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An Algebraic Completeness Proof for Kleene's 3-Valued Logic.

Maurizio Negri

Sunto. – *La logica trivalente di Kleene è presentata in un linguaggio che comprende, oltre alle costanti booleane, anche un simbolo per il valore di verità intermedio n. Parallelamente si sviluppa la semantica utilizzando la classe delle DMF-algebre al posto della classe più estesa costituita dalle algebra di De Morgan. Si introduce quindi un calcolo di sequenti che viene dimostrato completo rispetto alla semantica trivalente. La dimostrazione di completezza è fondata su una versione del teorema dell'ideale primo tipica delle DMF-algebre. La dimostrazione è interamente algebrica solo per quanto riguarda il teorema di completezza debole. La dimostrazione della versione forte del teorema di completezza utilizza metodi topologici, in particolare il teorema di Tychonoff sul prodotto di spazi compatti.*

Summary. – *We introduce Kleene's 3-valued logic in a language containing, besides the Boolean connectives, a constant n for the undefined truth value, so in developing semantics we can switch from the usual treatment based on DM-algebras to the narrower class of DMF-algebras (De Morgan algebras with a single fixed point for negation). A sequent calculus for Kleene's logic is introduced and proved complete with respect to threevalent semantics. The completeness proof is based on a version of the prime ideal theorem that is typical of DMF-algebras. Only for the weak completeness theorem the proof is fully algebrical, because in the proof of strong completeness we have been compelled to use topological methods (Tychonoff theorem on the product of compact spaces).*

1. – Introduction.

Kleene's 3-valued logic is usually developed in a language based on a set $K(BA) = \{\wedge, \vee, \neg, 0, 1\}$ of connectives, considering constants as 0-ary connectives. As a consequence of this linguistic choice, formulas must take values in algebras of type $K(BA)$ and a natural choice is the class of (normal) De Morgan algebras (see, for instance, [4] [par. 7.1] and [2] [ch. 8]). We recall that a De Morgan algebra is a distributive lattice with 0 and 1 equipped with an unary operation $\neg$ satisfying the double negation law $\neg\neg x = x$ and De Morgan laws $\neg(x \wedge y) = \neg x \vee \neg y$ and $\neg(x \vee y) = \neg x \wedge \neg y$. This amounts to say that $\neg$ is an involution and a dual automorphism. In this work we develop 3-valued logic in a language based on a set $K(DMF) = \{\wedge, \vee, \neg, 0, 1, n\}$ of connectives and consequently we choose DMF-algebras, i.e.

De Morgan algebras with a single fixed point for negation, as domains of truth values. On the linguistic side, a name for the intermediate truth value n is now available, what is needed to make indeterminacy expressible; axioms for n make also available weakened forms of deduction theorem and of modus ponens (see the end of paragraph 3). On the semantic side, the class of truth values domains is now restricted, because every DMF-algebra is obviously a DM-algebra, but not every DM-algebra can be expanded to a DMF-algebra. (For instance, every Boolean algebra is a DM-algebra, but there is no Boolean algebra with an element x such that $\neg x = x$.)

In the following paragraph the semantics of 3-valued logic is presented and contrasted with classical semantics; the role of Boolean algebra and fields of sets is now taken by DMF-algebras and fields of partial sets. In the third paragraph we introduce a sequent calculus for 3-valued logic that will be proved complete in the last paragraph. The fourth paragraph is devoted to prove a version of the prime ideal theorem for DMF-algebras that will be the main tool in the completeness proof. It should be noted that only the proof of the weak completeness theorem of 3-valued logic is of a fully algebraic character, because the proof of the full completeness theorem depends essentially on a topological argument based on Tychonoff theorem.

2. – Semantics: 2-valued vs. 3-valued.

Let Fm be the algebra of formulas of $\mathcal{L} = E \cup K(BA)$, where $E = \{p_i : i \in \omega\}$ is the set of propositional variables; Fm is free in the class of all structures of type $K(BA)$, having E as a set of free generators. Classical (2-valued) semantics can be seen as a morphism M from Fm to an algebra whose elements are to be considered as meanings of formulas: M being a morphism guarantees that the meaning of a complex formula can be computed from the meaning of its subformulas. We choose as a codomain of M the field of sets

$$\mathcal{P}(2^\omega) = (P(2^\omega),\ \cap\,,\ \cup\,,\ -\,,\ \emptyset,\ 2^\omega)$$

and we define $M : Fm \to \mathcal{P}(2^\omega)$ as the (unique) morphism induced by $g : E \to P(2^\omega)$, where $g(p_i) = \{s \in 2^\omega : s(i) = 1\}$. Then we have

$$M(\alpha \wedge \beta) = M(\alpha) \cap M(\beta)\,,$$
$$M(\alpha \vee \beta) = M(\alpha) \cup M(\beta)\,,$$
$$M(\neg \alpha) = -M(\alpha)\,,$$
$$M(0) = \emptyset\,,$$
$$M(1) = 2^\omega.$$

We say that s is a *model* of α iff $s \in M(\alpha)$ and write $s \models \alpha$. So the meaning of a formula α is the set of its models and a model of α is an element of its meaning.

Classical semantics is usually presented through the concept of «truth with respect to an assignment». An assignment is a function $\nu : E \to 2$ and every assignment ν induces a morphism $Val_\nu : Fm \to 2$: we say that α is *true with respect to* ν iff $Val_\nu(\alpha) = 1$. These different approaches to classical semantics can be related as follows. First we observe that there is an obvious bijection between E and ω: so we can associate to ν the sequence s_ν such that $s_\nu(i) = p_i$, and to s the assignment ν_s such that $\nu_s(p_i) = s(i)$, for all $i \in \omega$. Then we can easily prove that

$$s \in M(\alpha) \quad \text{iff} \quad Val_{\nu_s}(\alpha) = 1 \,.$$

So the meaning of α can be defined from the concept of truth with respect to ν. But we can also prove that

$$Val_\nu(\alpha) = 1 \quad \text{iff} \quad s_\nu \in M(\alpha)\,,$$

so the truth of α with respect to ν can be defined from the concept of meaning.

Now we can define the concept of logical consequence setting $\alpha \models \beta$ iff $M(\alpha) \subseteq M(\beta)$. The case of an infinte set of premisses can be handled as follows: first we extend M to set of formulas setting $M(\Sigma) = \bigcap\{M(\sigma) : \sigma \in \Sigma\}$, then we say that $\Sigma \models \alpha$ iff $M(\Sigma) \subseteq M(\alpha)$. It can be easily proved from the equations above that $M(\alpha) \subseteq M(\beta)$ iff for all ν, $Val_\nu(\alpha) \leqslant Val_\nu(\beta)$ and $M(\Sigma) \subseteq M(\alpha)$ iff for all ν, $\bigwedge\{Val_\nu(\sigma) : \sigma \in \Sigma\} \leqslant Val_\nu(\alpha)$, thus showing that logical consequence can be equivalently defined in terms of meaning and in terms of truth assignments. We say that α is a *tautology* iff $1 \models \alpha$, that amounts to say that a tautology is a consequence of the empty set of assumptions, as $M(\emptyset) = \bigcap \emptyset = 2^\omega = M(1)$.

Classical semantics can be generalized to Boolean semantics by substituting 2 with any Boolean algebra $\mathcal{A}$. Then we can define the concepts of meaning $M^{\mathcal{A}}$ and logical consequence $\models_{\mathcal{A}}$ relativized to $\mathcal{A}$ and we can prove that, for any Boolean algebra $\mathcal{A}$, $\alpha \models \beta$ iff $\alpha \models_{\mathcal{A}} \beta$, showing that classical logic is equivalent to Boolean logic.

In Kleene's 3-valued logic the role of Boolean algebra is played by DMF-algebra (see [5] and [6]). We remember that a DMF-algebra is a De Morgan algebra with a single fixed point for negation. We generally think DMF-algebras as structures of type $K(DMF)$ satisfying the following set of equational axioms: axioms for De Morgan algebras, normality axiom $x \wedge \neg x \leqslant y \vee \neg y$ and the fixed point axiom $\neg n = n$. We denote with 3 both the set $\{0, n, 1\}$ and the DMF-algebra having 3 as domain, whose operations are defined by Kleene's strong tables. (If we give 3 the partial order $0 \leqslant n \leqslant 1$, then Kleene's connectives $\wedge$ and $\vee$ agree with inf and sup, while negation is the involution $\neg 1 = 0$, $\neg 0 = 1$, $\neg n = n$.)

From any set X we can form the set $D(X)$ of all partial sets on X (see [6]). A partial set on X is a couple (A, B) with $A, B \subseteq X$ and $A \cap B = \emptyset$, A representing the positive cases and B the negative cases of a partial property (i.e. a property undefined for elements of $X - A \cup B$). We denote with $\mathcal{D}(X)$ the DMF-algebra whose operations and constants are as follows:

$$(A, B) \wedge (A', B') = (A \cap A', B \cup B'),$$

$$(A, B) \vee (A', B') = (A \cup A', B \cap B'),$$

$$\neg(A, B) = (B, A),$$

$$0 = (X, \emptyset),$$

$$1 = (\emptyset, X),$$

$$n = (\emptyset, \emptyset).$$

We introduce a partial order relation on partial sets setting $(A, B) \leqslant (A', B')$ iff $(A, B) \wedge (A', B',) = (A, B)$. So we have $(A, B) \leqslant (A', B')$ iff $A \subseteq A'$ and $B' \subseteq B$. Any subalgebra $\mathcal{A}$ of $\mathcal{D}(X)$ is called a *field of partial sets on* X.

Now we can develope 3-valued semantics along the same way followed with 2-valued semantics, keeping in mind that the roles played by 2 and $\mathcal{P}(2^\omega)$ are to be taken by 3 and $\mathcal{D}(3^\omega)$. Let Fm be the algebra of formulas from $\mathcal{L} = E \cup K(DMF)$, we define the meaning of formulas as the morphism $M : Fm \to \mathcal{D}(3^\omega)$ induced by $g : E \to \mathcal{D}(3^\omega)$, where

$$g(p_i) = (\{s \in 3^\omega : s_i = 1\}, \{s \in 3^\omega : s_i = 0\}).$$

Then we have

$$M(\alpha \wedge \beta) = M(\alpha) \wedge M(\beta) = (M(\alpha)_0 \cap M(\beta)_0, M(\alpha)_1 \cup M(\beta)_1),$$

$$M(\alpha \vee \beta) = M(\alpha) \vee M(\beta) = (M(\alpha)_0 \cup M(\beta)_0, M(\alpha)_1 \cap M(\beta)_1),$$

$$M(\neg\alpha) = \neg M(\alpha) = (M(\alpha)_1, M(\alpha)_0),$$

$$M(0) = (\emptyset, 3^\omega),$$

$$M(1) = (3^\omega, \emptyset),$$

$$M(n) = (\emptyset, \emptyset).$$

So the meaning of α in 3-valued semantics is a partial set $M(\alpha)$ on 3^ω. Elements of $M(\alpha)_0$ and of $M(\alpha)_1$ are respectively said *positive* and *negative models* of α.

We can also introduce 3-valued semantics through the concept of truth with respect to an assignment $\nu : E \to 3$. Every such ν can be extended to an unique morphism $Val_\nu : Fm \to 3$ and we'll say that α is true in ν if $Val_\nu(\alpha) = 1$, false if $Val_\nu(\alpha) = 0$, undefined if $Val_\nu(\alpha) = n$. As in classical semantics these two approaches can be proved to be equivalent. Firstly we observe that there is a bijection between assignments in E^ω and sequences in 3^ω, associating to ν

the sequence s_ν such that $s_\nu(i) = \nu(i)$ and associating to s the assignment ν_s such that $\nu_s(p_i) = s(i)$. Then we can easily prove by induction on α the following theorems.

THEOREM 1. – $M(\alpha) = (\{s \in 3^\omega: Val_{\nu_s}(\alpha) = 1\}, \{s \in 3^\omega: Val_{\nu_s}(\alpha) = 0\})$.
When α is a variable we have:

$$M(p_i) = (\{s : s(i) = 1\}, \{s : s(i) = 0\}) = (\{s : Val_{\nu^s}(p_i) = 1\}, \{s : Val_{\nu^s}(p_i) = 0\}).$$

When α is $\beta \wedge \gamma$ we have $M(\beta \wedge \gamma) = M(\beta) \wedge M(\gamma) = (M(\beta)_0 \cap M(\gamma)_0, M(\beta)_1 \cup M(\gamma)_1)$. By inductive hypothesis

$$M(\beta)_0 \cap M(\gamma)_0 = \{s : Val_{\nu^s}(\beta) = 1\} \cap \{s : Val_{\nu^s}(\gamma) = 1\}$$
$$= \{s : Val_{\nu^s}(\beta \wedge \gamma) = 1\}$$

and

$$M(\beta)_0 \cup M(\gamma)_0 = \{s : Val_{\nu^s}(\beta) = 0\} \cap \{s : Val_{\nu^s}(\gamma) = 0\}$$
$$= \{s : Val_{\nu^s}(\beta \wedge \gamma) = 0\}.$$

The other cases are proved in the same way.

THEOREM 2. – $Val_\nu(\alpha) = \begin{cases} 1 & \textit{if} \quad s_\nu \in M(\alpha)_0, \\ 0 & \textit{if} \quad s_\nu \in M(\alpha)_1, \\ n & \quad \textit{otherwise}. \end{cases}$

When α is a variable, $Val_\nu(p_i) = 1$ iff $\nu(p_i) = 1$ iff $s_\nu(i) = 1$ iff $s_\nu \in M(p_i)_0$. In the same way $Val_\nu(p_i) = 0$ iff $s_\nu \in M(p_i)_1$ and $Val_\nu(p_i) = n$ iff $s_\nu \notin M(p_i)_0$ and $s_\nu \notin M(p_i)_1$. When α is $\beta \wedge \gamma$ we have $Val_\nu(\beta \wedge \gamma) = 1$ iff $Val_\nu(\beta) = 1$ and $Val_\nu(\gamma) = 1$ iff $s_\nu \in M(\beta)_0$ and $s_\nu \in M(\gamma)_0$ iff $s_\nu \in M(\beta)_0 \cap M(\gamma)_0$ iff $s_\nu \in M(\beta \wedge \gamma)_0$; in the same way $Val_\nu(\beta \wedge \gamma) = 0$ iff $Val_\nu(\beta) = 0$ or $Val_\nu(\gamma) = 0$ iff $s_\nu \in M(\beta)_1$ or $s_\nu \in M(\gamma)_1$ iff $s_\nu \in M(\beta)_1 \cup M(\gamma)_1$ iff $s_\nu \in M(\beta \wedge \gamma)_1$. If $Val_\nu(\beta \wedge \gamma) = n$ then $s_\nu \notin M(\beta \wedge \gamma)_0$ and $s_\nu \notin M(\beta \wedge \gamma)_1$, because we have just proved that $s_\nu \in M(\beta \wedge \gamma)_0$ implies $Val_\nu(\beta \wedge \gamma) = 1$ and $s_\nu \in M(\beta \wedge \gamma)_1$ implies $Val_\nu(\beta \wedge \gamma) = 0$. In the other direction, if $s_\nu \notin M(\beta \wedge \gamma)_0$ and $s_\nu \notin M(\beta \wedge \gamma)_1$ then $Val_\nu(\beta \wedge \gamma)$ must be n because it is different from 1 and 0 and the codomain of Val_ν is 3. An analogous proof can be given for the remaining connectives.

We introduce the concept of logical consequence in 3-valued logic setting $\alpha \models \beta$ iff $M(\alpha) \leqslant M(\beta)$, i.e. $M(\alpha)_0 \subseteq M(\beta)_0$ and $M(\beta)_1 \subseteq M(\alpha)_1$. In the case of an infinite set of premisses firstly we extend M to set of formulas setting

$$M(\Sigma) = \bigwedge \{M(\sigma) : \sigma \in \Sigma\} = (\bigcap \{M(\sigma)_0 : \sigma \in \Sigma\}, \bigcup \{M(\sigma)_1 : \sigma \in \Sigma\}).$$

The following theorem shows that logical consequence can be equivalently defined in terms of meaning of formulas and in terms of truth assignments.

THEOREM 3. – $M(\Sigma) \leqslant M(\alpha)$ iff for all $\nu : E \to 3$, $\bigwedge\{Val_\nu(\sigma) : \sigma \in \Sigma\} \leqslant Val_\nu(\alpha)$.

Assume $M(\Sigma) \leqslant M(\alpha)$. Case 1), $\bigwedge\{Val_\nu(\sigma) : \sigma \in \Sigma\} = 0$, there is nothing to prove. Case 2), $\bigwedge\{Val_\nu(\sigma) : \sigma \in \Sigma\} = 1$, then $Val_\nu(\sigma) = 1$ for all $\sigma \in \Sigma$. By theorem 2 $s_\nu \in M(\sigma)_0$ for all $\sigma \in \Sigma$, so $s_\nu \in \bigcap\{M(\sigma)_0 : \sigma \in \Sigma\}$. From our hypothesis $\bigcap\{M(\sigma)_0 : \sigma \in \Sigma\} \subseteq M(\alpha)_0$, so $s_\nu \in M(\alpha)_0$ and $Val_\nu(\alpha) = 1$ by theorem 2, remembering that $\nu = \nu_{s_\nu}$. Case 3), $\bigwedge\{Val_\nu(\sigma) : \sigma \in \Sigma\} = n$ then, for all $\sigma \in \Sigma$, $Val_\nu(\sigma) \in \{n, 1\}$. We can limit ourselves to prove that $Val_\nu(\alpha)$ is different from 0. If $Val_\nu(\alpha) = 0$ then $s_\nu \in M(\alpha)_1$ by theorem 2. But from our hypothesis $M(\alpha)_1 \subseteq \bigcup\{M(\sigma)_1 : \sigma \in \Sigma\}$, so $s_\nu \in M(\sigma)_1$ for some $\sigma \in \Sigma$. Then $Val_\nu(\sigma) = 0$, by theorem 2, which is absurd.

In the other direction, we firstly show that $\bigcap\{M(\sigma)_0 : \sigma \in \Sigma\} \subseteq M(\alpha)_0$. If $s \in \bigcap\{M(\sigma)_0 : \sigma \in \Sigma\}$ then $s \in M(\sigma)_0$ for all $\sigma \in \Sigma$. By theorem 2 $Val_{\nu_s}(\sigma) = 1$ for all $\sigma \in \Sigma$, so $\bigwedge\{Val_{\nu_s}(\sigma) : \sigma \in \Sigma\} = 1$ and $Val_{\nu_s}(\alpha) = 1$, from our hypothesis. So $s \in M(\alpha)_0$ follows from theorem 2, remembering that $s = s_{\nu_s}$. Then we show that $M(\alpha)_1 \subseteq \bigcup\{M(\sigma)_1 : \sigma \in \Sigma\}$. Suppose $s \in M(\alpha)_1$, then by theorem 2 $Val_{\nu_s}(\alpha) = 0$ and from our hypothesis $\bigwedge\{Val_{\nu_s}(\sigma) : \sigma \in \Sigma\} = 0$. Then $Val_{\nu_s}(\sigma) = 0$ for some $\sigma \in \Sigma$. As $s_{\nu_s} = s$, from theorem 2 we have $s \in M(\sigma)_1$ and $s \in \bigcup\{M(\sigma)_1 : \sigma \in \Sigma\}$.

In particular we have $\alpha \models \beta$ iff for all $\nu : E \to 3$, $Val_\nu(\alpha) \leqslant Val_\nu(\beta)$. We note that Blamey's definition of logical consequence, $\alpha \models_b \beta$ iff for all $\nu : E \to 3$, $Val_\nu(\alpha) = 1$ implies $Val_\nu(\beta) = 1$ and $Val_\nu(\beta) = 0$ implies $Val_\nu(\alpha) = 0$ (see [1] [p. 5]), is equivalent to ours. Obviously $\alpha \models \beta$ implies $\alpha \models_b \beta$. Conversely, if $Val_\nu(\alpha) = 1$ then $Val_\nu(\beta) = 1$ by $\alpha \models_b \beta$. If $Val_\nu(\alpha) = n$ then $Val_\nu(\beta) \in \{n, 1\}$, because $Val_\nu(\beta) = 0$ implies $Val_\nu(\alpha) = 0$ by $\alpha \models_b \beta$.

The following points of difference between classical and 3-valued semantics deserve some attention. We can define tautologies as formulas α satisfying $1 \models \alpha$, i.e. $(3^\omega, \emptyset) \leqslant M(\alpha)$, but tautologies in 3-valued logic are not very interesting. In fact every tautology must contain a constant 0 or 1. For suppose there is neither 0 nor 1 in α, then $Val_\nu(\alpha) = n$ when $\nu(i) = n$ for all $i \in \omega$, so $s_\nu \notin M(\alpha)_0$ and α is not a tautology. In particular *tertium non datur* (tnd) is not a tautology because $M(p_i \vee \neg p_i)$ is $(M(p_i)_0 \cup M(p_i)_1, \emptyset)$, but $M(p_i)_0 \cup M(p_i)_1 \neq 3^\omega$. (Consider s such that $s(i) = n$ for all i.) So $1 \not\models p_i \vee \neg p_i$ and dually $p_i \wedge \neg p_i \not\models 0$. Other differences can be found in the relations between $\models$ and $\to$. If we define $\alpha \to \beta$ as $\neg \alpha \vee \beta$, the classical equivalence

$$\Sigma \cup \{\alpha\} \models \beta \text{ iff } \Sigma \models \alpha \to \beta$$

is no more true. The implication from left to right, generally known as deduction theorem, doesn't hold because $p_i \models p_i$ and $\emptyset \not\models p_i \to p_i$. In fact $M(\emptyset) = (3^\omega, \emptyset)$, but $M(\neg p_i \vee p_i) \neq (3^\omega, \emptyset)$, as shown above. The implication from right to left, known as modus ponens, doesn't hold because $\neg p_i \models p_i \to 0$, and

$\{\neg p_i, p_i\} \not\models 0$. In fact

$$M(\neg p_i \vee 0) = (M(p_i)_1, M(p_i)_0) \vee (\emptyset, 3^\omega) = (M(p_i)_1, M(p_i)_0) = M(\neg p_i),$$

but $M(\neg p_i) \wedge M(p_i) = (\emptyset, M(p_i)_0 \cup M(p_i)_1) \nleqslant (\emptyset, 3^\omega)$.

Finally we can develope DMF-valued semantics by substituting 3 with any DMF-algebra $\mathcal{A}$, as we did with classical logic by substituting 2 with any Boolean logic $\mathcal{A}$, and in full analogy with classical logic we can prove that 3-valued logic is equivalent to DMF-valued logic.

For every DMF-algebra $\mathcal{A}$ we consider the morphism $M^{\mathcal{A}}: Fm \to \mathcal{D}(A^\omega)$ induced by $g_{\mathcal{A}}: E \to \mathcal{D}(A^\omega)$, where

$$g_{\mathcal{A}}(p_i) = (\{s \in A^\omega: s_i = 1\}, \{s \in A^\omega: s_i = 0\}).$$

Then we can define the concept of logical consequence relativized to $\mathcal{A}$ setting $\alpha \models^{\mathcal{A}} \beta$ iff $M^{\mathcal{A}}(\alpha) \leqslant M^{\mathcal{A}}(\beta)$. In order to prove the equivalence between 3-valued consequence $\models$ and $\mathcal{A}$-valued consequence $\models_{\mathcal{A}}$ we have to state relations between 3 -valued semantics, given by M, and $\mathcal{A}$-valued semantics, given by $M^{\mathcal{A}}$. Firstly we note that $\mathcal{D}(3^\omega)$, the codomain of M, is just the closed interval $[(\emptyset, 3^\omega), (3^\omega, \emptyset)]$ taken in $\mathcal{D}(A^\omega)$, the codomain of $M^{\mathcal{A}}$. We recall that in any lattice $\mathcal{A}$, given a and b such that $a \leqslant b$, the closed inteval $[a, b]$ is a sublattice. If $\mathcal{A}$ is bounded then $[a, b]$ is no more a sublattice, because 0 and 1 generally don't belong to $[a, b]$, but when $\mathcal{A}$ is distributive we can define a morphism of bounded lattices f_a^b from $\mathcal{A}$ onto $[a, b]$ setting $f_a^b(x) = (x \vee a) \wedge b$. If $\mathcal{A}$ is a DMF-algebra and $b = \neg a$ then f_a^b is a morphism of DMF-algebras (see [5] [par. 3]). In particular we can set $a = (\emptyset, 3^\omega)$ and $b = (3^\omega, \emptyset)$ and consider the morphism f_a^b from $\mathcal{D}(A^\omega)$ to $\mathcal{D}(3^\omega)$. It can be easily verified that

$$\begin{aligned} f_a^b(X, Y) &= ((X \cup \emptyset) \cap 3^\omega, (Y \cap 3^\omega) \cup \emptyset) \\ &= (X \cap 3^\omega, Y \cap 3^\omega). \end{aligned}$$

THEOREM 4. – *If* $M^{\mathcal{A}}(\alpha) \leqslant M^{\mathcal{A}}(\beta)$ *then* $M(\alpha) \leqslant M(\beta)$.

Firstly we prove that $M = f_a^b \circ M^{\mathcal{A}}$, when $a = (\emptyset, 3^\omega)$ and $b = (3^\omega, \emptyset)$. As $f_a^b \circ M^{\mathcal{A}}$ is a morphism, we can limit ourselves to prove that $f_a^b \circ M^{\mathcal{A}}$ and M agree on the generators of Fm. This is proved by observing that

$$\begin{aligned} f_a^b(M^{\mathcal{A}}(p_i)) &= (\{s \in A^\omega: s_i = 1\} \cap 3^\omega, \{s \in A^\omega: s_i = 0\} \cap 3^\omega) \\ &= (\{s \in 3^\omega: s_i = 1\}, \{s \in 3^\omega: s_i = 0\}) \\ &= M(p_i). \end{aligned}$$

Then $M^{\mathcal{A}}(\alpha) \leqslant M^{\mathcal{A}}(\beta)$ implies $f_a^b(M^{\mathcal{A}}(\alpha)) \leqslant f_a^b(M^{\mathcal{A}}(\beta))$ and $M(\alpha) \leqslant M(\beta)$.

In order to prove the converse we associate to every $s \in A^\omega$ the sequence

$s^* \in 3^\omega$ defined as follows:

$$s_i^* = \begin{cases} 0 & \text{if } s_i = 0 \\ 1 & \text{if } s_i = 1 \\ n & \text{otherwise.} \end{cases}$$

LEMMA 5. – *For any* $s \in A^\omega$ *and any formula* α, $s \in M^{\mathfrak{a}}(\alpha)_0$ *iff* $s^* \in M(\alpha)_0$ *and* $s \in M^{\mathfrak{a}}(\alpha)_1$ *iff* $s^* \in M(\alpha)_1$.

By induction on α. Let $\alpha = p_i$. Then $s \in M^{\mathfrak{a}}(p_i)_0$ iff $s_i = 1$ iff $s_i^* = 1$ iff $s^* \in M(\alpha)_0$. In the same way $s \in M^{\mathfrak{a}}(p_i)_1$ iff $s_i = 0$ iff $s_i^* = 0$ iff $s^* \in M(\alpha)_1$. We consider only the case of $\alpha = \beta \wedge \gamma$, as the other cases follow in the same way. On one side we have $s \in M^{\mathfrak{a}}(\beta \wedge \gamma)_0$ iff $s \in (M^{\mathfrak{a}}(\beta) \wedge M^{\mathfrak{a}}(\gamma))_0$ iff $s \in M^{\mathfrak{a}}(\beta)_0 \cap M^{\mathfrak{a}}(\gamma)_0$ iff $s^* \in M(\beta)_0 \cap M(\gamma)_0$ iff $s^* \in (M(\beta) \wedge M(\gamma))_0$ iff $s^* \in M(\beta \wedge \gamma)_0$. On the other side, $s \in M^{\mathfrak{a}}(\beta \wedge \gamma)_1$ iff $s \in (M^{\mathfrak{a}}(\beta) \wedge M^{\mathfrak{a}}(\gamma))_1$ iff $s \in M^{\mathfrak{a}}(\beta)_1 \cup M^{\mathfrak{a}}(\gamma)_1$ iff $s^* \in M(\beta)_1 \cup M(\gamma)_1$ iff $s^* \in (M(\beta) \wedge M(\gamma))_1$ iff $s^* \in M(\beta \wedge \gamma)_1$.

THEOREM 6. – *If* $M(\alpha) \leqslant M(\beta)$ *then* $M^{\mathfrak{a}}(\alpha) \leqslant M^{\mathfrak{a}}(\beta)$.

Let $s \in A^\omega$. Then $s \in M^{\mathfrak{a}}(\alpha)_0$ and, by the lemma above, $s^* \in M(\alpha)_0$. By hypothesis $M(\alpha)_0 \subseteq M(\beta)_0$ so $s^* \in M(\beta)_0$ and $s \in M^{\mathfrak{a}}(\beta)_0$, by the lemma above. This proves that $M^{\mathfrak{a}}(\alpha)_0 \subseteq M^{\mathfrak{a}}(\beta)_0$. In the same way we can prove that $M^{\mathfrak{a}}(\beta)_1 \subseteq M^{\mathfrak{a}}(\alpha)_1$ and then $M^{\mathfrak{a}}(\alpha) \leqslant M^{\mathfrak{a}}(\beta)$.

3. – Calculi of sequents.

A sequent is an ordered pair $(\Gamma; \alpha)$ where $\Gamma \subseteq Fm$ and $\alpha \in Fm$. If we denote with Sq the set of all sequents, the product $\mathcal{P}(Fm) \times Fm$, then every logical consequence relation $\models$ is a subset of Sq. Our task is to describe the relations $\models$ introduced in the preceding paragraph in purely syntactical terms, so we introduce the concept of sequent calculus. We say that a n-ary rule, $n > 0$, is a subset ϱ of $Sq^n \times Sq$. A sequent calculus is an ordered pair $S = (B, R)$, where $B \subseteq Sq$ and $R = \{\varrho_i : i \in I\}$, where every ϱ_i is a n-ary rule for some $n \in \omega$. Every ordered pair $((\sigma_0, \ldots, \sigma_{n-1}), \sigma)$ in ϱ is said to be an application of ϱ, where $(\sigma_0, \ldots, \sigma_{n-1})$ are the premisses and σ the conclusion of the application. Every application of ϱ can be represented with

$$\frac{\sigma_0, \ldots, \sigma_{n-1}}{\sigma}.$$

Given a sequent calculus S, we can inductively define the set of theorems of S as the least subset of Sq containing B and closed with respect to rules in R.

When $(\Gamma; \alpha)$ is a theorem of S we write $\Gamma \vdash_S \alpha$, or simply $\Gamma \vdash \alpha$ if S is clear from the context. Only finitary rules are admitted, so every theorem has a finitary construction process, starting from axioms and growing through a finite number of applications of rules: such a process is called a proof of the theorem and is generally (but not necessarily) represented in tree form. Let $\models$ be a given relation of logical consequence. We say that a sequent calculus is *sound* if $\Gamma \vdash \alpha$ implies $\Gamma \models \alpha$ and *complete if* $\Gamma \models \alpha$ implies $\Gamma \vdash \alpha$. Then our task may be newly formulated as follows: to define a sound and complete calculus for the consequence relations of the preceding paragraph.

Given a calculus S and a set of formulas Σ, we define a relation R on Fm setting $\alpha R \beta$ iff $\Sigma, \alpha \vdash \beta$. (As usual, we often write Σ, α instead of $\Sigma \cup \{\alpha\}$.) If we assume as axioms in S all sequents $(\Gamma; \alpha)$, with $\alpha \in \Gamma$, called *identity axioms*, then R can be proved reflexive. If we assume as a rule of S

$$\frac{(\Gamma; \alpha) \ (\Delta, \alpha; \beta)}{(\Gamma, \Delta; \beta)}$$

the *cut rule*, then R can be proved transitive. We say that two formulas α and β are syntactically equivalent with respect to Σ iff $\alpha R \beta$ and $\beta R \alpha$ iff $\Sigma, \alpha \vdash \beta$ and $\Sigma, \beta \vdash \alpha$. We write $\alpha \equiv_\Sigma \beta$ to denote syntactical equivalence between α and β with respect to Σ: as R is a preorder, $\equiv_\Sigma$ is an equivalence. It can be easily verified that $\equiv_\Sigma$ is also a congruence with respect to R, so we can extend R to a relation $\leqslant$ between equivalence classes modulo $\equiv_\Sigma$ setting $|\alpha| \leqslant |\beta|$ iff $\alpha R \beta$ iff $\Sigma, \alpha \vdash \beta$. The set $Fm/\equiv_\Sigma$ with the partial order $\leqslant$ is the *Lindenbaum algebra of* Σ. We denote with $Fm/\equiv_\Sigma$ both the set of equivalence classes and the Lindenbaum algebra. When $\Sigma = \emptyset$ we denote with $\equiv$ the relation $\equiv_\Sigma$ and call $Fm/\equiv$ the *Lindenbaum algebra of logic*. Which axioms are true in the Lindenbaum algebra depends strictly on the axioms and rules in S. We shall define three different calculi, S_{DL}, S_{BA} and S_{DMF}: the first gives place to bounded distributive lattices, the second to Boolean algebras, the third to DMF-algebras. Axioms and rules of S_{DL} are as follows:

- identity axioms: $\Gamma \vdash \alpha$, where $\alpha \in \Gamma$,
- empty set axiom: $\emptyset \vdash 1$,
- maximum axiom: $\alpha \vdash 1$,
- minimum axiom: $0 \vdash \alpha$,
- cut rule,
- first rule of introduction of $\wedge$ in the antecedent or $\wedge \vdash_0$,

$$\frac{(\Gamma, \alpha_0; \beta)}{(\Gamma, \alpha_0 \wedge \alpha_1; \beta)}$$

- second rule of introduction of $\wedge$ in the antecedent or $\wedge\vdash_1$,

$$\frac{(\Gamma, \alpha_1; \beta)}{(\Gamma, \alpha_0 \wedge \alpha_1; \beta)}$$

- rule of introduction of $\wedge$ in the consequent or $\vdash\wedge$,

$$\frac{(\Gamma; \alpha)\ (\Gamma; \beta)}{(\Gamma; \alpha \wedge \beta)}$$

- rule of introduction of $\vee$ in the antecedent or $\vee\vdash$,

$$\frac{(\Gamma, \alpha_0; \beta)\ (\Gamma, \alpha_1; \beta)}{(\Gamma, \alpha_0 \vee \alpha_1; \beta)}$$

- first rule of introduction of $\vee$ in the consequent or $\vdash\vee_0$,

$$\frac{(\Gamma; \alpha_0)}{(\Gamma; \alpha_0 \vee \alpha_1)}$$

- second rule of introduction of $\vee$ in the consequent or $\vdash\vee_1$,

$$\frac{(\Gamma; \alpha_1)}{(\Gamma; \alpha_0 \vee \alpha_1)}.$$

In the following lemmas some basic properties of $\vdash_{DL}$ are collected.

LEMMA 7. – *In $S(DL)$ we have:*

1) $\alpha \wedge \beta \vdash \alpha$ *and* $\alpha \wedge \beta \vdash \beta$,

2) $\alpha \vdash \alpha \vee \beta$ *and* $\beta \vdash \alpha \vee \beta$.

1) Immediate from identity axioms and rules $\wedge\vdash_0$ and $\wedge\vdash_1$. 2) Immediate from identity axioms and rules $\vdash\vee_0$ and $\vdash\vee_0$.

LEMMA 8. – *In $S(DL)$ monotony rule holds*:

$$\frac{(\Gamma; \alpha)}{(\Gamma, \Delta; \alpha)}$$

From cut rule and identity axioms we have

$$\frac{(\Gamma; \alpha)\ (\Delta, \alpha; \alpha)}{(\Gamma, \Delta; \alpha)}$$

LEMMA 9. – *In S(DL) the following rule holds*:

$$\frac{(\Gamma, \alpha, \beta; \gamma)}{(\Gamma, \alpha \wedge \beta; \gamma)}$$

From $\wedge \vdash_1$ and $\wedge \vdash_0$ we have

$$\cfrac{\cfrac{(\Gamma, \alpha, \beta; \gamma)}{(\Gamma, \alpha, \alpha \wedge \beta; \gamma)}}{\Gamma, \alpha \wedge \beta, \alpha \wedge \beta \vdash \gamma}$$

that is what we need, because $\Gamma, \alpha \wedge \beta, \alpha \wedge \beta$ is the same as $\Gamma, \alpha \wedge \beta$.

THEOREM 10. – *For every set of formulas Σ, if $\equiv_\Sigma$ is the equivalence generated by $\vdash_{DL}$ then $Fm/\equiv_\Sigma$ is a bounded distributive lattice.*

We know that $Fm/\equiv_\Sigma$ is a partial order when $|\alpha| \leqslant |\beta|$ is defined as $\Sigma, \alpha \vdash \beta$. We introduce in $Fm/\equiv_\Sigma$ two operation inf and sup as follows: $\inf(|\alpha|, |\beta|) = |\alpha \wedge \beta|$ and $\sup(|\alpha|, |\beta|) = |\alpha \vee \beta|$. In the first place we have $|\alpha \wedge \beta| \leqslant |\alpha|, |\beta|$ because from lemma 3 and monotony rule we have $\Sigma, \alpha \wedge \beta \vdash \alpha$ and $\Sigma, \alpha \wedge \beta \vdash \beta$. Then for all $|\gamma|$, if $|\gamma| \leqslant |\alpha|$ and $|\gamma| \leqslant |\beta|$ then $|\gamma| \leqslant |\alpha \wedge \beta|$, because from $\Sigma, \gamma \vdash \alpha$ and $\Sigma, \gamma \vdash \beta$, with an application of $\vdash \wedge$ we have $\Sigma, \gamma \vdash \alpha \wedge \beta$. In the second place we have $\sup(|\alpha|, |\beta|) = |\alpha \vee \beta|$ because $|\alpha|, |\beta| \leqslant |\alpha \vee \beta|$ follows from $\Sigma, \alpha \vdash \alpha \vee \beta$ and $\Sigma, \beta \vdash \alpha \vee \beta$, which is easily proved by lemma 3 and monotony rule. Then for all $|\gamma|$, if $|\alpha|, |\beta| \leqslant |\gamma|$ then $|\alpha \vee \beta| \leqslant |\gamma|$, because $\Sigma, \alpha \vee \beta \vdash \gamma$ follows from $\Sigma, \alpha \vdash \gamma$ and $\Sigma, \beta \vdash \gamma$ with an application of $\vee \vdash$. From now on we shall write $\wedge$ instead of inf and $\vee$ instead of sup.

The lattice $Fm/\equiv_\Sigma$ is bounded because from the maximum axiom and the monotony rule we have $\Sigma, \alpha \vdash 1$ and so $|\alpha| \leqslant |1|$ for all α. In the same way, from the minimum axiom we have $\Sigma, 0 \vdash \alpha$ and so $|0| \leqslant |\alpha|$. In order to prove distributivity, we can limit ourselves to show that $|\alpha| \vee (|\beta| \wedge |\gamma|) = (|\alpha| \vee |\beta|) \wedge (|\alpha| \vee |\gamma|)$. If we denote with Π_0 the proof tree

$$\cfrac{\Sigma, \alpha \vdash \alpha \vee \beta \qquad \cfrac{\Sigma, \beta \wedge \gamma \vdash \beta \quad \beta \vdash \alpha \vee \beta}{\Sigma, \beta \wedge \gamma \vdash \alpha \vee \beta}}{\Sigma, \alpha \vee (\beta \wedge \gamma) \vdash \alpha \vee \beta}$$

and with Π_1 the proof tree

$$\cfrac{\Sigma, \alpha \vdash \alpha \vee \gamma \qquad \cfrac{\Sigma, \beta \wedge \gamma \vdash \gamma \quad \gamma \vdash \alpha \vee \gamma}{\Sigma, \beta \wedge \gamma \vdash \alpha \vee \gamma}}{\Sigma, \alpha \vee (\beta \wedge \gamma) \vdash \alpha \vee \gamma}$$

then with an application of $\vdash \wedge$ we have

$$\frac{\Pi_0 \qquad \Pi_1}{\Sigma,\ \alpha \vee (\beta \wedge \gamma) \vdash (\alpha \vee \beta) \wedge (\alpha \vee \gamma)}\,.$$

If we denote with Ξ_0 the proof tree

$$\frac{\Sigma,\ \alpha \vee \beta,\ \alpha \vdash \alpha}{\Sigma,\ \alpha \vee \beta,\ \alpha \vdash \alpha \vee (\beta \wedge \gamma)}$$

and with Ξ_1 the proof tree

$$\frac{\dfrac{\Sigma,\ \gamma,\ \alpha \vdash \alpha}{\Sigma,\ \gamma,\ \alpha \vdash \alpha \vee (\beta \wedge \gamma)} \quad \dfrac{\dfrac{\Sigma,\ \gamma,\ \beta \vdash \beta \quad \Sigma,\ \gamma,\ \beta \vdash \gamma}{\Sigma,\ \gamma,\ \beta \vdash \beta \wedge \gamma}}{\Sigma,\ \gamma .\ \beta \vdash \alpha \vee (\beta \wedge \gamma)}}{\Sigma,\ \gamma,\ \alpha \vee \beta \vdash \alpha \vee (\beta \wedge \gamma)}$$

then with an application of $\vee \vdash$ we have

$$\frac{\dfrac{\Xi_0 \qquad \Xi_1}{\Sigma, (\alpha \vee \beta), (\alpha \vee \gamma) \vdash \alpha \vee (\beta \wedge \gamma)}}{\Sigma, (\alpha \vee \beta) \wedge (\alpha \vee \gamma) \vdash \alpha \vee (\beta \wedge \gamma)}$$

where the last line follows from lemma 9.

From $S(DL)$ we obtain $S(BA)$ by adjoining the following axioms:

- tertium non datur or tnd axiom: $1 \vdash \alpha \vee \neg \alpha$,
- contradiction axiom: $\alpha \wedge \neg \alpha \vdash 0$.

An easy argument by induction shows that $S(BA)$ is sound with respect to the relation $\models$ of classical logic.

THEOREM 11. – *For every set of formulas* Σ, *if* $\equiv_\Sigma$ *is the equivalence generated by* $\vdash_{BA}$ *then* $Fm/\equiv_\Sigma$ *is a Boolean algebra in which* $|\alpha| = |1|$ *iff* $\Sigma \vdash \alpha$.

By the preceding theorem we know that $Fm/\equiv_\Sigma$ is a bounded distributive lattice, so we need only to prove that $|\neg\alpha|$ is the complement of $|\alpha|$, i.e. $|\alpha| \vee |\neg\alpha| = |1|$ and $|\alpha| \wedge |\neg\alpha| = |0|$. By monotony and maximum axiom we have $\Sigma,\ \alpha \vee \neg\alpha \vdash 1$ and by monotony and tnd we have $\Sigma,\ 1 \vdash \alpha \vee \neg\alpha$, so $\alpha \vee \neg\alpha \equiv_\Sigma 1$. In the same way we can prove that $\alpha \wedge \neg\alpha \equiv_\Sigma 0$.

From the hypothesis $\Sigma \vdash \alpha$ we have $\Sigma, 1 \vdash \alpha$ by monotony; on the other side we have

$$\frac{\Sigma, \alpha \vdash \alpha \quad \alpha \vdash 1}{\Sigma, \alpha \vdash 1}$$

by identity axioms, maximum axioms and cut rule, so $|\alpha| = |1|$. Now suppose $|\alpha| = |1|$, then $\alpha \equiv_\Sigma \beta$ and $\Sigma, 1 \vdash \alpha$. By the empty set axiom $\emptyset \vdash 1$ and an application of the cut rule we have $\Sigma \vdash \alpha$.

From $S(DL)$ we obtain $S(DMF)$ by adjoining the following axioms and rules:

- first double negation axiom: $\neg\neg\alpha \vdash \alpha$,
- second double negation axiom: $\alpha \vdash \neg\neg\alpha$,
- first undefined element axiom: $n \vdash \neg n$,
- second undefined element axiom $\neg n \vdash n$,
- normality axiom: $\alpha \wedge \neg\alpha \vdash \beta \vee \neg\beta$.
- contraposition rule:

$$\frac{\alpha; \beta}{\neg\beta; \neg\alpha}$$

An easy argument by induction shows that $S(DMF)$ is sound when $\models$ is the logical consequence of 3-valued logic.

THEOREM 12. – *For every set of formulas Σ, if $\equiv$ is the equivalence generated by $\vdash_{DMF}$ then $Fm/\equiv$ is a DMF-algebra.*

We know that $Fm/\equiv$ is a bounded distributive lattice by the above theorem. By contraposition rule, $\alpha \equiv \beta$ implies $\neg\beta \equiv \neg\alpha$, so we can introduce an operation $\neg|\alpha| = |\neg\alpha|$ in $Fm/\equiv$. We show that DMF-algebras axioms hold in $Fm/\equiv$. From the double negation axioms we have $|\alpha| = \neg\neg|\alpha|$. From the undefined element axioms we have $|n| = \neg|n|$. From the normality axiom we have $|\alpha| \wedge \neg|\alpha| \leqslant |\beta| \vee \neg|\beta|$. We prove the first De Morgan law $\neg(|\alpha| \wedge |\beta|) = \neg|\alpha| \vee \neg|\beta|$ as follows. We denote with Π_0 the proof tree

$$\frac{\dfrac{\neg\alpha \vdash \neg\alpha \vee \neg\beta}{\neg(\neg\alpha \vee \neg\beta) \vdash \neg\neg\alpha} \quad \neg\neg\alpha \vdash \alpha}{\neg(\neg\alpha \vee \neg\beta) \vdash \alpha}$$

and with Π_1 an analogous proof tree for $\neg(\neg\alpha\vee\neg\beta)\vdash\beta$. Then we have

$$\dfrac{\dfrac{\dfrac{\Pi_0,\quad \Pi_1}{\neg(\neg\alpha\vee\neg\beta)\vdash\alpha\wedge\beta}}{\neg(\alpha\wedge\beta)\vdash\neg\neg(\neg\alpha\vee\neg\beta)}\quad \neg\neg(\neg\alpha\vee\neg\beta)\vdash\neg\alpha\vee\neg\beta}{(\alpha\wedge\beta)\vdash\neg\alpha\vee\neg\beta}$$

In the other direction we have

$$\dfrac{\dfrac{\alpha\wedge\beta\vdash\alpha\qquad \alpha\wedge\beta\vdash\beta}{\neg\alpha\vdash\neg(\alpha\wedge\beta)\quad \neg\beta\vdash\neg(\alpha\wedge\beta)}}{\neg\alpha\vee\neg\beta\vdash\neg(\alpha\wedge\beta)}$$

The other De Morgan law can be proved in the same way.

We don't introduce the Lindenbaum algebra $Fm/\equiv_\Sigma$ in correspondence of $S(DMF)$, because $\equiv_\Sigma$ is not generally a $\neg$-congruence. Let $\Sigma=\{p_i\}$, we have $\Sigma,\ p_i\vdash 1$,by the maximum axiom and monotony, and $\Sigma,\ 1\vdash p_i$, by the identity axiom, so $p_i\equiv_\Sigma 1$. If we had also $\neg p_i\equiv_\Sigma 0$, then we could prove $\Sigma,\ \neg p_i\vdash 0$, i.e. $p_i,\ \neg p_i\vdash 0$, in $S(DMF)$, but this is impossible because $S(DMF)$ is sound with respect to the relation $\models$ of 3-valued logic and $p_i,\ \neg p_i\not\models 0$ (as we have seen in the preceding paragraph). For the same reason the generalized contraposition rule

$$\dfrac{(\Gamma,\ \alpha;\ \beta)}{(\Gamma,\ \neg\beta;\ \neg\alpha)}$$

is not a derived rule of $S(DMF)$.

Finally we observe that a weak form of deduction theorem and of modus ponens holds in $S(DMF)$. In fact, $\Sigma,\ \alpha\vdash\beta$ implies $\Sigma,\ n\vdash\neg\alpha\vee\beta$ because

$$\dfrac{n\vdash\alpha\vee\neg\alpha\qquad \dfrac{\dfrac{\Sigma,\ \alpha\vdash\beta}{\Sigma,\ \alpha\vdash\neg\alpha\vee\beta}\quad \neg\alpha\vdash\neg\alpha\vee\beta}{\Sigma,\ \alpha\vee\neg\alpha\vdash\neg\alpha\vee\beta}}{\Sigma,\ n\vdash\neg\alpha\vee\beta}$$

and $\Sigma \vdash \neg\alpha \vee \beta$ implies $\Sigma, \alpha \vdash \beta \vee n$ because

$$\frac{\Sigma \vdash \neg\alpha \vee \beta \qquad \dfrac{\dfrac{\Sigma, \alpha, \neg\alpha \vdash n \qquad \Sigma, \alpha, \beta \vdash \beta}{\Sigma, \alpha, \neg\alpha \vdash \beta \vee n \quad \Sigma, \alpha, \beta \vdash \beta \vee n}}{\Sigma, \alpha, \neg\alpha \vee \beta \vdash \beta \vee n}}{\Sigma, \alpha \vdash \beta \vee n}$$

4. – Prime pairs and morphisms onto 3.

The main tool in the proof of completeness theorem for 3-valued logic is given by a modified version of the prime ideal theorem in DMF-algebras. We recall that an ideal I is prime iff it is proper and $x \wedge y \in I$ implies $x \in I$ or $y \in I$. Duallly, a filter F is prime iff it is a proper filter and $x \vee y \in I$ implies $x \in I$ or $y \in I$. There is a bijection between prime ideals and prime filters, because an ideal is prime iff its complement is a prime filter. We assume the following formulation of the prime ideal theorem (see [3] [theorem 9.13]).

THEOREM 13. – *If $\mathcal{A}$ is a distributive lattice and I and F are respectively an ideal and a filter in $\mathcal{A}$ such that $I \cap F = \emptyset$, then*

1) *there is a prime ideal J such that $I \subseteq J$ and $J \cap F = \emptyset$,*

2) *there is a prime filter G such that $F \subseteq G$ and $G \cap I = \emptyset$.*

COROLLARY 14. – *If x and y are points of a distributive lattice $\mathcal{A}$ and $x \not\leq y$, then there is a prime ideal I in $\mathcal{A}$ such that $y \in I$ and $x \notin I$, and there is a prime filter F in $\mathcal{A}$ such that $x \in F$ and $y \not\leq F$.*

Let $x\downarrow$ be the ideal $\{a \in A : a \leq x\}$ and $y\uparrow$ be the filter $\{a \in A : y \leq a\}$. They have no common element, because $x \not\leq y$, so by the prime ideal theorem there is a prime ideal I such that $y\downarrow \subseteq I$ and $I \cap x\uparrow = \emptyset$, and there is a prime filter F such that $x\uparrow \subseteq F$ $F \cap y\downarrow = \emptyset$.

There is a tight connection between prime ideals and epimorphisms $\phi : \mathcal{A} \to 2$. On one side we observe that, if ϕ is an epimorphism, then the set $I_\phi = \phi^{-1}\{0\}$ is a prime ideal in $\mathcal{A}$. On the other side we can define, for every prime ideal I in $\mathcal{A}$, a function ϕ_I from $\mathcal{A}$ to 2 as follows:

$$\phi_I(a) = \begin{cases} 0 & \text{if } a \in I \\ 1 & \text{if } a \notin I . \end{cases}$$

Then we can easily prove the following result (see [3] [ex. 9.2]).

THEOREM 15. – *If $\mathcal{A}$ is a bounded lattice and I is a prime ideal in $\mathcal{A}$, then $\phi_I\colon \mathcal{A} \to 2$ is a morphism of bounded lattices, and if $\mathcal{A}$ is a Boolean algebra then ϕ_I is a Boolean morphism.*

A dual proposition holds for filters.

Now we prove similar results for DMF-algebras, where the role of a single prime ideal I (or filter F) separating points $x \not\leq y$ is kept by a couple (I, F), with $I \cap F = \emptyset$ and $F = \neg I$, and where the role of morphisms $\phi_I\colon \mathcal{A} \to 2$ is taken by $\phi_I\colon \mathcal{A} \to 3$. We recall some results obtained in [5]. There we have shown (T. 4.2) that if $\mathcal{A}$ is a bounded distributive lattice and I and F are respectively an ideal and a filter in $\mathcal{A}$, then the relation $a \equiv b$ iff there are $i \in I$ and $f \in F$ such that $(a \vee i) \wedge f = (b \vee i) \wedge f$ is a congruence on $\mathcal{A}$. Then we have shown (T. 4.2) that the quotient $\mathcal{A}/I, F$ is a bounded lattice that is not trivial iff $I \cap F = \emptyset$. If $\mathcal{A}$ is a De Morgan algebra, I is an ideal in $\mathcal{A}$ and $F = \neg I$, where $\neg I = \{\neg i\colon i \in I\}$, then $\equiv$ is also a $\neg$-congruence and $\mathcal{A}/I, \neg I$ is a De Morgan algebra (T.5.2). This result can be immediately extended to DMF-algebras, so for every couple $(I, \neg I)$ in a DMF-algebra $\mathcal{A}$ there is a homomorphism from $\mathcal{A}$ to the DMF-algebra $\mathcal{A}/I, \neg I$ that is not trivial iff $I \cap \neg I = \emptyset$. We observe that in a DMF-algebra the last condition is equivalent to $n \notin I$. In fact, $n \in I$ implies $\neg n \in -I$, but $n = \neg n$ so $I \cap \neg I \neq \emptyset$. In the other direction, if $x \in I \cap \neg I$ then from $x \in \neg I$ we have $x = \neg i$, for some $i \in I$, and from $x \in I$ we have $i \vee -i \in I$. As in DMF-algebras $n \leqslant i \vee \neg i$, we have $n \in I$. In the following we shall consider only $\mathcal{A}/I, \neg I$ with I prime. If I is prime so is also $\neg I$ (T. 5.1) and the zero and unit of $\mathcal{A}/I, \neg I$ are respectively I and $\neg I$ (T. 5.5). The following theorem is analogous to the corollary of the prime ideal theorem.

THEOREM 16. – *If $\mathcal{A}$ is a DMF-algebra and $a \not\leq b$, then there is a pair (I, F) such that:*

1) *I is a prime ideal in $\mathcal{A}$ and F is a prime filter in $\mathcal{A}$, with $I \cap F = \emptyset$,*

2) *$F = -I$,*

3) *$a \notin I$ and $b \in I$ or $a \in F$ and $b \notin F$.*

If $a \not\leq b$ then $a\uparrow \cap b\downarrow = \emptyset$. By corollary 14, there is a prime ideal I such that $b\downarrow \subseteq I$ and $I \cap a\uparrow = \emptyset$. So $b \in I$ and $a \notin I$. If $n \notin I$ then $I \cap \neg I = \emptyset$ by the remark above, $\neg I$ is a prime filter and so $(I, \neg I)$ is the couple we are looking for. If $n \in I$ we distinguish two cases.

Case 1, $n \in b\downarrow$. Then $n \leqslant b$ and so $n \notin a\uparrow$ because $a \not\leq b$. By 14 there is a prime filter F such that $a\uparrow \subseteq F$ and $F \cap b\downarrow = \emptyset$. Then $a \in F$ and $b \notin F$. We have also $n \notin F$, because $n \in F$ implies $b \in F$ and from the hypothesis $n \in b\downarrow$ we could derive $F \cap b\downarrow \neq \emptyset$, that is absurd. So from $n \notin F$ we have $\neg F \cap F = \emptyset$. As $\neg F$ is a prime ideal, $(\neg F, F)$ is the required couple.

Case 2, $n \in I - b\downarrow$. By 14 there is a prime filter F such that $a\uparrow \subseteq F$ and $F \cap b\downarrow = \emptyset$. If $n \notin F$ still holds, then $(\neg F, F)$ is the required couple. So we suppose $n \in F$. As a and n belong to F, also $a \wedge n$ belongs to F. Then we have $a \wedge n \nleq b$, because from $a \wedge n \leq b$ we could infer $b \in F$, but $F \cap b\downarrow = \emptyset$. By 14 there is a prime ideal J such that $b\downarrow \subseteq J$ and $J \cap (a \wedge n)\uparrow = \emptyset$. Then we have $b \in J$ and $a \notin J$. In fact, from $a \in J$ we could derive $a \wedge n \in J$, but $J \cap (a \wedge n)\uparrow = \emptyset$. Finally $n \notin J$, because $n \in J$ implies $a \wedge n \in J$. So $(J, \neg J)$ is the required couple.

The next result illustrates the relations between morphisms $\phi : \mathcal{A} \to 3$ and couples $(I, \neg I)$ where I is a prime ideal in $\mathcal{A}$. For every morphism $\phi : \mathcal{A} \to 3$ we define the ideal $I_\phi = \phi^{-1}\{0\}$ and the filter $F_\phi = \phi^{-1}\{0\}$. We can easily prove that both I_ϕ and F_ϕ are prime and $F_\phi = -I_\phi$. Summing up, for every morphism $\phi : \mathcal{A} \to 3$ there is a couple $(I_\phi, \neg I_\phi)$ in which I_ϕ is a prime ideal, $\neg I_\phi$ is a prime filter and $I_\phi \cap \neg I_\phi = \emptyset$.

Now we suppose that $\mathcal{A}$ is a DMF-algebra and I a prime ideal in $\mathcal{A}$ such that $n \notin I$: we define $\phi_I : A \to 3$ setting

$$\phi_I(a) = \begin{cases} 0 & \text{if} \quad a \in I, \\ 1 & \text{if} \quad a \in \neg I, \\ n & \text{if} \quad a \in A - (I \cup \neg I). \end{cases}$$

The following theorem is analogous to 15.

THEOREM 17. – *If $\mathcal{A}$ is a DMF-algebra and I is a prime ideal in $\mathcal{A}$ such that $n \notin I$, then $\phi_I : \mathcal{A} \to 3$ is a morphism of DMF-algebras.*

Obviously $\phi_I(0) = 0$, $\phi_I(1) = 1$ and $\phi_I(n) = n$, because $n \notin I$ and $n \notin \neg I$.

We show that ϕ_I preserves $\wedge$. If $\phi_I(x \wedge y) = 0$ then $x \wedge y \in I$ and $x \in I$ or $y \in I$, because I is prime. So $\phi_I(x) = 0$ or $\phi_I(y) = 0$ and then $\phi_I(x) \wedge \phi_I(y) = 0$. If $\phi_I(x \wedge y) = 1$ then $x \wedge y \in \neg I$ and both x and y belongs to $\neg I$, because $\neg I$ is a filter. Then $\phi_I(x) = \phi_I(y) = 1$ and $\phi_I(x) \wedge \phi_I(y) = 1$. Finally we suppose $\phi_I(x \wedge y) = n$, then $x \wedge y \notin I$ e $x \wedge y \notin \neg I$. As I is an ideal, we have $x \notin I$ and $y \notin I$, otherwise we could derive $x \wedge y \in I$. As $\neg I$ is a filter, we have $x \notin \neg I$ or $y \notin \neg I$, otherwise we could derive $x \wedge y \in \neg I$. We can distinguish the following three cases. Case 1, x and y belong to $A - (I \cup \neg I)$. Then $\phi_I(x) = \phi_I(y) = n$ and so $\phi_I(x) \wedge \phi_I(y) = n$. Case 2, x belongs to $A - (I \cup \neg I)$ and y belongs to $\neg I$. Then $\phi_I(x) = n$, $\phi_I(y) = 1$ and so $\phi_I(x) \wedge \phi_I(y) = n$. Case 3, y belongs to $A - (I \cup \neg I)$ and x belongs to $\neg I$. Then $\phi_I(x) = 1$, $\phi_I(y) = n$ and so $\phi_I(x) \wedge \phi_I(y) = n$.

In the same way we can verify that ϕ_I preserves $\vee$.

Finally we verify that ϕ_I preserves $\neg$. If $\phi_I(\neg x) = 0$ then $\neg x \in I$ and so $x \in \neg I$ and $\phi_I(x) = 1$, i.e. $\phi_I(\neg x) = \neg \phi_I(x)$. If $\phi_I(\neg x) = 1$ then $\neg x \in -I$

and then $x \in I$ and $\phi_I(x) = 0$, i.e. $\phi_I(\neg x) = \neg \phi_I(x)$. If $\phi_I(\neg x) = n$ then $\neg x \notin I$ and $\neg x \notin \neg I$, so $x \notin I$ and $x \notin \neg I$. Then $\phi_I(x) = n$. As $n = -n$ we have $\phi_I(\neg x) = \neg \phi_I(x)$.

5. – Completeness and compactness.

Now we are in a good position to prove the completeness of $S(DMF)$ with respect to the relation $\models$ of 3-valued logic. The proof closely parallels the completeness proof for classical logic, as far as weak completeness is concerned. For the full completeness result we need a topological detour through the compactness of the space of models.

We shortly recall the lines of the completeness proof for classical logic. We have to prove that $\Sigma \models \alpha$ implies $\Sigma \vdash \alpha$, when $\models$ is the consequence relation of classical logic and $\vdash$ is the proof relation generated by $S(BA)$. If we prove that $\Sigma, \xi \models \alpha$ implies $\Sigma, \xi \vdash \alpha$, then the result follows. In fact, when ξ is 1, we have on the semantic side $\Sigma, 1 \models \alpha$ iff $\Sigma \models \alpha$. On the syntactic side, from $\Sigma, 1 \vdash \alpha$ and the empty set axiom $\emptyset \vdash 1$ we derive $\Sigma \vdash \alpha$ by cut, and from $\Sigma \vdash \alpha$ we derive $\Sigma, 1 \vdash \alpha$ by monotony. Now suppose $\Sigma, \xi \models \alpha$ and $\Sigma, \xi \nvdash \alpha$, then in the Lindenbaum algebra of Σ we have $|\xi| \nleq |\alpha|$ and by the corollary to the prime ideal theorem there is a prime ideal I such that $|\alpha| \in I$ and $|\xi| \notin I$. By theorem 15 there is a Boolean morphism ϕ_I from $Fm/\equiv_\Sigma$ to 2. Then we can define a classical evaluation $\nu : E \to 2$ setting $\nu(p_i) = \phi_I(|p_i|)$. Every ν induces an unique morphism Val_ν from the algebra of formulas Fm to 2. For every γ, $Val_\nu(\gamma) = \phi(|\gamma|)$ because Val_ν and $\phi_I \circ |\,|$ agree on the generators E of Fm. From $|\alpha| \in I$ and the definition of ϕ_I we have $\phi_I(|\alpha|) = 0$ and $Val_\nu(\alpha) = 0$. From $|\xi| \notin I$ and the definition of ϕ_I we have $\phi_I(|\xi|) = 1$ and $Val_\nu(\xi) = 1$. As $\Sigma \vdash \sigma$ for all $\sigma \in \Sigma$, we have $|\sigma| = |1|$ by theorem 11 and so $Val_\nu(\sigma) = \phi(|\sigma|) = 1$. Summing up, we have a valuation ν making true ξ and all $\sigma \in \Sigma$ and making false α, so $\Sigma, \xi \not\models \alpha$. The same kind of proof gives the following weak completeness theorem for 3-valued logic.

THEOREM 18. – *If $\models$ is the consequence relation of 3-valued logic and $\vdash$ is the proof relation generated by $S(DMF)$, then $\alpha \models \beta$ implies $\alpha \vdash \beta$.*

If $\alpha \nvdash \beta$ then $|\alpha| \nleq |\beta|$ in the Lindenbaum algebra of logic $Fm/\equiv$. By theorem 16 there is a couple (I, F) in $Fm/\equiv$ such that $|\alpha| \notin I$ and $|\beta| \in I$ or $|\alpha| \in F$ and $|\beta| \notin F$. By the same theorem I and F are prime, $F = \neg I$ and $I \cap F = \emptyset$, so $n \notin I$. (If $n \in I$ then $\neg n \in F$, but $n = \neg n$ so $n \in F$ and $I \cap F \neq \emptyset$.) By theorem 17 there is a morphism ϕ from $Fm/\equiv$ to 3 such that: for all $x \in I$, $\phi(x) = 0$, for all $x \in F$, $\phi(x) = 1$ and for all $x \notin I \cup F$, $\phi(x) = n$. Then we can define a valuation $\nu : E \to 3$ setting $\nu(p_i) = \phi(|p_i|)$. As Val_ν and $\phi \circ |\,|$ agree on the generators E of Fm, for every formula ξ we have $Val_\nu(\xi) = \phi(|\xi|)$. Now we can distinguish two cases. Case 1, $|\alpha| \notin I$ and $|\beta| \in I$, then $Val_v(\alpha) \in \{n, 1\}$ and $Val_v(\beta) = 0$, so $Val_v(\alpha) \nleq Val_v(\beta)$. Case 2, $|\alpha| \in F$ and $|\beta| \notin F$, then $Val_v(\alpha) = 1$ and $Val_v(\beta) \in \{0, n\}$, so $Val_v(\alpha) \nleq Val_v(\beta)$. In both cases $\alpha \not\models \beta$.

COROLLARY 19. – *$\alpha_0, \ldots, \alpha_{n-1} \models \beta$ implies $\alpha_0, \ldots, \alpha_{n-1} \vdash \beta$.*

If $\alpha_0, \ldots, \alpha_{n-1} \models \beta$ then $\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \models \beta$ and $\alpha_0 \wedge \ldots \wedge \alpha_{n-1} \vdash \beta$ by the above theorem. Now we observe that by the identity axioms $\alpha_0, \ldots, \alpha_{n-1} \vdash \alpha_i$, for all $i < n$, so with n applications of rule $\vdash \wedge$ we have $\alpha_0, \ldots, \alpha_{n-1} \vdash \alpha_0 \wedge \ldots \wedge \alpha_{n-1}$. Finally we get $\alpha_0, \ldots, \alpha_{n-1} \vdash \beta$ by cut.

Before proving the full completeness theorem, we must take a closer look at the space of models. We denote with $(3, \mathcal{T})$ the topological space in which $3 = \{0, n, 1\}$ and $\mathcal{T}$ is the discrete topology over 3. We denote with $(3^\omega, \mathcal{T}^*)$ the product space where $\mathcal{T}^*$ is the topology induced by the projections π_i. A subbase for $\mathcal{T}^*$ is given by $\mathcal{S} = \{\pi_i^{-1}(A) : A \in \mathcal{T}, i \in \omega\}$. By Tychonoff theorem $(3, \mathcal{T}^*)$ is a compact space, as a product of compact spaces.

LEMMA 20. – *For every $\alpha \in Fm$, $M(\alpha)_0$ and $M(\alpha)_1$ are clopen in $\mathcal{T}^*$.*

$M(p_i)_0 = \{s \in 3^\omega : s(i) = 1\} = \pi_i^{-1}(\{1\})$, so $M(p_i)_0 \in \mathcal{T}^*$ because $\{1\} \in \mathcal{T}$.

$-M(p_i)_0 = \{s \in 3^\omega : s(i) = 0 \text{ or } s(i) = n\} = \pi_i^{-1}(\{0, n\})$, so $-M(p_i)_0 \in \mathcal{T}^*$ because $\{0, n\} \in \mathcal{T}$.

$M(p_i)_1 = \{s \in 3^\omega : s(i) = 0\} = \pi_i^{-1}(\{0\})$, so $M(p_i)_1 \in \mathcal{T}^*$ because $\{0\} \in \mathcal{T}$.

$-M(p_i)_1 = \{s \in 3^\omega : s(i) = 1 \text{ or } s(i) = n\} = \pi_i^{-1}(\{1, n\})$, so $-M(p_i)_1 \in \mathcal{T}^*$ because $\{1, n\} \in \mathcal{T}$.

Let $\alpha = \beta \wedge \gamma$, then $M(\alpha)_0 = M(\beta)_0 \cap M(\gamma)_0$ and by inductive hypothesis $M(\beta)_0$ and $M(\gamma)_0$ are clopen so $M(\alpha)_0$ is a clopen too. In the same way we show that $M(\alpha)_1 = M(\beta)_1 \cup M(\gamma)_1$ is clopen. If $\alpha = \beta \vee \gamma$ then the same kind of proof shows that $M(\alpha)_0$ and $M(\alpha)_1$ are clopen. If $\alpha = \neg\beta$ then $M(\neg\beta)_0 = M(\beta)_1$ and $M(\neg\beta)_1 = M(\beta)_0$, where $M(\beta)_0$ and $M(\beta)_1$ are clopens by inductive hypothesis. Finally, if $\alpha = 1$ then $M(\alpha)_0 = 3^\omega$ and $M(\alpha)_1 = \emptyset$, and both 3^ω and $\emptyset$ are clopens in $\mathcal{T}^*$. The same proof works when $\alpha = 0$ and $\alpha = n$.

THEOREM 21. – *If $\models$ is the consequence relation of 3-valued logic, then $\Sigma \models \alpha$ implies $\Sigma' \models \alpha$ for some finite subset $\Sigma' \subseteq \Sigma$.*

By hypothesis $M(\Sigma) \leqslant M(\alpha)$, then $\bigwedge\{M(\sigma) : \sigma \in \Sigma\} \leqslant M(\alpha)$. So we have

$$\bigcap\{M(\sigma)_0 : \sigma \in \Sigma\} \subseteq M(\alpha)_0 \tag{1}$$

and

$$M(\alpha)_1 \subseteq \bigcup\{M(\sigma)_1 : \sigma \in \Sigma\}. \tag{2}$$

From 1 we have $-M(\alpha)_0 \subseteq \bigcup\{-M(\sigma)_0: \sigma \in \Sigma\}$. By the lemma above, $-M(\alpha)_0$ is closed and $\{-M(\sigma)_0: \sigma \in \Sigma\}$ is an open cover. As $(3^\omega, \mathcal{C}^*)$ is compact and every closed subset of a compact space is compact, $-M(\alpha)_0$ is compact too, so there are $\sigma_1^0, \ldots, \sigma_n^0$ in Σ such that $-M(\alpha)_0 \subseteq -M(\sigma_1^0)_0 \cup \ldots \cup -M(\sigma_n^0)_0$ and

$$M(\sigma_1^0)_0 \cap \ldots \cap M(\sigma_n^0)_0 \subseteq M(\alpha)_0. \tag{3}$$

From 2 and the preceding lemma we have an open cover $\{M(\sigma)_1: \sigma \in \Sigma\}$ of a closed set $M(\alpha)_1$, so by the compactness of $(3^\omega, \mathcal{C}^*)$ there are $\sigma_1^1, \ldots, \sigma_m^1$ in Σ such that

$$M(\alpha)_1 \subseteq M(\sigma_1^1)_1 \cup \ldots \cup M(\sigma_m^1)_1. \tag{4}$$

From 3 we have

$$M(\sigma_1^0)_0 \cap \ldots \cap M(\sigma_n^0)_0 \cap M(\sigma_1^1)_0 \cap \ldots \cap M(\sigma_m^1)_0 \subseteq M(\alpha)_0$$

and from 4 we have

$$M(\alpha)_1 \subseteq M(\sigma_1^0)_1 \cup \ldots \cup M(\sigma_n^0)_1 \cup M(\sigma_1^1)_1 \cup \ldots \cup M(\sigma_m^1)_1.$$

This proves that $\Sigma' \models \alpha$ when $\Sigma' = \{\sigma_1^0, \ldots, \sigma_n^0, \sigma_1^1, \ldots, \sigma_m^1\}$.

Now we can easily derive the full completeness theorem.

THEOREM 22. – *If* $\models$ *is the consequence relation of 3-valued logic and* $\vdash$ *is the proof relation generated by S(DMF), then* $\Sigma \models \alpha$ *implies* $\Sigma \vdash \alpha$.

If $\Sigma \models \alpha$ then $\sigma_0, \ldots, \sigma_{n-1} \models \alpha$ for some $\sigma_0, \ldots, \sigma_{n-1}$ in Σ, by the preceding theorem. Then $\sigma_0 \wedge \ldots \wedge \sigma_{n-1} \models \alpha$ and so $\sigma_0 \wedge \ldots \wedge \sigma_{n-1} \vdash \alpha$ by the weak completeness theorem. Now $\Sigma \vdash \sigma_i$ for all $i < n$, by identity axioms, and $\Sigma \vdash \sigma_0 \wedge \ldots \wedge \sigma_{n-1}$ by n applications of rule $\vdash \wedge$, so we get $\Sigma \vdash \alpha$ by cut.

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