $q(F) < \alpha$, either $s_{kk} > 0$ and we apply case i), or there exist $k \neq l$ such that $s_{kl} \neq 0$. In the latter case, we take $a = (a_i) \in \mathbb{R}^m$ and $b = (b_j) \in \mathbb{R}^n$ with $a_i = \delta_{im}$ and $b_j = \delta_{jk} + \operatorname{sign} s_{kl} \delta_{jl}$. Then, $q(a \otimes b) = |s_{kl}| > 0$ and the conclusion follows as before.

Finally, if $q(F) > \alpha$, we apply the previous step to the quadratic form $-q$ and the value $-\alpha$. ■

REMARK 2.3. – The only quadratic forms of the form above that are not included in the hypotheses of part ii) are those for which the matrix S is diagonal and all its coefficients are of the same sign. ■

A very similar result is as follows. We assume here that $n \leqslant m$.

COROLLARY 2.4. – *Assume that S is such that there exists $k \neq l$ with $s_{kk} s_{ll} < 0$. Let $\alpha \in \mathbb{R}$ and $F \in M_{mn}$ be such that $q(F) \neq \alpha$. Then there exists $\lambda \in [0, 1]$, $A, B \in M_{mn}$ such that* $\operatorname{rank}(A - B) \leqslant 1$ *with*

$$F = \lambda A + (1-\lambda) B \text{ and } q(A) = q(B) = \alpha . \tag{2.7}$$

In addition, $A^j \cdot A^p = B^j \cdot B^p = F^j \cdot F^p$ for all $j \neq p$.

PROOF. – Assume first that $q(F) < \alpha$ and $s_{kk} > 0$. Let G be the vector subspace of $\mathbb{R}^m$ spanned by the vectors F^j for all $j \neq k$. Since $n \leqslant m$, we have $\dim G \leqslant m-1$. We can thus choose $a \in G^{\perp} \setminus \{0\}$ and $b = (b_j)$ with $b_j = \delta_{jk}$, so that $q(a \otimes b) = s_{kk} |a|^2 > 0$. Applying Theorem 2.1, we obtain our matrices A and B. Moreover,

$$\begin{aligned} A^j \cdot A^p &= (F^j + \lambda t \delta_{jk} a) \cdot (F^p + \lambda t \delta_{pk} a) \\ &= F^j \cdot F^p + \lambda t (\delta_{jk} a \cdot F^p + \delta_{pk} a \cdot F^j) + \lambda^2 t^2 |a|^2 \delta_{jk} \delta_{pk}. \end{aligned}$$

First of all, when $p \neq j$, we always have $\delta_{jk} \delta_{pk} = 0$ so that the quadratic term disappears. Secondly, if $j \neq k$ and $p \neq k$, then $\delta_{jk} = \delta_{pk} = 0$. On the other hand, if $j = k$ (and thus $p \neq k$), then $a \cdot F^p = 0$, and if $p = k$ (and thus $j \neq k$), then $a \cdot F^j = 0$. Therefore, we have shown that $A^j \cdot A^p = F^j \cdot F^p$. The same argument applies to B.

In the case when $q(F) > \alpha$, we apply the previous argument to $-q$ and $-\alpha$ by exchanging the roles of k and l. ■

EXAMPLES 2.5. – Keeping the case of the James-Ericksen energy density in mind, we can for example apply Corollary 2.2 for $m = n = 2$ and $q(F) = F^1 \cdot F^2$. This yields rank-1 decomposition matrices A and B such that $A^1 \cdot A^2 = B^1 \cdot B^2 = \alpha$ and $A_{11} = B_{11} = F_{11}$ and $A_{12} = B_{12} = F_{12}$. Similarly, Corollary 2.4 applies for $m = n = 2$ and $q(F) = |F^1|^2 - |F^2|^2$ to obtain a rank-1 decomposition such that $|A^1|^2 - |A^2|^2 = |B^1|^2 - |B^2|^2 = \alpha$ and $A^1 \cdot A^2 = B^1 \cdot B^2 = F^1 \cdot F^2$.

3. – Relaxation results.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a function such that

a) there exists $\alpha > 0$, with $\min\limits_{\mathbb{R}_+} \varphi = \varphi(\alpha)$,

b) $\varphi(t) = \varphi^{**}(t)$ for all $t \geqslant \alpha$. Let $g : M_{m-1, n} \to \mathbb{R}$ be a convex function. For all $F \in M_{mn}$, we denote by $\widetilde{F}$ the $(m-1) \times n$ matrix obtained by erasing the m-th line of F.

THEOREM 3.1. – *Let q be a nonzero nonnegative quadratic form on M_{mn}, of the form $q(F) = s_{kl} F^k \cdot F^l$, and define $W : M_{mn} \to \mathbb{R}$ by $W(F) = \varphi(q(F)) + g(\tilde{F})$. Then*

$$QW(F) = \begin{cases} \varphi(\alpha) + g(\tilde{F}) & \text{if } q(F) \leqslant \alpha\,, \\ \varphi(q(F)) + g(\tilde{F}) & \text{if } q(F) > \alpha\,. \end{cases} \tag{3.1}$$

PROOF. – Since q is nonnegative, it follows that all diagonal coefficients s_{kk} are also nonnegative (take $F^j = 0$ for $j \neq k$ and $(F^k)_i = \delta_{1i}$). Assume for contradiction that they all vanish. Let $s_{kl} \neq 0$ be a nonzero off-diagonal coefficient. If we take $(F^k)_i = \delta_{1i}$, $(F^l)_i = \varepsilon \delta_{1i}$, with $\varepsilon = \pm 1$, and $F^j = 0$ otherwise, then $q(F) = \varepsilon s_{kl}$, and therefore q changes sign. We have thus shown that q satisfies the hypothesis of Corollary 2.2 i). Consequently, if $q(F) \leqslant \alpha$, we can find two matrices A and B with $\operatorname{rank}(A - B) \leqslant 1$, $q(A) = q(B) = \alpha$ and $F = \lambda A + (1 - \lambda) B$. Moreover, $\tilde{A} = \tilde{B} = \tilde{F}$. In this case, since QW is rank-1-convex, we see that

$$QW(F) \leqslant \lambda W(A) + (1 - \lambda)\, W(B) = \varphi(\alpha) + g(\tilde{F})\,.$$

If we introduce a function $\tilde{\varphi}$: $\mathbb{R}_+ \to \mathbb{R}$ by $\tilde{\varphi}(t) = \varphi(\alpha)$ for $t \leqslant \alpha$ and $\tilde{\varphi}(t) = \varphi(t)$ for $t > \alpha$, we can rewrite the above as the following statement:

$$\forall F \in M_{mn}, \qquad QW(F) \leqslant \tilde{\varphi}(q(F)) + g(\tilde{F})\,.$$

Note now that by condition b), φ is nondecreasing on $[\alpha, +\infty[$. As it coincides with its convex envelope on this interval, it follows that the function $\tilde{\varphi}$ is convex, nondecreasing on $\mathbb{R}_+$ and $\tilde{\varphi} \leqslant \varphi$. Since the quadratic form q is nonnegative, it is a convex function on M_{mn}. Therefore, the function $F \mapsto \tilde{\varphi}(q(F)) + g(\tilde{F})$ is convex and below W, thus

$$\tilde{\varphi}(q(F)) + g(\tilde{F}) \leqslant CW(F) \leqslant QW(F)\,,$$

and the result follows. ■

REMARK 3.2. – We could have assumed as well that g is quasiconvex, instead of convex. ■

EXAMPLE 3.3. – A classical example of function φ that satisfies a) and b) is $\varphi(t) = |t - 1|^p$ for $1 \leqslant p < +\infty$. If we take as quadratic form $q(F) = \operatorname{tr}(F^T F) = \sum_{j=1}^{n} |F^j|^2$ and $g = 0$, we recover a well-known result, see for example [5] and [3] for slightly different versions. ■

Let us now turn to relaxation results that are more specifically related to the James-Ericksen energy. We assume here that $\varphi : \mathbb{R} \to \mathbb{R}$ is bounded below and let $m = \inf_{t \in \mathbb{R}} \varphi(t)$. We do not make any convexity assumption on φ. Let $g : M_{m-1,n} \to \mathbb{R}$ be a quasiconvex function.

THEOREM 3.4. – *Let q be a quadratic form on M_{mn} that satisfies the hypotheses of Corollary 2.2 ii). The quasiconvex envelope of the function $W : M_{mn} \to \mathbb{R}$,*

$$W(F) = \varphi(q(F)) + g(\widetilde{F}) \tag{3.2}$$

is given by

$$QW(F) = m + g(\widetilde{F}). \tag{3.3}$$

PROOF. – First of all, the function $Z : F \mapsto m + g(\widetilde{F}F)$ is quasiconvex because for all $v \in W_0^{1,\infty}(D; \mathbb{R}^m)$, the function $\tilde{v}$ obtained by deleting the m-th component of v is in $W_0^{1,\infty}(D; \mathbb{R}^{m-1})$ and $\nabla \tilde{v} = \widetilde{\nabla v}$. Furthermore, the function Z is below W. Therefore, for all $F \in M_{mn}$,

$$m + g(\widetilde{F}) \leqslant QW(F). \tag{3.4}$$

Next, for all $\varepsilon > 0$, we choose $\alpha \in \mathbb{R}$ such that $m \leqslant \varphi(\alpha) \leqslant m + \varepsilon$. If F is such that $q(F) = \alpha$, then $QW(F) \leqslant m + g(\widetilde{F}) + \varepsilon$. If F is such that $q(F) \neq \alpha$, then by Corollary 2.2, we can find $\lambda \in [0, 1]$ and $A, B \in M_{mn}$ such that $\operatorname{rank}(A - B) \leqslant 1$ with $F = \lambda A + (1 - \lambda) B$, $q(A) = q(B) = \alpha$, and $\widetilde{A} = \widetilde{B} = \widetilde{F}$. Therefore, by the rank-1-convexity of QW, we obtain

$$QW(F) \leqslant \lambda W(A) + (1 - \lambda) W(B) = \varphi(\alpha) + g(\widetilde{F}) \leqslant m + g(\widetilde{F}) + \varepsilon , \tag{3.5}$$

from which the conclusion follows at once. ■

The following is an easy consequence of Theorem 3.4.

PROPOSITION 3.5. – *Let φ, q and q be as in Theorem 3.4. A necessary condition for the minimization problem: Find $u \in W_F^{1,\infty}$ such that*

$$\int_\Omega W(\nabla u(x))\, dx = \inf_{v \in W_F^{1,\infty}} \int_\Omega W(\nabla v(x))\, dx$$

where

$$W_F^{1,\infty} = \{v \in W^{1,\infty}(\Omega; \mathbb{R}^m);\ v(x) = Fx \text{ on } \partial\Omega\},$$

to have a solution, is that the infimum of φ be attained.

PROOF. – Indeed, by the previous result and by Dacorogna's representation formula for the quasiconvex envelope of a function, see [5], we have

$$\inf_{v \in W_F^{1,\infty}} \int_\Omega W(\nabla v(x))\, dx = (\text{meas } \Omega)(m + g(\tilde{F}))\,.$$

Since m is the infimum of φ, it is clear that for all v in $W_F^{1,\infty}$,

$$\int_\Omega \varphi(q(\nabla v(x)))\, dx \geqslant (\text{meas } \Omega)\, m\,,$$

and that by the quasiconvexity of g,

$$\int_\Omega g(\widetilde{\nabla v(x)})\, dx \geqslant (\text{meas } \Omega)\, g(\tilde{F})\,.$$

Let us assume that there exists u in $W_F^{1,\infty}$ such that

$$\int_\Omega W(\nabla u(x))\, dx = (\text{meas } \Omega)(m + g(\tilde{F}))\,.$$

Then necessarily,

$$\int_\Omega \varphi(q(\nabla u(x)))\, dx = (\text{meas } \Omega)\, m \quad \text{and} \quad \int_\Omega g(\widetilde{\nabla u(x)})\, dx = (\text{meas } \Omega)\, g(\tilde{F})\,.$$

In particular, the first inequality implies that $W(\nabla u(x)) = \varphi(q(\nabla u(x))) = m = \inf_{t \in \mathbb{R}} \varphi(t)$ almost everywhere, so that the infimum of φ is attained. ■

EXAMPLE 3.6. – Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a convex function. Then the quasiconvex envelope of the function $W : M_{2,2} \to \mathbb{R}$, $W(F) = \varphi(F^1 \cdot F^2) + g(F_{11}, F_{12})$ is given by $QW(F) = m + g(F_{11}, F_{12})$. ■

Here is another example in a similar spirit. Assume that $n \leqslant m$ and consider a quadratic form q that satisfies the hypotheses of Corollary 2.4. Let us be given coefficients β_{jp} for all $1 \leqslant j < p \leqslant n$, at least one of which is nonzero, a fonction $\varphi : \mathbb{R} \to \mathbb{R}$ bounded below, with $m = \inf_{\mathbb{R}} \varphi$, and a function $f : M_{mn} \to \mathbb{R}$ bounded below and such that there exists $\alpha \in \mathbb{R}$ such that for all $F \in M_{mn}$ satisfying $q(F) = \alpha$,

$$f(F) = I = \inf_{G \in M_{mn}} f(G)\,.$$

THEOREM 3.7. – *The quasiconvex envelope of the function* $W : M_{mn} \to \mathbb{R}$,

$$W(F) = \varphi(\beta_{jp} F^j \cdot F^p) + f(F) \tag{3.6}$$

is given by

$$QW(F) = m + I\,. \tag{3.7}$$

PROOF. – Again, it is obvious that

$$m + I \leqslant QW(F)\,. \tag{3.8}$$

If F is such that $q(F) = \alpha$, then $f(F) = I$, so that $QW(F) \leqslant \varphi(\beta_{jp} F^j \cdot F^p) + I$. If F is such that $q(F) \neq \alpha$, then by Corollary 2.4, we can find $\lambda \in [0, 1]$, $A, B \in M_{mn}$ with rank $(A - B) \leqslant 1$ such that $F = \lambda A + (1 - \lambda) B$, $q(A) = q(B) = \alpha$ and $A^j \cdot A^p = B^j \cdot B^p = F^j \cdot F^p$ for all $j \neq p$. Hence, by rank-1-convexity of QW, we obtain

$$QW(F) \leqslant \lambda W(A) + (1 - \lambda)\, W(B) = \varphi(\beta_{jp} F^j \cdot F^p) + I\,, \tag{3.9}$$

and the above inequality is established for all F. Let now $Z(F) = \varphi(\beta_{jp} F^j \cdot F^p) + I$. By inequality (3.9), it follows immediately that

$$QW(F) \leqslant QZ(F) = m + I\,, \tag{3.10}$$

by Theorem 3.4 applied to Z. ■

EXAMPLE 3.8. – Take the James-Ericksen density

$$W(F) = \kappa_2 (F^1 \cdot F^2)^2 + \kappa_3 \left(\left(\frac{|F^1|^2 - |F^2|^2}{2} \right)^2 - \varepsilon^2 \right)^2$$

with $\kappa_1 = 0$. We recover the result of [3] that in this case $QW(F) = 0$. ■

Let us now consider the general case $\kappa_1 \kappa_2 \kappa_3 > 0$. We can apply the rank-one decompositions of Theorem 2.1 and its various corollaries to the quadratic forms appearing in the expression of the James-Ericksen stored energy function. This yields diverse upper bounds for its quasiconvex envelope, depending on the order in which we try to relax each of the three terms. In the end, we will obtain a set of matrices where $QW(F) = 0$, that is to say, a region in the deformation gradient space where the relaxed energy is degenerate. If the deformation gradient of a solution of the relaxed minimization problem takes such values in a subset of the domain Ω, this indicates that relaxation is occurring in this subset and that minimizing sequences for the initial minimization problem will develop oscillations and exhibit microstructure in the same subset.

PROPOSITION 3.9. – *Let F be such that $|F^1|^2 + |F^2|^2 \leqslant 2$. Then we have*

(3.11) $$QW(F) \leqslant \kappa_2 (F^1 \cdot F^2)^2 + \kappa_3 \inf \{((1 - |F^2|^2)^2 - \varepsilon^2)^2, ((1 - |F^1|^2)^2 - \varepsilon^2)^2\}.$$

PROOF. – Let us take $q(F) = |F^1|^2 + |F^2|^2$ and F such that $q(F) \leqslant 2$ and $F^2 \neq 0$. Let us choose $a = (F^2)^\perp = (F_{22}, -F_{21})^T$ and $b = (1, 0)^T$. Then $a \otimes b = ((F^2)^\perp | 0)$ and we obtain

$$q(a \otimes b) = |F^2|^2 > 0$$

Consequently, as in Theorem 2.1, we construct a pair of matrices $A = F + \lambda t a \otimes b$ and $B = F + (\lambda - 1) t a \otimes b$ with $\lambda \in [0, 1]$, such that $q(A) = q(B) = 2$ and $A^1 \cdot A^2 = B^1 \cdot B^2 = F^1 \cdot F^2$. In addition, we see that, since $q(A) = q(B) = 2$ and $A^2 = B^2 = F^2$,

$$|A^1|^2 - |A^2|^2 = |B^1|^2 - |B^2|^2 = 2(1 - |F^2|^2).$$

Therefore, for all F such that $q(F) \leqslant 2$ and $F^2 \neq 0$,

$$QW(F) \leqslant \lambda W(A) + (1 - \lambda) W(B) = \kappa_2 (F^1 \cdot F^2)^2 + \kappa_3 \big((1 - |F^2|^2)^2 - \varepsilon^2\big)^2.$$

This inequality extends to the case $F^2 = 0$ by the continuity of both sides.

If we choose now $a = (F^1)^\perp$ and $b = (0, 1)^T$, we likewise obtain

$$QW(F) \leqslant \kappa_2 (F^1 \cdot F^2)^2 + \kappa_3 ((1 - |F^1|^2)^2 - \varepsilon^2)^2,$$

for all F such that $q(F) \leqslant 2$, hence the result. ■

Let us now relax the second term in the right-hand side of estimate (3.11).

PROPOSITION 3.10. – *Let F be such that $|F^1|^2 \leqslant 1 - \varepsilon$, $|F^2|^2 \leqslant 1 + \varepsilon$ or $|F^1|^2 \leqslant 1 + \varepsilon$, $|F^2|^2 \leqslant 1 - \varepsilon$. Then we have*

(3.12) $$QW(F) \leqslant \kappa_2 (F^1 \cdot F^2)^2.$$

PROOF. – Let us take $q'(F) = |F^2|^2$ and let F be such that $q'(F) \leqslant 1 + \varepsilon$ and $|F^1| \leqslant 1 - \varepsilon$ (note that then $q(F) \leqslant 2$). If we choose $a = (F^1)^\perp$ and $b = (0, 1)^T$, we obtain two matrices A and B such that rank $(A - B) = 1$, $F = \lambda A + (1 - \lambda) B$ with $\lambda \in [0, 1]$, $q'(A) = q'(B) = 1 + \varepsilon$ and $A^1 \cdot A^2 = B^1 \cdot B^2 = F^1 \cdot F^2$. Moreover, since $A^1 = F^1$ and $q'(A) = 1 + \varepsilon$, we see that

$$|A^1|^2 + |A^2|^2 = |F^1|^2 + 1 + \varepsilon \leqslant 2,$$

and the same holds true for B. Estimate (3.11) can thus be applied to A and B.

Therefore, we see that, for all F such that $|F^2|^2 \leqslant 1+\varepsilon$ and $|F^1|^2 \leqslant 1-\varepsilon$,

$$QW(F) \leqslant \lambda QW(A) + (1-\lambda)\, QW(B) \leqslant \kappa_2 (F^1 \cdot F^2)^2,$$

since $((1-|A^2|^2)^2 - \varepsilon^2)^2 = ((1-|B^2|^2)^2 - \varepsilon^2)^2 = 0$.

For F such that $|F^1|^2 \leqslant 1+\varepsilon$ and $|F^2|^2 \leqslant 1-\varepsilon$, we use the same argument with the quadratic form $q''(F) = |F^1|^2$, which completes the proof. ■

The final relaxation step is easier to carry out if we first perform a simple change of variables.

LEMMA 3.11. – *For any function* $W: M_{2,2} \to \mathbb{R}$, *we define a second function* $\widehat{W}: M_{2,2} \to \mathbb{R}$ *by* $\widehat{W}(G) = W(F)$ *with* $G = (G^1 | G^2) = \left(\frac{F^1+F^2}{\sqrt{2}} \middle| \frac{F^1-F^2}{\sqrt{2}} \right)$. *Then*

$$QW(F) = Q\widehat{W}(G).$$

PROOF. – Let $R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. This is an orthogonal matrix and $G = FR$ and $F = GR$. Let D be the unit disk in $\mathbb{R}^2$. By Dacorogna's representation formula, see [5], we have

$$Q\widehat{W}(G) = \inf_{\psi \in W_0^{1,\infty}(D;\mathbb{R}^2)} \frac{1}{\pi} \int_D \widehat{W}(G + \nabla\psi)\, dx$$

$$= \inf_{\psi \in W_0^{1,\infty}(D;\mathbb{R}^2)} \frac{1}{\pi} \int_D W(GR + \nabla\psi R)\, dx .$$

For any $\psi \in W_0^{1,\infty}(D;\mathbb{R}^2)$, define $\theta(y) = \psi(Ry)$. Since R is orthogonal and D is the unit disk, it follows that $\theta \in W_0^{1,\infty}(D;\mathbb{R}^2)$ and $\nabla\theta(y) = \nabla\psi(Ry)\, R$. Therefore, we see that

$$Q\widehat{W}(G) = \inf_{\theta \in W_0^{1,\infty}(D;\mathbb{R}^2)} \frac{1}{\pi} \int_D W(GR + \nabla\theta)\, dy = QW(GR) = QW(F). \qquad ■$$

THEOREM 3.12. – *Let* F *be such that*

$$\left\{ \begin{array}{l} |F^1|^2 + |F^1 \cdot F^2| \leqslant 1 - \varepsilon , \\ |F^2|^2 + |F^1 \cdot F^2| \leqslant 1 + \varepsilon , \end{array} \right. \tag{3.13}$$

or

$$\left\{\begin{array}{l} |F^1|^2+|F^1\cdot F^2|\leqslant 1+\varepsilon\,, \\ |F^2|^2+|F^1\cdot F^2|\leqslant 1-\varepsilon\,. \end{array}\right. \tag{3.14}$$

Then we have

$$QW(F)=0\,. \tag{3.15}$$

PROOF. – Let us rewrite estimate (3.12) in view of Lemma 3.11. This yields that for all G such that

$$\frac{|G^1|^2+|G^2|^2}{2}+G^1\cdot G^2\leqslant 1-\varepsilon \text{ and } \frac{|G^1|^2+|G^2|^2}{2}-G^1\cdot G^2\leqslant 1+\varepsilon$$

or

$$\frac{|G^1|^2+|G^2|^2}{2}+G^1\cdot G^2\leqslant 1+\varepsilon \text{ and } \frac{|G^1|^2+|G^2|^2}{2}-G^1\cdot G^2\leqslant 1-\varepsilon\,,$$

we have

$$Q\widehat{W}(G)\leqslant \frac{\kappa_2}{4}(|G^1|^2-|G^2|^2)^2. \tag{3.16}$$

Let now F satisfy condition (3.13) with $F^1\cdot F^2<0$. Expressed in terms of the associated matrix G, condition (3.13) reads

$$|G^2|^2+G^1\cdot G^2\leqslant 1-\varepsilon \text{ and } |G^2|^2-G^1\cdot G^2\leqslant 1+\varepsilon\,.$$

Moreover, G is such that $q'''(G)=|G^1|^2-|G^2|^2<0$ (which implies $G^2\neq 0$). A by now familiar argument leads us to pose $a=(G^2)^\perp$, $b=(1,0)^T$ and we obtain

$$q'''(a\otimes b)=|G^2|^2>0\,.$$

We thus find A and B with $q'''(A)=q'''(B)=0$ and $A^1\cdot A^2=B^1\cdot B^2=G^1\cdot G^2$. Therefore, as in the proof of Proposition 3.9,

$$|A^1|^2=|B^1|^2=|A^2|^2=|B^2|^2=|G^2|^2.$$

Consequently,

$$\frac{|A^1|^2+|A^2|^2}{2}\pm A^1\cdot A^2=\frac{|B^1|^2+|B^2|^2}{2}\pm B^1\cdot B^2=|G^2|^2\pm G^1\cdot G^2,$$

and estimate (3.16) applies to A and B. Hence,

$$Q\widehat{W}(G) \leqslant \lambda Q\widehat{W}(A) + (1-\lambda)\, Q\widehat{W}(B) = 0\,.$$

If $q'''(G) > 0$, we simply exchange the roles of G^1 and G^2. ■

REMARKS 3,13. – i) Let $\mathcal{E}_{QW} = \{F \in M_{2,2};\, QW(F) = 0\}$ be the zero set of QW. We have shown that $\mathcal{E}_{QW}$ contains the set of all matrices F that satisfy conditions (3.13) or (3.14). In the case $\kappa_3 = 0$ and $\kappa_2 > 0$, the quasiconvex envelope of W was computed in [3]:

$$QW(F) = \begin{cases} 0 & \text{if } \operatorname{tr} C \leqslant 2 \text{ and } 2|c_{12}| \leqslant 2 - \operatorname{tr} C, \\ \kappa_1(\operatorname{tr} C - 2)^2 + \kappa_2 c_{12}^2 & \text{if } \operatorname{tr} C \geqslant 2 \text{ and } \kappa_2|c_{12}| \leqslant 2\kappa_1(\operatorname{tr} C - 2), \\ \kappa_1(\operatorname{tr} C - 2)^2 + \kappa_2 c_{12}^2 - & \\ \dfrac{(2\kappa_1(\operatorname{tr} C - 2) - \kappa_2|c_{12}|)^2}{4\kappa_1 + \kappa_2} & \text{if } \operatorname{tr} C \geqslant 2 \text{ and } \kappa_2|c_{12}| \geqslant 2\kappa_1(\operatorname{tr} C - 2), \\ & \text{or } \operatorname{tr} C \leqslant 2 \text{ and } 2|c_{12}| \geqslant 2 - \operatorname{tr} C, \end{cases}$$

with $C = F^T F$. If we perform our computations in this particular case, we obtain first as in Proposition 3.9 that for all F with $\operatorname{tr} C = |F^1|^2 + |F^2|^2 \leqslant 2$,

$$QW(F) \leqslant \kappa_2 (F^1 \cdot F^2)^2.$$

Then we can skip Proposition 3.10 and go directly to Theorem 3.12. We thus obtain that for all G with $|G^1|^2 + |G^2|^2 \leqslant 2$,

$$Q\widehat{W}(G) \leqslant \frac{\kappa_2}{4}(|G^1|^2 - |G^2|^2)^2,$$

and in the case $F^1 \cdot F^2 < 0$, the condition on G that makes it possible to relax the upper bound to zero using the matrices A and B is simply $|G^2| \leqslant 1$. Expressed in terms of F this condition reads $\frac{|F^1|^2 + |F^2|^2}{2} - F^1 \cdot F^2 \leqslant 1$. The case $F^1 \cdot F^2 > 0$ leads to the condition $\frac{|F^1|^2 + |F^2|^2}{2} + F^1 \cdot F^2 \leqslant 1$, so that we find that $QW(F) = 0$ for all F such that

$$|F^1|^2 + |F^2|^2 \leqslant 2 \quad \text{and} \quad 2\,|F^1 \cdot F^2| \leqslant 2 - (|F^1|^2 + |F^2|^2)\,.$$

These are exactly the same conditions as those found in [3], so that we conclude that our method is optimal in this particular case. Unfortunately, when $\kappa_1 \kappa_2 \kappa_3 > 0$, the results of [2], obtained via entirely different techniques, show that this is not the case. The zero-set of the quasiconvex envelope contains matrices that are not captured by our method. It is not clear that such matrices can be recovered by using our kind of arguments.

ii) Instead of first relaxing the first term in the expression of the James-

Ericksen stored energy function, we can start with the third term. In this way, we obtain the bounds

$$QW(F) \leqslant 4\kappa_1(|F^2|^2+\varepsilon-1)^2+\kappa_2(F^1\cdot F^2)^2 \text{ for all } F \text{ such that } |F^1|^2-|F^2|^2 \leqslant 2\varepsilon\,,$$

and

$$QW(F) \leqslant 4\kappa_1(|F^1|^2+\varepsilon-1)^2+\kappa_2(F^1\cdot F^2)^2 \text{ for all F such that } |F^2|^2-|F^1|^2 \leqslant 2\varepsilon\,.$$

These bounds are slightly different from the previously obtained bounds in that they cover the whole of $M_{2,2}$, and not just a bounded region thereof. Unfortunately, it does not seem that the above bounds are much useful for the determination of the whole quasiconvex envelope of W. One interesting feature of these bounds is that they do not depend on κ_3. This indicates that the dependence of QW on κ_3 has to be quite mild, even though letting κ_3 tend to infinity results in W tending to infinity almost everywhere.

A similar bound, which is valid for all F, is obtained by first relaxing the second term:

$$QW(F) \leqslant \kappa_1(|F^1|^2+|F^2|^2+2|F^1\cdot F^2|-2)^2+\kappa_3\left(\left(\frac{|F^1|^2-|F^2|^2}{2}\right)^2-\varepsilon^2\right)^2.$$

Note that this bound does not depend on κ_2. Roughly speaking, this tends to show that the asymptotic behavior of QW for large F is governed by the terms involving κ_1.

We can further relax these bounds along the same lines as above, but the results do not improve those of Proposition 3.10 and Theorem 3.12. In fact, it can be checked that all six different ways of successively relaxing the three quadratic terms basically yield the same result in the end.

Let us conclude this article by showing how the case of functions depending on certain homogeneous functions of degree $p \neq 2$ can be handled along similar lines. Consider for example the function on $M_{2,2}$

$$h(F) = |F_{11}|^p + |F_{21}|^p - (|F_{12}|^p + |F_{22}|^p)\,, \tag{3.17}$$

with $p > 0$. Then we have an adapted rank-one decomposition.

PROPOSITION 3.14. – *Let $\alpha \in \mathbb{R}$ and F be such that $\varphi(F) \neq \alpha$. Then there exists two matrices A and B and a scalar $\lambda \in [0, 1]$ such that $F = \lambda A + (1-\lambda)B$,* $\operatorname{rank}(A-B) \leqslant 1$ *and $h(A) = h(B) = \alpha$.*

PROOF. – Assume to start with that $h(F) \leqslant \alpha$, $\alpha \geqslant 0$ and that $F^1 \neq 0$. Let

$$X = |F_{11}|^p + |F_{21}|^p \quad \text{and} \quad Y = |F_{12}|^p + |F_{22}|^p,$$

so that

$$X > 0 \quad \text{and} \quad z = \left(\frac{\alpha + Y}{X}\right)^{1/p} \geqslant 1 .$$

We take $A = ((1 + \lambda t)\, F^1 \,|\, F^2)$ and $B = ((1 + (\lambda - 1)\, t)\, F^1 \,|\, F^2)$ with

$$t = 2z \quad \text{and} \quad \lambda = \frac{z-1}{2z} .$$

If now $F^1 = 0$, we take $A = F + \frac{1}{2}((0, t)^T \,|\, 0)$ and $B = F - \frac{1}{2}((0, t)^T \,|\, 0)$ with $t = 2(Y + \alpha)^{1/p}$.

If $h(F) \geqslant \alpha$, we proceed as above, exchanging the roles of F^1 and F^2.

Finally, if $\alpha \leqslant 0$, we apply the above to $-h$. ■

COROLLARY 3.15. – *Let $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\inf_{t \in \mathbb{R}} \varphi(t) = m > -\infty$ and define $W(F) = \varphi(h(F))$. Then, for all $F \in M_{2,2}$,*

$$QW(F) = m .$$

PROOF. – Clear. ■

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