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Some Results on Existence and Structure of Solution sets to Differential Inclusions on the Halfline.

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Sunto. – *Si studia la struttura topologica dell'insieme delle soluzioni di inclusioni differenziali sulla semiretta per mezzo di risultati tipo Scorza-Dragoni e del metodo dei sistemi inversi. Sono anche presentati alcuni nuovi risultati di esistenza di soluzioni per problemi al bordo asintotici.*

Summary. – *A topological structure of solution sets to multivalued differential problems on the halfline is studied by the use of Scorza-Dragoni type results and by the inverse systems approach. Some new existence results for asymptotic boundary value problems are also presented.*

1. – Introduction.

The paper is mainly devoted to studying a topological structure of solution sets to differential inclusions defined on the halfline. This research was motivated by [1], where authors have proved some general existence results for such multivalued problems by the use of a generalized Schauder linearization method.

Some results on the structure of the solution sets have been obtained in [1], [2] and [3]. Three main techniques have been used in proofs. The first one relies on Scorza-Dragoni type results used for the Fréchet space of continuous functions defined on the halfline (see e.g. [1]). The second one uses some properties of multivalued limit maps of inverse systems (see e.g. [2]). It is also possible to apply some multivalued generalizations of the Browder-Gupta result (see [3]).

The paper developes the ideas presented in [1,2]. Some new applications are given.

In Section 2 we recall the method presented in [1] and illustrate it by a new existence result for a second order boundary value problem on the halfline. In Section 3 we deal with target problems. The existence result for the second order one is presented and a topological structure of the solution set to the first order problem is also investigated. As a consequence, Theorems 4.2 and 5.4 in

[2] are essentially generalized. Finally, in Section 4 we give some applications of the inverse systems approach. Differential inclusions with retards and discontinuous autonomous differential problems are studied. The results give a more precise characterization of a topological structure of the solution sets of such problems.

2. – Existence result for second order boundary value problem on the halfline.

In this section we are interested in existence of solutions to the following boundary value problem

$$(1)\qquad \begin{cases} \ddot{x}(t) + \dot{x}(t) \in F(t, x(t)), & \text{for a.a. } t \in J = [0, \infty) \\ x(0) = 0, \quad \lim\limits_{t \to \infty} x(t) = 0, \end{cases}$$

where $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ satisfies the following assumptions:

(A) F is a u-Carathéodory map, that is, it has nonempty, compact and convex values, $F(\cdot, x)$ is measurable for every $x \in \mathbb{R}^n$ and $F(t, \cdot)$ is upper semicontinuous for almost all $t \in J$;

(B) $\lim\limits_{t \to \infty} \alpha(t) = 0$ and $\int\limits_0^\infty \alpha(t)\, dt = K < \infty$, where $\alpha(t) = \sup\limits_{v \in \mathbb{R}^n} |F(t, v)|$ (by $|F(t, v)|$ we mean the number $\sup \{y \,|\, y \in F(t, v)\}$).

We will use the technique applied to differential equations on noncompact intervals in [5, 6] and developed in [1] for multivalued problems.

The following result is a slight modification of the one obtained in [1] for scalar problems.

PROPOSITION 2.1 ([1], Corollary 2.37). – *Consider the problem*

$$(2)\qquad \begin{cases} x^{(n)}(t) + \sum\limits_{i=0}^{n-1} a_i(t, x(t), \ldots, x^{(n-1)}(t))x^{(i)}(t) \\ \qquad\qquad \in F(t, x(t), \ldots, x^{(n-1)}(t)) \quad \textit{for a.a. } t \in J \\ x \in S, \end{cases}$$

where $J \subset \mathbb{R}$, $S \subset C(J, \mathbb{R}^m)$, and a_i, F are u-Carathéodory maps on $J \times \mathbb{R}^{mn}$ into $\mathbb{R}$ and $\mathbb{R}^m$, respectively.

Suppose that there exists a u-Carathéodory map $G : J \times \mathbb{R}^{mn} \times \mathbb{R}^{mn} \multimap \mathbb{R}^m$ such that, for every $c \in \mathbb{R}^{mn}$, $G(t, c, c) \subset F(t, c)$ a.e. in J.

The problem (2) *has a solution, if the following conditions are satisfied:*

(i) *There is a retract Q of the space $C^{n-1}(J, \mathbb{R}^m)$ such that, for every $q \in Q$, the following problem*

$$
(3) \qquad \begin{cases} x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, q(t), \dots, q^{(n-1)}(t)) x^{(i)}(t) \\ \qquad \in G(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t)) \quad \text{for a.a. } t \in J \\ x \in S \cap Q , \end{cases}
$$

has R_δ-set of solutions;

(ii) *There is a locally integrable function $\alpha : J \to \mathbb{R}$ such that, for every $i = 0, \dots, n-1$*:

$$
|a_i(t, q(t), \dots, q^{(n-1)}(t))| \leqslant \alpha(t) \quad \text{a.e. in } J
$$

and

$$
|G(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t))| \leqslant \alpha(t) \quad \text{a.e. in } J
$$

for each $(q, x) \in Q \times C^{n-1}(J)$ satisfying (3);

(iii) *For the multivalued map $T : Q \multimap C(J, \mathbb{R}^m)$, which assigns to any $q \in Q$ the set of solutions of* (3), *the set $T(Q)$ is bounded in $C(J, \mathbb{R}^m)$ and its closure in $C^{n-1}(J, \mathbb{R}^m)$ is contained in S (in particular, this holds if $S \cap C^{n-1}(J, \mathbb{R}^m)$ is closed in $C^{n-1}(J, \mathbb{R}^m)$).*

THEOREM 2.2. – *Under assumptions* (A) *and* (B) *problem* (1) *has a solution.*

PROOF. – Define $Q = \{q \in C^1(J, \mathbb{R}^n) \mid |q(t)| \leqslant 3K\}$. This set is convex and closed in $C^1(J, \mathbb{R}^n)$.

For each $q \in Q$ we can consider the problem

$$
(4) \qquad \begin{cases} \ddot{x}(t) + \dot{x}(t) \in F(t, q(t)), \quad \text{for a.a. } t \in J \\ x(0) = 0, \quad \lim_{t \to \infty} x(t) = 0 . \end{cases}
$$

Denote $F_q = F(\cdot, q(\cdot))$. Assumptions on F imply that there is a measurable selection f_q of F_q. Thus we have the problem

$$
(5) \qquad \begin{cases} \ddot{x}(t) + \dot{x}(t) = f_q(t), \quad \text{for a.a. } t \in J \\ x(0) = 0, \quad \lim_{t \to \infty} x(t) = 0 \end{cases}
$$

which has a unique solution of the form

$$x(t) = -\int_t^\infty f_q(s)\,ds - e^{-t}\left(\int_0^t e^s f_q(s)\,ds - \int_0^\infty f_q(s)\,ds\right). \tag{6}$$

Notice that

$$|x(t)| \leqslant \int_t^\infty |f_q(s)|\,ds + e^{-t}\left(\int_0^t e^s\,|f_q(s)|\,ds + \int_0^\infty |f_q(s)|\,ds\right) \leqslant \tag{7}$$

$$\leqslant (1 - e^{-t})\int_t^\infty \alpha(s)\,ds + e^{-t}\int_0^t (e^s + 1)\,\alpha(s)\,ds = \gamma(t) \leqslant 3K.$$

Moreover,

$$\dot{x}(t) = e^{-t}\left(\int_0^t e^s f_q(s)\,ds - \int_0^\infty f_q(s)\,ds\right),$$

which gives

$$|\dot{x}(t)| \leqslant e^{-t}\int_0^t e^s\,\alpha(s)\,ds + e^{-t}\int_0^\infty \alpha(s)\,ds = \gamma_1(t).$$

Define $S_1 = \{x \in C^1(J, \mathbb{R}^n) \mid |x(t)| \leqslant \gamma(t),\ |\dot{x}(t)| \leqslant \gamma_1(t) \text{ for } t \in J\}$. From (7) it follows that $S_1 \subset Q$. Moreover, (6) and (7) imply that, for the map T which assigns to each $q \in Q$ the set of solutions to (4), we have $T(Q) \subset S_1$.

Existence of a solution to problem (5) implies that each $T(q)$ is nonempty. Moreover, by the convexity assumption on values of F, each $T(q)$ is also compact (since S_1 is bounded in $C^1(J, \mathbb{R}^n)$) and convex.

Finally, notice that $\gamma(0) = 0$ and

$$\lim_{t\to\infty} \gamma(t) = \lim_{t\to\infty} \frac{\int_0^t e^s\,\alpha(s)\,ds}{e^t} = 0.$$

Therefore, $S_1 \subset S$ and, since S_1 is also closed in $C^1(J, \mathbb{R}^n)$, T satisfies assumption (iii) of Proposition 2.1. Applying this proposition we end the proof. ■

Note that problem (1) with single-valued right-hand side F was studied by using the linearization method in [7]. Above we have enlarged the class of considering problems (we include also a multivalued case) under stronger boundedness assumptions on F. The problem of weakening these assumptions is open.

3. – Target problems.

The method presented in the previous section can be also applied for obtaining an existence result for the following second order target problem:

$$
\begin{cases} \ddot{x} \in F(t, x) & \text{for a.a. } t \in J = [0, \infty), \\ \lim\limits_{t \to \infty} x(t) = v, \end{cases} \tag{8}
$$

where $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a u-Carathéodory map and $v \in \mathbb{R}^n$.

Instead of (B) in Section 2, assume that there exists $r > 0$ such that

$$
\int_0^\infty t\alpha_r(t)\, dt < r - |v|,
$$

where $\alpha_r(t) = \sup\limits_{|c| \leq r} |F(t, c)|$. Assume also that the function α_r is essentially bounded on $[0, 1]$.

THEOREM 3.1. – *Under the above assumptions, problem* (8) *has a solution.*

PROOF. – Consider the convex closed set

$$
Q = \{q \in C^1(J, \mathbb{R}^n) \mid |q(t)| \leq r \text{ for every } t \in J\}
$$

and

$$
S = \{x \in C(J, \mathbb{R}^n) \mid \lim_{t \to \infty} x(t) = v\}.
$$

It is easy to see that S is not closed. Define the closed bounded subset S_1 of $C^1(J, \mathbb{R}^n)$ as follows

$$
S_1 = \left\{ x \in C^1(J, \mathbb{R}^n) \;\middle|\; |x(t) - v| \leq \int_t^\infty s\alpha_r(s)\, ds,\ |\dot{x}(t)| \leq t\alpha_r(t),\ t \geq 0 \right\}.
$$

It is easy to check that $S_1 \subset S \cap Q$.

Now we can follow like in the proof of Theorem 2.2 and associate with (8) the family of problems

$$
\begin{cases} \ddot{x}(t) \in F_q(t), & \text{for a.a. } t \in J \\ x \in S \cap Q \end{cases} \tag{9}
$$

and the family of single-valued problems

$$
\begin{cases} \ddot{x}(t) = f_q(t), & \text{for a.a. } t \in J \\ x \in S \cap Q\,, \end{cases} \tag{10}
$$

where f_q is a measurable selection of F_q.

The above problem (10) has a unique solution of the form

$$
x_q(t) = v + \int_t^{\infty} (s-t)\, f_q(s)\, ds\,.
$$

Therefore, the set of solutions to problem (9) is nonempty. Moreover, it is convex and compact (like in the proof of Theorem 2.2). To end the proof it is sufficient to notice that all solutions belong to S_1, but this follows from the form of solutions and a simple calculation. ■

Now we are interested in a topological structure of the solution set to the following first order target problem:

$$
\begin{cases} \dot{x}(t) \in F(t, x(t)), & \text{for a.a. } t \in [0, \infty), \\ \lim\limits_{t \to \infty} x(t) = x_\infty\,, & x_\infty \in \mathbb{R}^n, \end{cases} \tag{11}
$$

where $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a u-Carathéodory map satisfying the following growth condition:

$$
|F(t, x)| \leq \alpha(t)(1 + |x|), \qquad \text{for all } (t, x) \in J \times \mathbb{R}^n,
$$

where $\alpha : J \to J$ is a globally integrable function.

In [2] authors proved that problem (11) admits a bounded solution (comp. [2], Theorem 5.4). Below we give a more precise information on the set of all solutions to (11).

THEOREM 3.2. – *Under the above assumptions on the right-hand side F, the set of solutions to* (11) *is a compact* R_δ *set* (1). *Moreover, all solutions are bounded by the same constant.*

(1) We say that a metric space is an R_δ-set, if it is homeomorphic to the intersection of countable decreasing sequence of absolute retracts.

PROOF. – Consider the u-Carathéodory map $G : J \times \mathbb{R}^n \multimap \mathbb{R}^n$,

$$G(t, x) = \begin{cases} F(t, x), & \text{for } |x| \leqslant D \text{ and } t \in J, \\ F\left(t, D\dfrac{x}{|x|}\right), & \text{for } |x| \geqslant D \text{ and } t \in J, \end{cases}$$

where

$$D \geqslant (|x_\infty| + A) \exp A, \qquad A = \int_0^\infty \alpha(t)\, dt < \infty .$$

It is easy to see that G is (globally) integrably bounded by the function $\eta(t) = \alpha(t)(1 + D)$. Using the Gronwall inequality one can check that problem (11) is equivalent to the following one:

$$\text{(12)} \qquad \begin{cases} \dot{x}(t) \in G(t, x(t)), & \text{for a.a. } t \in J, \\ \lim\limits_{t \to \infty} x(t) = x_\infty, & x_\infty \in \mathbb{R}^n, \end{cases}$$

This means that the set of solutions $S(F, x_\infty)$ to (11) is equal to the set of solutions $S(G, x_\infty)$ to (12).

By Proposition 4.1 in [1] there exists an almost upper semicontinuous (a.u.s.c.) map $\overline{G} : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ with nonempty compact convex values and such that

(i) $\overline{G}(t, x) \subset G(t, x)$ for every $(t, x) \in J \times \mathbb{R}^n$;

(ii) If $\Delta \subset J$ is measurable, $u : \Delta \to \mathbb{R}^n$ and $v : \Delta \to \mathbb{R}^n$ are measurable maps and $u(t) \in G(t, v(t))$ for almost all $t \in \Delta$, then $u(t) \in \overline{G}(t, v(t))$ for almost all $t \in \Delta$.

This implies that $S(G, x_\infty) = S(\overline{G}, x_\infty)$, where $S(\overline{G}, x_\infty)$ is a solution set to the target problem with $\overline{G}$ as a right-hand side.

Theorem 4.2 in [1] implies an existence of the sequence $\{G_k : J \times \mathbb{R}^n \multimap \mathbb{R}^n\}$ of a.u.s.c. maps such that

(1) each G_k has measurable-locally lipschitz selector g_k,

(2) $G_{k+1}(t, x) \subset G_k(t, x)$,

(3) $G(t, x) = \bigcap\limits_{k=1}^{\infty} G_k(t, x)$ for every $(t, x) \in J \times \mathbb{R}^n$,

(4) each G_k is integrably bounded by η.

By a standard calculation one can obtain that

(13) $$S(\overline{G}, x_\infty) = \bigcap_{k=1}^{\infty} S(G_k, x_\infty).$$

We show that $S(G_k, x_\infty)$ is compact and contractible for every $k \geq 1$.

STEP 1. – At first, we prove that $S(g_k, x_\infty)$ is nonempty, where g_k is a measurable-locally lipschitz selector of G_k mentioned above.

To show this, consider the family of linear problems

(14) $$\begin{cases} \dot{x}(t) = g_k(t, q(t)), & \text{for a.a. } t \in J, \\ x \in Q \cap S, \end{cases}$$

where

$$S = \{x \in C(J, \mathbb{R}^n) \mid \lim_{t \to \infty} x(t) = x_\infty\}$$

and

$$q \in Q = \{q \in C(J, \mathbb{R}^n) \mid |q(t)| \leq |x_\infty| + A(1+D) \text{ for } t \geq 0\}.$$

Put

$$S_1 = \left\{ x \in Q \,\middle|\, |x(t) - x_\infty| \leq (1+D) \int_t^\infty \alpha(s)\, ds \text{ for } t \geq 0 \right\}.$$

It is evident that S_1 is a closed subset of S.

It is also easy to see that problem (14) has a unique solution of the form

$$x_q(t) = x_\infty + \int_\infty^t g_k(s, q(s))\, ds .$$

Denoting $T(q) = x_q$ we obtain a continuous compact operator T (see [1], Proposition 2.32 or [6], Theorem 1.1) from Q into $C(J, \mathbb{R}^n)$.

Notice that

$$|x_q(t)| \leq |x_\infty| + \int_0^\infty |g_k(s, q(s))|\, ds \leq |x_\infty| + A(1+D)$$

and

$$|x_q(t) - x_\infty| \leq \int_t^\infty |g_k(s, q(s))|\, ds \leq (1+D) \int_t^\infty \alpha(s)\, ds ,$$

what implies that, in fact, $T(Q) \subset S_1 \subset Q$ and, by the Schauder-Tychonoff fixed point theorem, there is a fixed point of T which belongs to $S(g_k, x_\infty)$.

STEP 2. – Now, we check that $S(g_k, x_\infty)$ is a singleton.

Notice that, if $x \in S(g_k, x_\infty)$, then $x(t) \in B = \{y \in \mathbb{R}^n \mid |y| \leq |x_\infty| + A(1+D)\}$ for a.a. $t \in J$, and B is compact. Since that, we may consider g_k as a measurable-lipschitz map and hence (comp. the construction of g_k in [1], Proposition 4.13) there exists a globally integrable function $\gamma : J \to J$ such that

$$|g_k(t, x) - g_k(t, y)| \leq \gamma(t)\,|x - y| \tag{15}$$

for each $x, y \in B$ and a.a. $t \in J$.

Suppose that $x, y \in S(g_k, x_\infty)$. We show that $x = y$. Since x and y are bounded, it is sufficient to check that

$$\|x - y\|_B = \sup_{t \in J} e^{-2r(t)}\,|x(t) - y(t)| = 0\,,$$

where $r(t) = \int_t^\infty \gamma(s)\,ds$ for every $t \in J$.

We have

$$\begin{aligned}
& e^{-2r(t)}\,|x(t) - y(t)| \leq \\
&\quad \leq e^{-2r(t)} \int_t^\infty |g_k(s, x(s)) - g_k(s, y(s))|\,ds \leq \\
&\quad \leq e^{-2r(t)} \int_t^\infty \gamma(s)\,|x(s) - y(s)|\,ds = \\
&\quad = e^{-2r(t)} \int_t^\infty e^{2r(s)} \gamma(s)\, e^{-2r(s)}\,|x(s) - y(s)|\,ds \leq \\
&\quad \leq \|x - y\|_B\, e^{-2r(t)} \int_t^\infty e^{2r(s)} \gamma(s)\,ds = \\
& = \frac{1}{2} \|x - y\|_B\, e^{-2r(t)} (e^{2r(t)} - 1) = \\
& = \frac{1}{2} \|x - y\|_B (1 - e^{-2r(t)}) \leq \\
& \leq \frac{1}{2} \|x - y\|_B\,.
\end{aligned}$$

Therefore

$$\|x-y\|_B \leqslant \frac{1}{2}\|x-y\|_B,$$

what implies that $\|x-y\|_B = 0$.

STEP 3. – We are in a position to define a desired homotopy contracting $S(G_k, x_\infty)$ to a single point. We put for every $x \in S(G_k, x_\infty)$:

$$h(x, \lambda)(t) = \begin{cases} x(t), & \text{for } t \geqslant \dfrac{1}{\lambda} - \lambda,\ \lambda \neq 0 \\[2ex] \overline{z}(t), & \text{for } 0 \leqslant t < \dfrac{1}{\lambda} - \lambda,\ \lambda \neq 0 \\[2ex] \overline{x}(t), & \text{for } \lambda = 0, \end{cases}$$

where $\overline{x}$ is a unique solution $S(g_k, x_\infty)$ and $\overline{z}$ is a unique solution to the reverse Cauchy problem

$$\text{(16)} \qquad \begin{cases} \dot{z}(t) = g_k(t, z(t)), & \text{for a.a. } t \in \left[0, \dfrac{1}{\lambda} - \lambda\right], \\[2ex] z\left(\dfrac{1}{\lambda} - \lambda\right) = x\left(\dfrac{1}{\lambda} - \lambda\right). \end{cases}$$

One can see that $h(S(G_k, x_\infty) \times [0, 1]) \subset S(G_k, x_\infty)$ and $h(x, 0) = \overline{x}$, $h(x, 1) = x$, as required. Hence, the set $S(G_k, x_\infty)$ is contractible. Its compactness easily follows from the convexity of values of the right-hand side and from the growth condition. From (13) we obtain that $S(F, x_\infty)$ is R_δ which ends the proof. ∎

REMARK 3.3. – By the substitution $t = -\tau$, the conclusion of the above theorem is also true for the problem on the left halfline. Thus we can generalize Remark 5.5 and Corollary 5.6 in [2].

4. – Illustrations of inverse systems approach.

Applications of inverse systems to differential inclusions were initiated in [2], where problems on the halfline were considered. A motivation to such approach was taken from the fact that some spaces of functions defined on the halfline may be considered as limits of inverse systems of Banach spaces of functions defined on compact intervals. This observation allows us to obtain some useful properties of operators associated with differential problems on the halfline by the use of properties of suitable operators connected with problems on intervals $[0, m]$.

The results below give more precise informations on a topological structure of the solution sets to two multivalued differential problems on the halfline than that in [2]. To this aim we use results from [8] concerning with a topological structure of fixed point sets of multivalued limit maps. In the topological background below details are omitted. They may be found in [8] (see also [2]).

By an inverse system (countable) of topological spaces we mean a family $S=\{X_m, \pi^p_m, \mathbb{N}\}$, where X_m is a topological (Hausdorff) space for every $m \in \mathbb{N}$ and $\pi^p_m : X_p \to X_m$ is a continuous mapping for each two elements $m, p \in \mathbb{N}$ such that $m \leqslant p$. Moreover, for each $m \leqslant p \leqslant r$, the following conditions should hold: $\pi^m_m = id_{X_m}$ and $\pi^p_m \pi^r_p = \pi^r_m$. A limit of such inverse system is the set

$$\varprojlim S = \{(x_m) \in \Pi_{m \in \mathbb{N}} X_m \mid \pi^p_m(x_p) = x_m \text{ for all } m \leqslant p\}.$$

The following property of inverse systems is very useful in applications.

PROPOSITION 4.1 ([8], Proposition 3.2). – *Let $S = \{X_n, \pi^p_n, \mathbb{N}\}$ be an inverse system. If each X_n is a compact R_δ-set, then $\varprojlim S$ is also compact R_δ.*

In particular, a limit of an inverse system of compact AR-spaces is R_δ (it does not need to be an AR). On the other hand, every R_δ set can be obtained as a limit of an inverse system of some AR-spaces (induced by inclusion relations).

EXAMPLE 4.2. – $C([0, \infty), \mathbb{R}^n)$, $C^k([0, \infty), \mathbb{R}^n)$, $L^1([0, \infty), \mathbb{R}^n)$ are limits of inverse systems of resp. $C([0, m], \mathbb{R}^n)$, $C^k([0, m], \mathbb{R}^n)$, $L^1([0, m], \mathbb{R}^n)$, where the bonding maps are of the form $\pi^p_m(x) = x|_{[0, m]}$. We can easily increase the list of examples. It is interesting that every Fréchet space can be considered as a limit of an inverse system of Banach spaces.

By a map (self-map) of an inverse system S we mean a family of maps $\{\varphi_n\} : S \multimap S$ such that each $\varphi_n : X_n \multimap X_n$ is a multivalued one. Every such map induces a limit map $\varphi : \varprojlim S \multimap \varprojlim S$,

$$\varphi((x_n)) = \Pi^\infty_{n=1} \varphi_n(x_n) \cap \varprojlim S .$$

The following result is crucial.

THEOREM 4.3 ([8], Theorem 3.9). – *Let $S = \{X_n, \pi^p_n, \mathbb{N}\}$ be an inverse system and $\varphi : \varprojlim S \multimap \varprojlim S$ be a limit map induced by $\{\varphi_n\}$. If* $\mathrm{Fix}(\varphi_n)$ *are compact acyclic (or R_δ), then* $\mathrm{Fix}(\varphi)$ *is compact acyclic (resp. R_δ).*

As applications of the above theorem we study a topological structure of solution sets to two multivalued problems on the halfline. The first one extends the one described in [9] to the case of the halfline $[0, \infty)$. The second one is a multivalued generalization of the results obtained in [4].

At first, consider the problem

$$(17)\qquad \begin{cases} x^{(k)}(t) \in \varphi(t, A(t)x, A(t)\dot{x}, \ldots, A(t)x^{(k-1)}), & \text{for a.a. } t \in J_1 \\ x|_{J_2} = b, \\ \vdots \\ x^{(k-1)}|_{J_2} = b^{(k-1)}, \end{cases}$$

where $J_1 = [0, \infty)$, $J_2 = [r, 0]$, $r < 0$, $J = J_1 \cup J_2$, $A : J_1 \to C(C(J, \mathbb{R}^n), C(J_2, \mathbb{R}^n))$ is defined as $[A(t)x](s) = x(t+s)$ and $b \in C^{k-1}(J_2, \mathbb{R}^n)$ is any function.

Denote $E_2 = C(J_2, \mathbb{R}^n) \times \ldots \times C(J_2, \mathbb{R}^n)$, ($k$-times).

THEOREM 4.4 (comp. [2], Theorem 6.9). – *Assume that $\varphi : J_1 \times E_2 \multimap \mathbb{R}^n$ is a locally integrably bounded map with compact, convex values satisfying the following conditions:*

(A) *For all $x_0, \ldots, x_{k-1} \in C(J_2, \mathbb{R}^n)$, the map $\varphi(\cdot, x_0, \ldots, x_{k-1})$ is measurable*;

(B) $\exists L \geqslant 0 \forall x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1} \in C(J_2, \mathbb{R}^n)\ \forall t \in J_1$:

$$d_H(\varphi(t, x_0, \ldots, x_{k-1}), \varphi(t, y_0, \ldots, y_{k-1})) \leqslant L \sum_{i=0}^{k-1} \|x_i - y_i\|.$$

Then the set $S(\varphi, b)$ of all solutions to problem (17) *is a limit of an inverse system of compact absolute retracts. In particular, it is compact R_δ.*

PROOF. – Without loss of generality we assume that $L \geqslant 1$. Consider the map $l : L^1_{\text{loc}}(J_1, \mathbb{R}^n) \to C^{k-1}(J, \mathbb{R}^n)$,

$$l(z)(t) = \begin{cases} \displaystyle\sum_{j=0}^{k-1} \frac{t^j}{j!} b^{(j)}(0) + \int_0^t \int_0^{s_1} \ldots \int_0^{s_{k-1}} z(s)\, ds\, ds_{k-1} \ldots ds_1, & \text{for } t \in J_1, \\ b(t), & \text{for } t \in J_2, \end{cases}$$

and a sequence of maps $l_m\colon L^1([0, m], \mathbb{R}^n) \to C^{k-1}([r, m], \mathbb{R}^n)$,

$$l_m(z)(t) = \begin{cases} \sum_{j=0}^{k-1} \frac{t^j}{j!} b^{(j)}(0) + \int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} z(s)\, ds\, ds_{k-1} \dots ds_1, & \text{for } t \in [0, m], \\ b(t), & \text{for } t \in J_2. \end{cases}$$

Define the operator $\Phi : C^{k-1}(J, \mathbb{R}^n) \multimap C^{k-1}(J, \mathbb{R}^n)$ as follows:

$$\Phi(x) = \{ y \in C^{k-1}(J, \mathbb{R}^n) \mid y(t) = l(z)(t)$$

$$\text{and } z(t) \in \varphi(t, A(t)x, A(t)\dot{x}, \dots, A(t)x^{(k-1)}) \text{ a.a. in } J_1 \}.$$

We can similarly define a sequence of multivalued maps $\Phi_m\colon C^{k-1}([r, m], \mathbb{R}^n) \multimap C^{k-1}([r, m], \mathbb{R}^n)$ as follows:

$$\Phi_m(x) = \{ y \in C^{k-1}([r, m], \mathbb{R}^n) \mid y(t) = l_m(z)(t)$$

$$\text{and } z(t) \in \varphi(t, A(t)x, A(t)\dot{x}, \dots, A(t)x^{(k-1)}) \text{ a.a. in } [0, m] \}.$$

It is easy to check that each Φ_m is compact convex valued.

Consider in $C^{k-1}([r, m], \mathbb{R}^n)$ the equivalent norms

$$q_m(x) = \sum_{i=0}^{k-1} \max_{t \in [r, m]} (|x^{(i)}(t)| e^{-Lkt})$$

which induce the equivalent metric in $C^{k-1}(J, \mathbb{R}^n)$. Using such weighted norms one can check (see [2]) that each Φ_m is a multivalued contraction. By the result of Ricceri (see [10]) it follows that the fixed point sets $Fix(\Phi_m)$ are absolute retracts. Since all Φ_m are compact valued, each $Fix(\Phi_m)$ is also compact (see [11], Theorem 1).

Following Example 4.2 we can consider $C^{k-1}(J, \mathbb{R}^n)$ as a limit of an inverse system of $C^{k-1}([r, m], \mathbb{R}^n)$. Moreover the family of maps Φ_m forms a map of this system and Φ is its limit map. Hence, by Theorem 4.3, the set $\mathrm{Fix}(\Phi)$ is a limit of an inverse system of $\mathrm{Fix}(\Phi_m)$.

Now, to end the proof it is sufficient to observe that $\mathrm{Fix}(\Phi)$ is equal to $S(\varphi, b)$. ■

As the second illustration of the inverse systems approach consider the following autonomous multivalued Cauchy problem:

$$\begin{cases} \dot{x}(t) \in \varphi(x(t)), \text{ for a.a. } t \in J = [0, \infty), \\ x(0) = v. \end{cases} \tag{18}$$

Here $\varphi : \mathbb{R}^n \multimap \mathbb{R}^n$ is a multivalued map of the form $\varphi(x) = \psi(\tau(x), x)$, where $\tau : \mathbb{R}^n \to \mathbb{R}$ is a single-valued map and $\psi : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a multivalued one.

THEOREM 4.5 ([2], Theorem 7.1). – *Assume that*

(i) *The map* $\tau : \mathbb{R}^n \to \mathbb{R}$ *is continuously differentiable* (2) and $\psi : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a u-Carathéodory map,

(ii) *For some compact, convex set* $K \subset \mathbb{R}^n$, *at every point* x, *one has*

$$\varphi(x) \subset K\,, \qquad \nabla\tau(x)\cdot z > 0\,, \ \textit{for every}\ z \in K\,. \tag{19}$$

Then the Cauchy problem (18) *has a solution.*

If, additionally,

(iii) *The gradient* $\nabla\tau$ *has bounded directional variation* (3) *w.r.t. the cone* $\Gamma = \{\lambda z \,|\, \lambda \geqslant 0,\ z \in K\}$,

then the set of solutions to problem (18) *is compact* R_δ.

SKETCH OF THE PROOF. – Without loss of generality assume that $v = 0$. At first we notice that under assumptions (i)-(ii) our problem considered on every compact interval $[0, m]$ has nonempty compact and, under (i)-(iii), also R_δ set of solutions S_m (see [2]). Consider the family of maps $\Psi_m\colon C([0, m], \mathbb{R}^n) \multimap C([0, m], \mathbb{R}^n)$, $m \geqslant 1$,

$$\Psi_m(u)(t) = \left\{ \int_0^t v(s)\, ds \,\middle|\, v \in L^1([0, m], \mathbb{R}^n) \ \text{and}\ v(s) \in \psi(\tau(u(s)), u(s)), \right.$$

$$\left. \text{for a.a.}\quad t \in [0, m] \right\}.$$

Defining compact convex sets

$$C_m = \left\{ v \in C([0, m], \mathbb{R}^n) \,|\, v(0) = 0,\ \frac{v(t) - v(s)}{t - s} \in K,\ \ t > s \right\}$$

one can show that Ψ_m maps C_m into C_m.

(2) The statement is true also with weaker assumption on the regularity of τ, namely if τ is lipschitzian.

(3) The map u has a *bounded directional variation w.r.t. the cone* Γ *if*

$$\sup\left\{ \sum_{i=1}^{N} |u(p_i) - u(p_{i-1})| \,\middle|\, N \geqslant 1,\ p_i - p_{i-1} \in \Gamma,\ \text{for every}\ i \right\} < \infty\,.$$

Denote $\Phi_m\colon C_m \multimap C_m$, $\Phi_m(u) = \Psi_m(u)$. Of course, $\mathrm{Fix}(\Phi_m) = S_m$.
Notice that the set

$$C = \left\{ v \in C(J, \mathbb{R}^n) \,|\, v(0) = 0, \ \frac{v(t) - v(s)}{t - s} \in K, \ t > s \right\}$$

can be considered as a limit of the inverse system $\{C_m, \pi^p_m\}$, where $\pi^p_m\colon C_p \to C_m$ is a bonding map defined as follows $\pi^p_m(u) = u|_{[0,\,m]}$. Moreover, the map $\{\Phi_m\}$ of the above system induces the limit map $\Phi : C \multimap C$,

$$\Phi(u)(t) = \left\{ \int_0^t v(s)\,ds \,|\, v \in L^1_{\mathrm{loc}}(J, \mathbb{R}^n) \ \text{ and } \ v(s) \in \psi(\tau(u(s)), u(s)), \right.$$

$$\left. \text{for a.a. } \ t \in J \right\},$$

with fixed point set $\mathrm{Fix}(\Phi)$ which is equal to the solution set S of problem (18). Now, a nonemptiness of S follows from the compactness of S_m (comp. [2], Proposition 2.1) while the R_δ property is a consequence of Theorem 4. ∎

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