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Invariant Harmonic Unit Vector Fields on Lie Groups (*).

J. C. GONZÁLEZ-DÁVILA - L. VANHECKE

Sunto. – *In questo lavoro viene presentata una nuova caratterizzazione dei campi vettoriali armonici unitari sui gruppi di Lie dotati di metrica invariante a sinistra. Ciò permette di dedurre risultati di esistenza e nuovi esempi di tali campi, in particolare sui gruppi di Lie con metrica bi-invariante, sui gruppi di Lie di dimensione* 3*, sui gruppi di Heisenberg generalizzati, sugli spazi di Damek-Ricci e su particolari prodotti semi-diretti. In diversi casi si ottiene l'elenco completo di tutti i campi di questo tipo; in molti esempi vengono determinate le applicazioni armoniche associate, il cui dominio è il gruppo considerato e il codominio è il relativo fibrato tangente unitario, con metrica di Sasaki.*

Summary. – *We provide a new characterization of invariant harmonic unit vector fields on Lie groups endowed with a left-invariant metric. We use it to derive existence results and to construct new examples on Lie groups equipped with a bi-invariant metric, on three-dimensional Lie groups, on generalized Heisenberg groups, on Damek-Ricci spaces and on particular semi-direct products. In several cases a complete list of such vector fields is given. Furthermore, for a lot of the examples we determine associated harmonic maps from the considered group into its unit tangent bundle equipped with the associated Sasaki metric.*

1. – Introduction.

Let (M, g) be a compact oriented Riemannian manifold and $(T_1 M, g_S)$ its unit tangent bundle equipped with the corresponding Sasaki metric g_S. A unit vector field ξ on (M, g) determines, if it exists, a map from the manifold into this unit tangent bundle and the energy of this map is called the *energy* of the vector field [16].The critical point condition of this energy functional has been considered in [5] and also in [15] where the *total bending* of the vector field is studied. This total bending equals, up to constants, the energy of the vector field. The obtained critical point condition also makes sense for a general Riemannian manifold and a unit vector field which satisfies this condition is called a *harmonic vector field* [5]. It should be noted already that this condition does not automatically imply that the corresponding map is a *harmonic map*.

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The harmonicity of a unit vector field and of the corresponding map has already been considered in [3], [4], [5], [8], [14] where several examples are given. The main purpose of this paper is to continue this study. More precisely, we study the existence and classification of *invariant* harmonic unit vector fields on Lie groups equipped with a left-invariant metric. In Section 2, we recall some basic material and give some elementary examples of harmonic unit vector fields on Riemannian manifolds by relating this notion to the one of normal and strongly normal vector fields. Already here it will turn out that not every harmonic unit vector field determines a harmonic map since we show that this is the case on a surface if and only if the manifold is flat. In Section 3, we then consider Lie groups, derive a new characterization of invariant harmonic unit vector fields and specialize it to the case of unimodular and non-unimodular Lie groups. These considerations yield some existence results. For example, it is proved that every unimodular Lie group admits a left-invariant harmonic unit vector field, a result which remains true for all odd-dimensional Lie groups. Whether this also holds for even-dimensional non-unimodular Lie groups remains an open question.

In the rest of the paper we use the derived characterization to determine the full set of solutions of the critical point condition, or at least, a lot of special solutions for a series of particular cases. In Section 4, we start by considering Lie groups with a bi-invariant metric and also prove that any invariant harmonic unit vector field determines a harmonic map. Moreover, when the Lie group G is compact and semisimple with Killing form B, then we obtain that any invariant unit vector field determines a harmonic map of $(G, g = -B)$ into its unit tangent bundle $(T_1 G, g_S)$.

In Section 5, we determine all examples on three-dimensional Lie groups. It turns out that in the unimodular case, invariant harmonic unit vector fields always exist and they determine harmonic maps if and only if they are eigenvectors of the Ricci operator. This contrasts to the non-unimodular case where invariant harmonic unit vectors also exist in all cases but there are cases where no invariant unit vector fields exist which determine harmonic maps. Since several three-dimensional Lie groups are semi-direct products of the form $\mathbb{R} \times_\alpha \mathbb{R}^2$, we treat in Section 6 the general case of semi-direct products $\mathbb{R} \times_\alpha \mathbb{R}^n$.

In Section 7, we consider the critical point condition for the generalized Heisenberg groups of type $H(1, r)$, determine the complete set of solutions and derive when these solutions also determine harmonic maps. In Section 8, we consider similar problems for the generalized Heisenberg groups of Kaplan-type, that is, H-type groups. Here the problem turns out to be more difficult and we only provide the set of all solutions for some special cases, more precisely, when the dimension of the center of the two-step nilpotent Lie group equals one and also for the six- and seven-dimensional H-type groups.

The Damek-Ricci spaces are particular solvable extensions of H-type groups. Our study in Section 9 shows that on these spaces there never exist invariant unit vector fields such that the corresponding map into the unit tangent bundle is harmonic, although harmonic invariant unit vector fields always exist.

We note that this study is quite similar to the one done for the critical point condition for the volume functional, that is, the volume of submanifolds of $(T_1 M, g_S)$ determined by unit vector fields on a compact oriented Riemannian manifold. For a general Riemannian manifold, possibly non-compact or non-orientable, a solution of the corresponding critical point condition is called a *minimal* unit vector field [6]. This condition is satisfied if and only if the submanifold is minimal. We refer to [6], [7] and the already mentioned papers for more information and for further references about this topic of study. Here we only mention that our detailed study shows that on three-dimensional unimodular Lie groups, invariant unit vector fields are harmonic if and only if they are minimal. In contrast, it follows that for the three-dimensional non-unimodular Lie groups both notions are not always equivalent.

2. – Harmonic unit vector fields.

Let (M, g) be an n-dimensional smooth Riemannian manifold which we suppose to be connected and let $(T_1 M, g_S)$ be its unit tangent bundle equipped with the associated Sasaki metric g_S. ∇ denotes its Levi Civita connection and R the corresponding Riemannian curvature tensor taken with the sign convention $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ for all smooth X, Y. Furthermore, let $\mathfrak{X}^1(M)$ denote the set of all unit vector fields on M, supposed to be non-empty. We put $A_V = -\nabla V$ for $V \in \mathfrak{X}^1(M)$. Each unit vector field V can be regarded as the immersion $V: M \to T_1 M$, $x \mapsto V_x$, $x \in M$, of M into $T_1 M$. The pull-back metric $V^* g_S$ is given by

$$(V^* g_S)(X, Y) = g(X, Y) + g(A_V X, A_V Y).$$

This shows that V is an isometry if and only if V is parallel. For a closed and oriented manifold (M, g), the *energy* $E(V)$ of V, that is, the energy of the map $V: (M, g) \to (T_1 M, g_S)$ [16], is given by

$$E(V) = \frac{n}{2} \mathrm{Vol}(M, g) + \frac{1}{2} \int_M \|A_V\|^2 \, dv .$$

Here, $B(V) = \int_M \|A_V\|^2 \, dv$ is known as the *total bending* of the vector field V [15]. dv denotes the volume form on (M, g). In what follows we put

$$b(V) = \frac{1}{2} \|A_V\|^2 = \frac{1}{2} \mathrm{tr}\, A_V^t A_V.$$

The unit vector field V is a critical point for the energy functional E if and only if the one-form ν_V defined by

$$\nu_V(X) = \mathrm{tr}\,(Z \mapsto (\nabla_Z A_V^t)\,X)$$

vanishes on the distribution $\mathcal{H}^V$ determined by the vector fields orthogonal to V. For general Riemannian manifolds (M, g), a unit vector field satisfying this condition is called a *harmonic unit vector field*. It follows easily that when V is a Killing vector field, it is also harmonic if and only if it is an eigenvector of the Ricci operator [5]. Furthermore, the map $V:(M, g) \to (T_1 M, g_S)$ turns out to be a *harmonic map* if and only if V is a harmonic unit vector field such that the one-form $\tilde{\nu}_V$, defined by

$$\tilde{\nu}_V(X) = \mathrm{tr}\,(Z \mapsto R(A_V Z, V)\,X)\,,$$

vanishes for *all* vectors X [5].

Next, we recall some definitions from [7]. $V \in \mathfrak{X}^1(M)$ is said to be *normal* if $g(R(X, Y)\,Z, V) = 0$ for all $X, Y, Z \in \mathcal{H}^V$. It is called *strongly normal* if $g((\nabla_X A_V)\,Y, Z) = 0$ for all $X, Y, Z \in \mathcal{H}^V$. Since R and ∇A_V are related by

$$R_{XY} V = (\nabla_X A_V)\,Y - (\nabla_Y A_V)\,X\,,$$

it follows that any strongly normal vector field is also normal. Moreover, a unit Killing vector field is strongly normal if and only if it is normal and in this case V is harmonic. Now, we generalize this last result for geodesic vector fields, that is, for vector fields whose integral curves are geodesics.

PROPOSITION 2.1. – *Every strongly normal geodesic vector field $V \in \mathfrak{X}^1(M)$ is harmonic. Moreover, the corresponding map is harmonic if and only if $\tilde{\nu}_V(V) = 0$.*

PROOF. – Let $\{E_i,\ i = 1, \ldots, n\}$ be a local orthonormal basis with $E_n = V$ and let $X \in \mathcal{H}^V$. First, note that when V is a geodesic field, $(\nabla_V A_V^t)\,X = (\nabla_V A_V)^t X \in \mathcal{H}^V$ and $A_V X \in \mathcal{H}^V$. Then, we get

$$\nu_V(X) = \sum_{i=1}^{n-1} g((\nabla_{E_i} A_V)^t X, E_i) = \sum_{i=1}^{n-1} g(X, (\nabla_{E_i} A_V)\,E_i)\,.$$

Hence, $\nu_V(X) = 0$ because V is strongly normal. For the second part we apply the normality to get $\tilde{\nu}_V(X) = 0$ for all $X \in \mathcal{H}^V$. ■

As a consequence of this proposition and the results of [7], we obtain a lot of examples of Riemannian manifolds equipped with a harmonic unit vector field. Note also that the condition $\tilde{\nu}_V(V) = 0$ in Proposition 2.1 is satisfied when V is a Killing vector field. In general, this fact does not occur; that is, a strongly normal unit geodesic vector field does not always determine a harmonic map. To illustrate this, we consider the simplest case of a two-dimensional manifold (M, g). So, let $V \in \mathfrak{X}^1(M)$ and let $\{E_1, E_2 = V\}$ be a local orthonormal basis. Furthermore, we put $\lambda = g(\nabla_V V, E_1)$, $\mu = g(\nabla_{E_1} E_1, V)$. Then we have

$$A_V E_1 = \mu E_1, \qquad A_V V = -\lambda E_1.$$

Hence, V is harmonic if and only if $E_1(\mu) = V(\lambda)$ (see [5, Proposition 16]) and it is strongly normal if and only if $E_1(\mu) = -\lambda\mu$ (see [7, Proposition 3.1]). This implies that a geodesic unit vector field on (M, g) is harmonic if and only if it is strongly normal. Furthermore, a harmonic unit vector field on (M, g) determines a harmonic map if and only if $\mu K = \lambda K = 0$, where K denotes the Gauss curvature; and this occurs if and only if (M, g) is flat. Nevertheless, there exist examples of non-flat surfaces (M, g) equipped with a strongly normal geodesic unit vector field. For example, let (H^2, g) be the Poincaré half-plane $\{(y^1, y^2) \mid y^1 > 0\}$ with the metric

$$g = r^2 (y^1)^{-2} \{(dy^1)^2 + (dy^2)^2\},$$

where r is constant. Then $K = -r^{-2}$, and $V = r^{-1} y^1 \dfrac{\partial}{\partial y^1}$ satisfies the required conditions.

3. – Invariant harmonic unit vector fields on Lie groups.

Now we turn to the consideration of invariant harmonic unit vector fields. Let G be an n-dimensional connected Lie group equipped with a left-invariant metric and let $\mathfrak{g}$ denote its Lie algebra. Then a left-invariant metric g on G determines an associated inner product $\langle , \rangle$ on $\mathfrak{g}$. Furthermore, let $\mathcal{S}$ be the unit sphere of $\mathfrak{g}$ with respect to $\langle , \rangle$. For $V \in \mathcal{S}$, A_V, ν_V and $\tilde{\nu}_V$ are invariant by left translations. Therefore they can be viewed as tensors on $\mathfrak{g}$ and b as a function on $\mathcal{S}$. The distribution $\mathcal{H}^V$ is also invariant by left translation and is identified with the orthogonal complement $V^\perp$ of V in $\mathfrak{g}$. $V^\perp$ may also be naturally identified with the tangent space $T_V\mathcal{S}$ of the unit sphere $\mathcal{S}$ at V. Hence, a left-invariant unit vector field V is harmonic if and only if the one-form ν_V on $\mathfrak{g}$ vanishes on $V^\perp \cong T_V\mathcal{S}$ and it determines a harmonic map if and only if in addition $\tilde{\nu}_V$ vanishes on $\mathfrak{g}$.

Next, we determine a characterization of invariant harmonic vector fields by using the differential db of the function b on $\mathcal{S}$. For $X \in T_V\mathcal{S}$, we have

$$db_V(X) = -\frac{1}{2}\operatorname{tr}(A_V^t \nabla X + (\nabla X)^t A_V) = -\operatorname{tr}(A_V^t \nabla X).$$

On the other hand, we have

$$\begin{aligned}\nu_V(X) &= \sum_{i=1}^{n} \{\langle \nabla_{E_i}(A_V^t X), E_i\rangle - \langle A_V^t \nabla_{E_i} X, E_i\rangle\} \\ &= -\operatorname{tr}\operatorname{ad}_{A_V^t X} - \operatorname{tr}(A_V^t \nabla X)\end{aligned}$$

where $\{E_i, i = 1, \ldots, n\}$ is an orthonormal basis of $\mathfrak{g}$. Hence, we have

PROPOSITION 3.1. – *For $X \in T_V \mathcal{S}$, we have*

$$\nu_V(X) = db_V(X) - \operatorname{tr} \operatorname{ad}_{A_V^t X} \tag{3.1}$$

and V is harmonic if and only if

$$db_V(X) = \operatorname{tr} \operatorname{ad}_{A_V^t X} \tag{3.2}$$

for all $X \in T_V \mathcal{S}$.

First, let G be a unimodular Lie group, that is, $\operatorname{tr} \operatorname{ad}_X = 0$ for all $X \in \mathfrak{g}$ [11]. Then we have the following criterion.

PROPOSITION 3.2. – *A left-invariant unit vector field V on a unimodular Lie group G is harmonic if and only if V is a critical point of the function b on $\mathcal{S}$.*

This yields at once the following existence result.

COROLLARY 3.3. – *Any unimodular Lie group admits a left-invariant harmonic unit vector field.*

Next, let G be a non-unimodular Lie group with left-invariant metric and denote by $\mathfrak{u}$ its unimodular kernel, that is,

$$\mathfrak{u} = \{X \in \mathfrak{g} \,|\, \operatorname{tr} \operatorname{ad}_X = 0\}.$$

$\mathfrak{u}$ is an ideal of codimension one. Let H be a unit vector orthogonal to $\mathfrak{u}$. Then the linear transformation ad_H restricted to $\mathfrak{u}$ is a derivation of $\mathfrak{u}$ and for all $X \in \mathfrak{g}$ we have

$$\operatorname{tr} \operatorname{ad}_X = (\operatorname{tr} \operatorname{ad}_H)\langle H, X\rangle.$$

Moreover, we have

$$\nabla_H H = 0, \qquad \nabla_H X = \frac{1}{2}(\operatorname{ad}_H - \operatorname{ad}_H^t)\, X$$

for all $X \in \mathfrak{u}$. See [11] for the details. Note that $A_V H = 0$ when $\operatorname{ad}_{H|\mathfrak{u}}$ is symmetric. Hence, we have

PROPOSITION 3.4. – *A left-invariant unit vector field V on a non-unimodular Lie group is harmonic if and only if*

$$db_V(X) = (\operatorname{tr} \operatorname{ad}_H)\langle A_V H, X\rangle \tag{3.3}$$

for all $X \in T_V \mathcal{S}$. Moreover, if $\operatorname{ad}_{H|\mathfrak{u}}$ is a symmetric endomorphism of $\mathfrak{u}$ with respect to $\langle , \rangle$, then V is harmonic if and only if it is a critical point of the function b on $\mathcal{S}$.

Taking into account that H is a geodesic vector on (G, g), we get the following

COROLLARY 3.5. – *The unit vector H on a non-unimodular Lie group is harmonic if and only if it is a critical point of the function b on $\mathcal{S}$.*

In the next sections we shall provide several applications of these results. But before we do this, we prove another existence result.

PROPOSITION 3.6. – *Let G be an odd-dimensional Lie group with left-invariant Riemannian metric g. Then there exists a left-invariant harmonic unit vector field on G.*

PROOF. – Let $\mathcal{S}$ be the unit sphere in the corresponding Lie algebra $\mathfrak{g}$ of G. For $V \in \mathcal{S}$ we define the vector

$$N_V = \nu_V^{\#} - \nu_V(V)\, V$$

where $\nu_V^{\#}$ denotes the dual vector of ν_V with respect to the inner product $\langle , \rangle$ on $\mathfrak{g}$. Then N_V is orthogonal to V and hence it may be viewed as a tangent vector of $\mathcal{S}$ at V. In this way we obtain a smooth vector field on $\mathcal{S}$. Since dim $\mathcal{S}$ is even, there exists a point $V \in \mathcal{S}$ such that $N_V = 0$. At this point $\nu_V^{\#} = \nu_V(V)\, V$ and hence $\nu_V(X) = 0$ for $X \in V^{\perp}$. Hence, this V is harmonic. ∎

4. – Lie groups with bi-invariant metrics.

In this section we determine the full set of invariant harmonic unit vector fields on a (connected) Lie group G equipped with a bi-invariant metric. We refer to [12] for details about this kind of Lie groups and recall here some basic formulas.

An inner product $\langle , \rangle$ on the Lie algebra $\mathfrak{g}$ of G defines a bi-invariant metric if and only if ad_X is skew-symmetric for every X, or equivalently, if one of the following conditions hold:

$$\begin{aligned} \langle X, [Y, Z] \rangle &= \langle [X, Y], Z \rangle, \\ \nabla_X Y &= \frac{1}{2}[X, Y] \end{aligned} \tag{4.1}$$

for all $X, Y, Z \in \mathfrak{g}$. Hence, G is unimodular. Furthermore, the tensor A_V is given by

$$A_V = \frac{1}{2}\mathrm{ad}_V \tag{4.2}$$

for all $V \in \mathcal{S}$. From (4.1) it follows that A_V is skew-symmetric and hence, V determines a left-invariant unit Killing vector field. Moreover, the Riemannian curvature tensor is given by

$$R_{XY}Z = \frac{1}{4}[[X, Y], Z] \tag{4.3}$$

and hence, for the Ricci tensor ϱ of type $(0, 2)$ we get $\varrho = -\frac{1}{4}B$ where B denotes the Killing form given by $B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y)$. Using (4.2), we then get

$$b(V) = -\frac{1}{8}B(V, V). \tag{4.4}$$

First, we prove

PROPOSITION 4.1. – *Every harmonic left-invariant unit vector field on a Lie group with bi-invariant metric determines a harmonic map into its unit tangent bundle.*

PROOF. – From (4.1) and (4.3), and taking into account that ad_V is a Lie derivation, we get

$$\langle R_{XY}Z, V\rangle = \frac{1}{2}\langle [A_V X, Y] + [X, A_V Y], Z\rangle$$

for $X, Y, Z \in \mathfrak{g}$. Then, applying (4.1) and (4.2), it yields

$$\tilde{\nu}_V(X) = -\frac{1}{4}B(A_V X, V) \tag{4.5}$$

for all $X \in \mathfrak{g}$. Since $B([X, Y], Z) = B(X, [Y, Z])$ we then obtain $\tilde{\nu}_V(X) = 0$ and so the result follows. ∎

It is well-known that a connected Lie group G is compact and semisimple if and only if its Killing form B is negative definite. Then $-B$ provides a bi-invariant Riemannian metric on G, making $(G, g = -B)$ into an Einstein space of strictly positive scalar curvature. Note that when G is moreover simple, every bi-invariant metric g is essentially unique. Indeed, it takes the form $g = \beta B$ for some $\beta < 0$. Using the notes made in Section 2 and also Proposition 4.1, we then have

COROLLARY 4.2. – *On a compact and semisimple Lie group G with Killing form B, all unit left-invariant vector fields determine a harmonic map of $(G, g = -B)$ into its unit tangent bundle $(T_1 G, g_S)$.*

For a general Lie group with a bi-invariant metric, the orthogonal complement of any ideal in $\mathfrak{g}$ is itself an ideal. (See [11] for details.) So $\mathfrak{g}$ can be expressed as an orthogonal direct sum

$$\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l$$

where its center $Z(\mathfrak{g})$ is isomorphic to $\mathbb{R}^k$ for some k, and $\mathfrak{g}_1, \ldots, \mathfrak{g}_l$ are compact simple ideals. Then the inner product $\langle , \rangle$ on $\mathfrak{g}$ is of the form

$$\langle , \rangle = \langle , \rangle_0 \perp \beta_1 B_1 \perp \ldots \perp \beta_l B_l \tag{4.6}$$

where $\langle , \rangle_0$ is the standard inner product of $\mathbb{R}^k$, B_i $(i = 1, \ldots, l)$ is the restriction of the Killing form B to $\mathfrak{g}_i \times \mathfrak{g}_i$ and $\beta_i < 0$. Put $\dim \mathfrak{g}_i = n_i$ and let $\{e_{j_0}, e_{j_1}, \ldots, e_{j_l};\ 1 \leq j_0 \leq k,\ 1 \leq j_p \leq n_p,\ p = 1, \ldots, l\}$ be an orthonormal basis adapted to the above decomposition. Then V can be written as

$$V = \sum_{j_0 = 1}^{k} x_{j_0} e_{j_0} + \sum_{j_1 = 1}^{n_1} x_{j_1} e_{j_1} + \ldots + \sum_{j_l = 1}^{n_l} x_{j_l} e_{j_l}.$$

From (4.4) and applying Proposition 3.2 for the orthonormal vectors to V given by

$$U_{j_0}^{j_p} = -x_{j_0} e_{j_p} + x_{j_p} e_{j_0}, \qquad U_{j_p}^{j_q} = -x_{j_p} e_{j_q} + x_{j_q} e_{j_p}, \qquad p, q \in \{1, \ldots, l\},$$

we obtain that V is harmonic if and only if

$$x_{j_0} x_{j_p} = 0, \qquad \left(\frac{1}{\beta_p} - \frac{1}{\beta_q} \right) x_{j_p} x_{j_q} = 0$$

for all $1 \leq j_0 \leq k$, $1 \leq j_p \leq n_p$, $1 \leq j_q \leq n_q$, $p, q = 1, \ldots, l$. Hence, we have

PROPOSITION 4.3. – *Let G be a Lie group equipped with a bi-invariant metric $\langle , \rangle$ as in* (4.6). *Then we have*

(i) *if all the β_i are equal, then the set of left-invariant harmonic unit vector fields is $\{Z(\mathfrak{g}) \cup \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_l\} \cap \mathcal{S}$;*

(ii) *if all the β_i are different, then the set of left-invariant harmonic unit vector fields is $\{Z(\mathfrak{g}) \cup \mathfrak{g}_1 \cup \ldots \cup \mathfrak{g}_l\} \cap \mathcal{S}$.*

Any other case is obtained by combining (i) *and* (ii).

5. – Three-dimensional Lie groups.

The purpose of this section is to determine all invariant harmonic unit vector fields on three-dimensional Lie groups and to determine which of them determine harmonic maps. We use the treatment given in [11] and consider separately the unimodular and the non-unimodular cases.

A. *Unimodular Lie groups.*

Let G be a three-dimensional unimodular Lie group, $\mathfrak{g}$ its Lie algebra and g a left-invariant metric on G. Then there exists an orthonormal basis (e_1, e_2, e_3) of $\mathfrak{g}$ such that

$$[e_2, e_3] = \lambda_1 e_1, \qquad [e_3, e_1] = \lambda_2 e_2, \qquad [e_1, e_2] = \lambda_3 e_3 \tag{5.1}$$

where $\lambda_1, \lambda_2, \lambda_3$ are constants. Let θ^i, $i = 1, 2, 3$, be the dual one-forms of e_i, $i = 1, 2, 3$. Then the Levi Civita connection ∇ is determined by

$$\begin{aligned} \nabla e_1 &= \mu_3 e_2 \otimes \theta^3 - \mu_2 e_3 \otimes \theta^2, \\ \nabla e_2 &= -\mu_3 e_1 \otimes \theta^3 + \mu_1 e_3 \otimes \theta^1, \\ \nabla e_3 &= \mu_2 e_1 \otimes \theta^2 - \mu_1 e_2 \otimes \theta^1 \end{aligned} \tag{5.2}$$

where

$$\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i, \qquad i = 1, 2, 3 .$$

Furthermore, the curvature two-forms Ω^i_j, $i, j = 1, 2, 3$, are given by

$$\Omega^i_j(X, Y) = -\frac{1}{2}\theta^i(R_{XY} e_j), \qquad X, Y \in \mathfrak{g}$$

and the second structure equations yield

$$\begin{aligned} \Omega^1_2 &= \{\mu_3(\mu_1 + \mu_2) - \mu_1\mu_2\}\, \theta^1 \wedge \theta^2, \\ \Omega^1_3 &= \{\mu_2(\mu_1 + \mu_3) - \mu_1\mu_3\}\, \theta^1 \wedge \theta^3, \\ \Omega^2_3 &= \{\mu_1(\mu_2 + \mu_3) - \mu_2\mu_3\}\, \theta^2 \wedge \theta^3. \end{aligned} \tag{5.3}$$

Hence, the Ricci tensor ϱ is given by

$$\varrho = 2\{\mu_2\mu_3 \theta^1 \otimes \theta^1 + \mu_1\mu_3 \theta^2 \otimes \theta^2 + \mu_1\mu_2 \theta^3 \otimes \theta^3\},$$

which shows that (e_1, e_2, e_3) is a basis of eigenvectors for ϱ. (See [1] for more details.)

Now, let $V \in \mathcal{S}$ and put $V = x_1 e_1 + x_2 e_2 + x_3 e_3$. Then we get

$$A_V = \mu_1(x_3 e_2 - x_2 e_3) \otimes \theta^1 + \mu_2(x_1 e_3 - x_3 e_1) \otimes \theta^2 + \mu_3(x_2 e_1 - x_1 e_2) \otimes \theta^3$$

and this yields

$$b(V) = \frac{1}{2}\{(\mu_2^2 + \mu_3^2)\, x_1^2 + (\mu_1^2 + \mu_3^2)\, x_2^2 + (\mu_1^2 + \mu_2^2)\, x_3^2\}.$$

Using Proposition 3.2 and computing $db(U_j^i)=0$ for $U_2^1=-x_2\,e_1+x_1\,e_2$, $U_3^1=-x_3\,e_1+x_1\,e_3$ and $U_3^2=-x_3\,e_2+x_2\,e_3$, we find that V is harmonic if and only if

$$x_1x_2(\mu_1^2-\mu_2^2)=0\,,\qquad x_1x_3(\mu_1^2-\mu_3^2)=0\,,\qquad x_2x_3(\mu_2^2-\mu_3^2)=0\,.$$

From this, we then obtain

LEMMA 5.1. – *We have the following cases*:

(i) *if* $\mu_1^2=\mu_2^2=\mu_3^2$, *then every left-invariant unit vector field is harmonic*;

(ii) *if* $\mu_i^2=\mu_j^2\neq\mu_k^2$, *where* $i,j,k\in\{1,2,3\}$ *and* $i\neq j\neq k\neq i$, *then the set of left-invariant harmonic unit vector fields is given by* $\{\pm e_k\}\cup(S\cap\{e_i,e_j\}_{\mathbb{R}})$ *where* $\{e_i,e_j\}_{\mathbb{R}}$ *denotes the plane spanned by* e_i *and* e_j;

(iii) *if* $\mu_1^2\neq\mu_2^2\neq\mu_3^2\neq\mu_1^2$, *then the left-invariant harmonic unit vector fields are* $\pm e_i$, $i=1,2,3$.

In function of the curvature two-form Ω_j^i, $\tilde{\nu}_V$ is expressed as

$$\tilde{\nu}_V=2\sum_{\substack{i,j=1\\ i\neq j}}^{3}\Omega_j^i(A_Ve_j,V)\,\theta^i \tag{5.4}$$

and then, by using (5.3), we obtain

$$\begin{aligned}\tilde{\nu}_V=\;&x_2x_3\{\mu_2^2(\mu_1-\mu_3)+\mu_3^2(\mu_2-\mu_1)\}\otimes\theta^1\\&+x_1x_3\{\mu_1^2(\mu_3-\mu_2)+\mu_3^2(\mu_2-\mu_1)\}\otimes\theta^2\\&+x_1x_2\{\mu_1^2(\mu_3-\mu_2)+\mu_2^2(\mu_1-\mu_3)\}\otimes\theta^3.\end{aligned}\tag{5.5}$$

Following [11] and according to the signs of $\lambda_1,\lambda_2,\lambda_3$, we have six kinds of Lie algebras as described in Table I.

Using this classification, Lemma 5.1 and (5.5), we have the following result, the proof of which is now straightforward.

TABLE I.

signs of $\lambda_1,\lambda_2,\lambda_3$	associated Lie groups
$+,+,+$	$SU(2)$ or $SO(3)$
$+,+,-$	$SL(2,\mathbb{R})$ or $O(1,2)$
$+,+,0$	$E(2)$
$+,0,-$	$E(1,1)$
$+,0,0$	Heisenberg group
$0,0,0$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$

PROPOSITION 5.2. – *Let G be a three-dimensional unimodular Lie group with left-invariant metric and let $\{e_i,\ i=1,2,3\}$ be an orthonormal basis of the Lie algebra satisfying* (5.1). *Moreover, assume that the signs of $\lambda_1, \lambda_2, \lambda_3$ are chosen as in Table* I *and let $\lambda_1 \geqslant \lambda_2, \geqslant \lambda_3$. Then the left-invariant harmonic unit vector fields and those determining harmonic maps are given in Table* II.

TABLE II.

G	conditions for λ_i	the sets of invariant harmonic unit vector fields	harmonic maps $V:(G,g)\to(T_1G, g_S)$
$SU(2)$	$\lambda_1=\lambda_2=\lambda_3$	$\mathcal{S}$	$\mathcal{S}$
	$\lambda_1>\lambda_2=\lambda_3$	$\pm e_1,\ \mathcal{S}\cap\{e_2, e_3\}_{\mathbb{R}}$	$\pm e_1,\ \mathcal{S}\cap\{e_2, e_3\}_{\mathbb{R}}$
	$\lambda_1=\lambda_2>\lambda_3$	$\pm e_3,\ \mathcal{S}\cap\{e_1, e_2\}_{\mathbb{R}}$	$\pm e_3,\ \mathcal{S}\cap\{e_1, e_2\}_{\mathbb{R}}$
	$\lambda_1>\lambda_2>\lambda_3$	$\pm e_1,\ \pm e_2,\ \pm e_3$	$\pm e_1,\ \pm e_2,\ \pm e_3$
$SL(2,\mathbb{R})$	$\lambda_1=\lambda_2$	$\pm e_3,\ \mathcal{S}\cap\{e_1, e_2\}_{\mathbb{R}}$	$\pm e_3,\ \mathcal{S}\cap\{e_1, e_2\}_{\mathbb{R}}$
	$\lambda_1>\lambda_2$	$\pm e_1,\ \pm e_2,\ \pm e_3$	$\pm e_1,\ \pm e_2,\ \pm e_3$
$E(2)$	$\lambda_1=\lambda_2$	$\pm e_3,\ \mathcal{S}\cap\{e_1, e_2\}_{\mathbb{R}}$	$\pm e_3,\ \mathcal{S}\cap\{e_1, e_2\}_{\mathbb{R}}$
	$\lambda_1>\lambda_2$	$\pm e_3,\ \mathcal{S}\cap\{e_1, e_2\}_{\mathbb{R}}$	$\pm e_1,\ \pm e_2,\ \pm e_3$
$E(1,1)$		$\pm e_2,\ \mathcal{S}\cap\{e_1, e_3\}_{\mathbb{R}}$	$\pm e_1,\ \pm e_2,\ \pm e_3$
Heisenberg group		$\mathcal{S}$	$\pm e_1,\ \mathcal{S}\cap\{e_2, e_3\}_{\mathbb{R}}$
$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$		$\mathcal{S}$	$\mathcal{S}$

Using [13], we also get

COROLLARY 5.3. – *A left-invariant unit vector field on a three-dimensional unimodular Lie group is harmonic if and only if it is minimal.*

Another immediate consequence is as follows.

COROLLARY 5.4. – *Every three-dimensional unimodular Lie group admits left-invariant unit vector fields which determine harmonic maps into its unit tangent bundle. More precisely, a left-invariant unit vector field determines a harmonic map if and only if it is an eigenvector of the Ricci operator.*

B. *Non-unimodular Lie groups.*

We now turn to the case of a non-unimodular three-dimensional Lie group. In this case, let e_1 be a unit vector orthogonal to the unimodular kernel $\mathfrak{u}$ for which we choose an orthonormal basis (e_2, e_3) which diagonalizes the symmetric part of $\mathrm{ad}_{e_1|\mathfrak{u}}$. Then the bracket operation can be expressed as

(5.6) $$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = -\beta e_2 + \delta e_3, \quad [e_2, e_3] = 0$$

where α, β, δ are real constants. α and δ are the eigenvalues of the symmetric part of $\mathrm{ad}_{e_1|\mathfrak{u}}$. If necessary, by changing e_1 into $-e_1$ or alternating e_2 and e_3, we may assume

$$\alpha + \delta > 0, \qquad \alpha \geqslant \delta .$$

Furthermore, let θ^i, $i = 1, 2, 3$, be the dual one-forms of e_i, $i = 1, 2, 3$. Then ∇ is given by

$$\nabla e_1 = -\alpha e_2 \otimes \theta^2 - \delta e_3 \otimes \theta^3,$$

$$\nabla e_2 = \alpha e_1 \otimes \theta^2 + \beta e_3 \otimes \theta^1,$$

$$\nabla e_3 = \delta e_1 \otimes \theta^3 - \beta e_2 \otimes \theta^1$$

and for the curvature two-forms, we have

(5.7) $$\begin{aligned} \Omega^1_2 &= -\alpha^2 \theta^1 \wedge \theta^2, \\ \Omega^1_3 &= -\delta^2 \theta^1 \wedge \theta^3, \\ \Omega^2_3 &= -\alpha\delta \theta^2 \wedge \theta^3. \end{aligned}$$

See [1]. So, we obtain

$$\varrho = -\{(\alpha^2 + \delta^2)\, \theta^1 \otimes \theta^1 + \alpha(\alpha + \delta)\, \theta^2 \otimes \theta^2 + \delta(\alpha + \delta)\, \theta^3 \otimes \theta^3\}$$

and

$$\nabla_X \varrho = (\alpha - \delta)\{-\beta(\alpha + \delta)\, \theta^1(X)(\theta^2 \otimes \theta^3 + \theta^3 \otimes \theta^2) - \alpha\delta\theta^2(X)(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1) + \alpha\delta\theta^3(X)(\theta^1 \otimes \theta^3 + \theta^3 \otimes \theta^1)\}.$$

From this we get

(5.8) $$\|\nabla\varrho\|^2 = 2(\alpha - \delta)^2\{\beta^2(\alpha + \delta)^2 + 2\alpha^2\delta^2\}.$$

Now, let $V = x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathcal{S}$. Then A_V is given by

(5.9) $$A_V = \beta(x_3 e_2 - x_2 e_3) \otimes \theta^1 + \alpha(x_1 e_2 - x_2 e_1) \otimes \theta^2 + \delta(x_1 e_3 - x_3 e_1) \otimes \theta^3$$

and we have

$$b(V) = \frac{1}{2}\{(\alpha^2 + \delta^2)\, x_1^2 + (\alpha^2 + \beta^2)\, x_2^2 + (\beta^2 + \delta^2)\, x_3^2\}.$$

By using Proposition 3.4 we see that V is harmonic if and only if

$$(5.10)\qquad \begin{aligned} x_1\{(\beta^2-\delta^2)\,x_2-\beta(\alpha+\delta)\,x_3\} &= 0\,,\\ x_1\{\beta(\alpha+\delta)\,x_2-(\alpha^2-\beta^2)\,x_3\} &= 0\,,\\ \beta(x_2^2+x_3^2)-(\alpha-\delta)\,x_2x_3 &= 0\,. \end{aligned}$$

Furthermore, from (5.4), (5.7) and (5.9) we get

$$(5.11)\qquad \tilde{\nu}_V=\{(\alpha^3+\delta^3)\,x_1^2+\alpha^3x_2^2+\delta^3x_3^2\}\otimes\theta^1+\alpha x_1(\delta^2x_2-\alpha\beta x_3)\otimes\theta^2+\delta x_1(\beta\delta x_2+\alpha^2x_3)\otimes\theta^3.$$

These formulas yield

PROPOSITION 5.5. – *Let G be a three-dimensional non-unimodular Lie group with left-invariant metric and let $\{e_i,\ i=1,2,3\}$ be an orthonormal basis of the Lie algebra satisfying* (5.6). *Then the left-invariant harmonic unit vector fields on G and those determining harmonic maps are given in Table* III, *where U is the vector field given by*

$$U=(\alpha^3-\delta^3)^{-1/2}\{(-\delta^3)^{1/2}e_2+\operatorname{sign}\beta(\alpha^3)^{1/2}e_3\}$$

and the sets $\mathcal{A}$ and $\mathcal{B}$ are defined as follows:

$\mathcal{A}$ =*the set of unit vector fields of the plane determined by* $\beta x_2-\alpha x_3=0$;

$\mathcal{B}$ =*the set of unit vectors $x_2e_2+x_3e_3$ where x_2 and x_3 satisfy* $(\alpha-\delta)\,x_2x_3=\beta$.

TABLE III.

conditions for α and δ	conditions for β	the sets of invariant harmonic unit vector fields	harmonic maps
$\alpha=\delta$	$\beta=0$	$\pm e_1$, $\{e_2,e_3\}_{\mathbb{R}}\cap\mathcal{S}$	$\varnothing$
	$\beta\neq 0$	$\pm e_1$	
$\alpha>\delta>0$		$\pm e_1$, $\mathcal{B}$	$\varnothing$
$\alpha>\delta=0$	$\beta=0$	$\{e_1,e_2\}_{\mathbb{R}}\cap\mathcal{S}$, $\pm e_3$	$\pm e_3$,
	$\beta\neq 0$	$\pm e_1$, $\mathcal{B}$	$\varnothing$
$\alpha>0>\delta$	$\beta^2+\alpha\delta=0$	$\mathcal{A}$, $\mathcal{B}$	$\varnothing$
	$(\alpha^2+\delta^2+\alpha\delta)^2\beta^2+\alpha^3\delta^3=0$	$\pm e_1\mathcal{B}$	$\pm U$
	in the other cases	$\pm e_1$, $\mathcal{B}$	$\varnothing$

REMARKS 5.6. – A. For $\beta = \delta = 0$, the corresponding simply connected Lie group is isomorphic to the product $\mathbb{R} \times H^2(-\alpha^2)$ of Lie groups where $H^2(-\alpha^2)$ denotes the Poincaré half-plane with Gauss curvature equal to $-\alpha^2$. Here, the vector fields $\pm \frac{d}{dt}$ tangent to $\mathbb{R}$ are the unique left-invariant unit vector fields which define corresponding harmonic maps.

B. Following [13] we have that if $\alpha = \delta$ (that is, (G, g) has constant sectional curvature (see, for example, [1])), then a left-invariant unit vector field is minimal if and only if it is harmonic. Nevertheless, there exist non-unimodular Lie groups where these conditions are not equivalent.

Note that $\pm e_1$ are minimal and harmonic unit vector fields in all cases.

Finally, from Table III and (5.8) we obtain

PROPOSITION 5.7. – *A three-dimensional non-unimodular Lie group G admits a left-invariant unit vector field such that the corresponding map into its unit tangent bundle is harmonic if and only if*

$$\|\nabla \varrho\|^2 = 2(\alpha^2 + \delta^2 + \alpha\delta)^{-2}\alpha^2\delta^2(\alpha - \delta)^2(\alpha^2 + \delta^2)[2(\alpha^2 + \delta^2) + 3\alpha\delta]$$

and G is not of constant curvature.

6. – The semi-direct product $\mathbb{R} \times_\alpha \mathbb{R}^n$.

Several three-dimensional Lie groups are semi-direct products of the form $\mathbb{R} \times_\alpha \mathbb{R}^2$. In this section we consider the general case of Lie groups of type $\mathbb{R} \times_\alpha \mathbb{R}^n$.

Let $(\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n, g)$ be the standard Riemannian product on the vector space $\mathbb{R}^{n+1}$ and $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ a linear map. We consider an orthonormal basis $\{e_0, e_1, \ldots, e_n\}$ on $\mathbb{R} \times \mathbb{R}^n$ such that $e_1, \ldots, e_n$ diagonalize the symmetric part of α. Then $c_j^i = -c_i^j$, $i \neq j$, $i, j \in \{1, \ldots, n\}$, where $\alpha(e_i) = \sum_{j=1}^{n} c_i^j e_j$. We define a bracket operation by

$$[e_0, e_i] = \alpha(e_i), \qquad [e_i, e_j] = 0, \qquad i, j = 1, \ldots, n.$$

This bracket makes $\mathbb{R}^{n+1}$ into a Lie algebra $\mathbb{R} +_\alpha \mathbb{R}^n$ which is called the semi-direct sum of $\mathbb{R}$ and $\mathbb{R}^n$. The unique connected, simply connected Lie group $\mathbb{R} \times_\alpha \mathbb{R}^n$ associated to this Lie algebra is called the semi-direct product of the Lie groups $\mathbb{R}$ and $\mathbb{R}^n$. It is unimodular if and only if $\operatorname{tr} \alpha = 0$. For $\operatorname{tr} \alpha \neq 0$, the class of semi-direct products $\mathbb{R} \times_\alpha \mathbb{R}^n$ coincides with the set of the connected simply connected non-unimodular Lie groups whose unimodular kernel is

Abelian. Using the Koszul formula, we get for the Levi Civita connection ∇ of g:

$$\nabla_{e_0} e_0 = 0\,, \qquad \nabla_{e_0} e_i = \sum_{j=1,\, i\neq j}^{n} c_i^j e_j,$$
$$\nabla_{e_i} e_0 = -\,c_i^i e_i, \quad \nabla_{e_i} e_i = c_i^i e_0,$$

the other covariant derivatives being zero.

It follows that $V = e_0$ is a geodesic vector and from [7, Proposition 3.6] we know that it is strongly normal. So, from Proposition 2.1 we have

PROPOSITION 6.1. – *$V = e_0$ is a harmonic vector field on $\mathbb{R}\times_\alpha \mathbb{R}^n$. Moreover, it defines a harmonic immersion into its unit tangent sphere bundle if and only if $\sum_{i=1}^{n} (c_i^i)^3 = 0$.*

Next, we determine the complete set of left-invariant harmonic unit vector fields on $\mathbb{R}\times_\alpha \mathbb{R}^n$ for the case where α is a symmetric operator. Then $c_i^j = 0$ for all $i \neq j$. Let V be a unit vector of $\mathbb{R}+_\alpha \mathbb{R}^n$ given by $V = x_0 e_0 + \sum_{i=1}^{n} x_i e_i$. Then we have

$$b(V) = \frac{1}{2}\sum_{i=1}^{n} (c_i^i)^2 (x_0^2 + x_i^2)\,.$$

It follows from Proposition 3.2 and Proposition 3.4 that V is harmonic if and only if

$$\text{(6.1)} \qquad \begin{aligned} x_0 x_p \sum_{i=1,\, i\neq p}^{n} (c_i^i)^2 &= 0\,, \\ x_p x_q \{(c_p^p)^2 - (c_q^q)^2\} &= 0 \end{aligned}$$

for all $p, q \in \{1, \ldots, n\}$. In the Abelian case ($\alpha \equiv 0$), every left-invariant unit vector field is harmonic. Furthermore, for dim $V_0 \leq n-1$, where $V_0 = \operatorname{Ker}\alpha$, we denote by $\lambda_1, \ldots, \lambda_k$ the different non-vanishing eigenvalues of α put in order to satisfy

$$\lambda_1 = -\lambda_2, \ldots, \lambda_{2l-1} = -\lambda_{2l}, \lambda_{2l+1}, \ldots \lambda_k$$

where $\lambda_i \neq -\lambda_j$ for all $i, j \in \{2l+1, \ldots, k\}$ and $0 \leq l \leq \frac{k}{2}$. Here $l = 0$ means that all the $|\lambda_m|$, $m \in \{1, \ldots, k\}$, are different. Furthermore, let $V_1, \ldots, V_k$ denote the corresponding eigenspaces. With these notations, we get from (6.1):

PROPOSITION 6.2. – *Let $\mathbb{R}\times_\alpha \mathbb{R}^n$ be the semi-direct product of $\mathbb{R}$ and $\mathbb{R}^n$ where α is a linear symmetric map. Then we have*

(i) *every left-invariant unit vector field is harmonic when* $n = 1$ *or* $\alpha \equiv 0$;

(ii) *for* $n \geq 2$, *the set of left-invariant harmonic unit vector fields is* $(V_0 \cup \{e_0, e_1\}_{\mathbb{R}}) \cap \mathcal{S}$ *if* $\dim V_0 = n-1$ *and* $e_1 \in \operatorname{Im} \alpha$, *and*

$$\{\pm e_0\} \cup \{V_0 \cup (V_{\lambda_1} \oplus V_{\lambda_2}) \cup \ldots \cup (V_{\lambda_{2l-1}} \oplus V_{\lambda_{2l}}) \cup V_{\lambda_{2l+1}} \cup \ldots \cup V_{\lambda_k}\} \cap \mathcal{S}$$

if $\dim V_0 < n-1$.

7. – The generalized Heisenberg groups $H(1, r)$.

The three-dimensional Heisenberg group $H(1, 1)$ has been generalized in two different directions. On the one hand, one introduced the groups $H(1, r)$ [9] and on the other hand the groups $H(r, 1)$ appear as a special case of the generalized Heisenberg groups considered in [10]. Their geometries have different features. Here we shall study the harmonicity of invariant unit vector fields and we start with the groups $H(1, r)$. The other ones will be considered in the next section.

$H(1, r)$ is the Lie group of matrices of the form

$$a = \begin{pmatrix} I_r & A^t & B^t \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where I_r denotes the identity matrix of type (r, r), $A = (a_1, \ldots, a_r) \in \mathbb{R}^r$, $B = (b_1, \ldots, b_r) \in \mathbb{R}^r$ and $c \in \mathbb{R}$. It is a connected, simply connected nilpotent Lie group of dimension $2r + 1$ and the dimension of its center is r.

The following coordinates (x^i, y^i, z), $1 \leq i \leq r$, provide a system of global coordinates:

$$x^i(a) = a_i, \qquad y^i(a) = b_i, \qquad z(a) = c$$

and a basis of left-invariant one-forms is given by

$$\alpha_i = dx^i, \qquad \beta_i = dy^i - x^i dz, \qquad \gamma = dz\,.$$

For the dual left-invariant vector fields, we then have

$$X_i = \frac{\partial}{\partial x^i}, \qquad Y_i = \frac{\partial}{\partial y^i}, \qquad Z = \frac{\partial}{\partial z} + \sum_{j=1}^{r} x^j \frac{\partial}{\partial y^j}\,.$$

The Lie bracket is given by

$$[X_i, Z] = Y_i, \qquad i = 1, \ldots, r\,,$$

the other brackets being zero. On $H(1, r)$ we consider the Riemannian metric for which these vectors form an orthonormal basis at each point. Then $H(1, r)$ is a unimodular Lie group and the corresponding Levi Civita connection is determined by

$$\begin{aligned} \nabla_{X_i} Y_i = \nabla_{Y_i} X_i &= -\frac{1}{2} Z, \\ \nabla_{X_i} Z = -\nabla_Z X_i &= \frac{1}{2} Y_i, \\ \nabla_{Y_i} Z = \nabla_Z Y_i &= \frac{1}{2} X_i \end{aligned}$$

where the remaining covariant derivatives of the basic vectors vanish. For the non-vanishing components of the curvature tensor R we then obtain

$$\begin{aligned} R(X_i, X_j, Y_i, Y_j) &= -\frac{1}{4}, \qquad i \neq j, \\ R(X_i, Y_j, X_j, Y_i) &= \frac{1}{4}, \\ R(X_i, Z, X_i, Z) &= -\frac{3}{4}, \\ R(Y_i, Z, Y_i, Z) &= \frac{1}{4}. \end{aligned}$$

Now, let V be a unit vector of the Lie algebra. It can be written in the form

$$V = \sum_{i=1}^{r} (V_i X_i + V_{i+r} Y_i) + V_{2r+1} Z .$$

Then we obtain

$$A_V = \frac{1}{2} \sum_{i=1}^{r} \{(V_{i+r} Z - V_{2r+1} Y_i) \otimes \alpha_i + (V_i Z - V_{2r+1} X_i) \otimes \beta_i + (V_i Y_i - V_{r+i} X_i) \otimes \gamma\}$$

and this yields

$$b(V) = \frac{1}{4} \left\{ \sum_{i=1}^{r} (V_i^2 + V_{r+i}^2) + r V_{2r+1}^2 \right\}.$$

From this, it is easy to see that V is harmonic if and only if

$$(r-1) V_i V_{2r+1} = 0, \qquad (r-1) V_{r+i} V_{2r+1} = 0, \qquad i = 1, \ldots, r .$$

As has been seen already in Section 5, we see that for $r=1$, all left-invariant unit vector fields are harmonic. For $r \geqslant 2$, we have

PROPOSITION 7.1. – *The set of left-invariant harmonic unit vector fields on $H(1, r)$, $r \geqslant 2$, is given by $\{\pm Z\} \cup \{\mathcal{S} \cap Z^{\perp}\}$.*

Using the given expression for the curvature tensor and for A_V, we obtain

$$\tilde{\nu}_V = \frac{1}{4} \sum_{i=1}^{r} \{(2-r)\, V_{r+i} V_{2r+1} \otimes \alpha_i + (1-r)\, V_i V_{2r+1} \otimes \beta_i - V_i V_{r+i} \otimes \gamma\}\,.$$

As already proved, we see that for $r=1$ the set of left-invariant unit vectors which determine a harmonic map into the unit tangent bundle is $\{\pm Y\} \cup (\mathcal{S} \cap \{X, Z\}_{\mathbb{R}})$. For $r \geqslant 2$, we have

PROPOSITION 7.2. – *The set of left-invariant unit vector fields on $H(1, r)$, $r \geqslant 2$, for which the corresponding maps into the unit tangent bundles are harmonic is given by $\{\pm Z\} \cup \mathcal{A}$, where $\mathcal{A}$ is the set of the unit vectors $\sum_{i=1}^{r} (V_i X_i + V_{r+i} Y_i)$ such that $\sum_{i=1}^{r} V_i V_{r+i} = 0$.*

8. – H-type groups.

Now, we consider the class of generalized Heisenberg groups introduced in [10] and first recall some basic facts. We refer to [2] for more details about these H-type groups.

Let $\mathfrak{v}$ and $\mathfrak{z}$ be real vector spaces of dimensions $n, m \in \mathbb{N}$, respectively. On the direct sum $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ we consider an inner product $\langle , \rangle$ such that $\mathfrak{v}$ and $\mathfrak{z}$ are perpendicular, and an $\mathbb{R}$-algebra homomorphism $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$, $Z \mapsto J_Z$ which satisfies

$$\langle J_Z U, V\rangle + \langle U, J_Z V\rangle = 0\,, \qquad J_Z^2 = -\langle Z, Z\rangle id_{\mathfrak{v}} \tag{8.1}$$

for all $U, V \in \mathfrak{v}$, $Z \in \mathfrak{z}$. Then we have

$$\langle J_X U, J_Y U\rangle = \|U\|^2 \langle X, Y\rangle, \qquad \langle J_X U, J_X V\rangle = \|X\|^2 \langle U, V\rangle \tag{8.2}$$

and

$$J_X J_Y + J_Y J_X = -\,2\langle X, Y\rangle\, id_{\mathfrak{v}}, \tag{8.3}$$

$$\langle J_X U, J_Y V\rangle + \langle J_Y U, J_X V\rangle = 2\langle U, V\rangle \langle X, Y\rangle \tag{8.4}$$

for all $U, V \in \mathfrak{v}$ and $X, Y \in \mathfrak{z}$. Furthermore, we define a Lie algebra structure on $\mathfrak{n}$ by

$$[U+X, V+Y] = [U, V]$$

where $[\,,\,]: \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ is the bilinear map given by

$$\langle [U, V], Z \rangle = \langle J_Z U, V \rangle . \tag{8.5}$$

Then $\mathfrak{n}$ becomes a two-step nilpotent Lie algebra with center $\mathfrak{z}$. Such an $\mathfrak{n}$ is said to be a generalized Heisenberg algebra and the connected, simply connected Lie group N with Lie algebra $\mathfrak{n}$ and with the induced left-invariant metric g is called a generalized Heisenberg group. Clearly, N is unimodular.

From now on and as before, we suppose that $U, V, W \in \mathfrak{v}$ and $X, Y, Z \in \mathfrak{z}$. For the Levi Civita connection of (N, g) we obtain

$$\nabla_{V+Y}(U+X) = -\frac{1}{2} J_X V - \frac{1}{2} J_Y U - \frac{1}{2}[U, V] \tag{8.6}$$

and the corresponding curvature tensor R is given by

$$\begin{aligned} R_{(U+X)(V+Y)}(W+Z) = & -\frac{1}{4} J_{[V, W]} U + \frac{1}{4} J_{[U, W]} V + \frac{1}{2} J_{[U, V]} W \\ & -\frac{1}{4} J_Y J_Z U + \frac{1}{4} J_X J_Z V + \frac{1}{2} J_X J_Y W \\ & +\frac{1}{2} \langle X, Y \rangle W + \frac{1}{4}[V, J_X W] - \frac{1}{4}[U, J_Y W] \\ & -\frac{1}{2}[U, J_Z V] + \frac{1}{2} \langle U, V \rangle Z . \end{aligned} \tag{8.7}$$

Put $\xi = \xi_{\mathfrak{v}} + \xi_{\mathfrak{z}} \in \mathfrak{v} \oplus \mathfrak{z}$ for a unit vector ξ of $\mathfrak{n}$. Using (8.6), we then get

$$A_\xi (U+X) = \frac{1}{2}(J_X \xi_{\mathfrak{v}} + J_{\xi_{\mathfrak{z}}} U + [\xi, U]) . \tag{8.8}$$

On $\mathfrak{n}$ we consider an orthonormal basis adapted to the decomposition $\mathfrak{v} \oplus \mathfrak{z}$ and which is given by $\{U_1, \ldots, U_p, J_{X_1} U_1, \ldots, J_{X_1} U_p, X_1, \ldots, X_m\}$ where $n = 2p$.

Using (8.1)-(8.5), we then obtain

$$
\begin{aligned}
&[U_i, J_{X_1} U_i] = X_1, \\
(8.9) \qquad &[U_i, U_j] \quad = -[J_{X_1} U_i, J_{X_1} U_j] = \sum_{\alpha=2}^{m} c_{ij}^{\alpha} X_\alpha, \\
&[U_i, J_{X_1} U_j] = -[U_j, J_{X_1} U_i] \quad = \sum_{\alpha=2}^{m} c_{ip+j}^{\alpha} X_\alpha, \; i \neq j,
\end{aligned}
$$

where $i, j = 1, \ldots, p$ and $c_{ij}^{\alpha}, c_{ip+j}^{\alpha}$, $\alpha = 2, \ldots, m$, are real numbers which satisfy the relations $c_{ij}^{\alpha} = -c_{ji}^{\alpha}$, $c_{ip+j}^{\alpha} = -c_{jp+i}^{\alpha}$.

Now, we are ready to prove our results.

PROPOSITION 8.1. – *On a generalized Heisenberg group N with Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, every harmonic unit vector belonging to $\mathfrak{v} \cup \mathfrak{z}$ defines a harmonic map into its unit tangent bundle.*

PROOF. – First, let $\xi \in \mathfrak{v} \cap \mathcal{S}$. Then $A_\xi(U+X) = \frac{1}{2}(J_X \xi + [\xi, U])$ and from (8.5) and (8.7) we get

$$
\tilde{\nu}_\xi(U+X) = \tilde{\nu}_\xi(X) = \frac{1}{8} \operatorname{tr}(V \mapsto J_{\operatorname{ad}_\xi V} J_X \xi).
$$

Choosing an orthonormal basis as before with $U_1 = \xi$, from (8.5) we obtain

$$
\tilde{\nu}_\xi(X_1) = \frac{1}{8} \sum_{i=1}^{p} \{ \langle [J_{X_1} U_1, U_i], [U_1, U_i] \rangle + \langle [J_{X_1} U_1, J_{X_1} U_i], [U_1, J_{X_1} U_i] \rangle \}.
$$

Hence and by using (8.9), we have $\tilde{\nu}_\xi(X_1) = 0$ which shows that ξ determines a harmonic map.

Finally, let $\xi \in \mathfrak{z} \cap \mathcal{S}$. Then $A_\xi(U+X) = \frac{1}{2} J_\xi U$. Hence, using (8.1), (8.3) and (8.7), we obtain at once $\tilde{\nu}_\xi(U+X) = 0$. This completes the proof. ■

PROPOSITION 8.2. – *Any left-invariant unit vector field $\xi \in \mathfrak{z}$ defines a harmonic map into its unit tangent bundle.*

PROOF. – From (8.1) and (8.8) we get $A_\xi^t = -A_\xi$. Furthermore, we have

$$
(\nabla_{V+Y} A_\xi)(U+X) = \frac{1}{4}([J_\xi, J_Y]\, U + J_\xi J_X V - [J_\xi U, V])
$$

and from this, we derive

$$\langle(\nabla_V A_\xi)(U+X),\, V\rangle = -\frac{1}{4}\|V\|^2\langle X,\, \xi\rangle\,,$$

$$\langle(\nabla_Y A_\xi)(U+X),\, Y\rangle = 0\,.$$

This shows that $\nu_\xi(X) = 0$ for all X orthogonal to ξ. So ξ is harmonic and the result follows from Proposition 8.1. ■

Now, we put $\xi = \sum_{i=1}^{p}(x_i U_i + x_{p+i} J_{X_1} U_i) + \sum_{\alpha=1}^{m} x_{n+\alpha} X_\alpha$.

LEMMA 8.3. – *With the above notations, we have*

$$\begin{aligned}
8b(\xi) = n \sum_{\alpha=1}^{m} x_{n+\alpha}^2 + (m+1)\sum_{l=1}^{n} x_l^2 + \\
\sum_{\alpha=2}^{m}\sum_{i=1}^{p}\sum_{j=1,\, j\neq i}^{p}\Big\{[(c_{ij}^\alpha)^2 + (c_{ip+j}^\alpha)^2](x_j^2 + x_{p+j}^2) + \\
2\sum_{k=1,\, k\neq i,\, j}^{p}[(c_{ij}^\alpha c_{ik}^\alpha + c_{ip+j}^\alpha c_{ip+k}^\alpha)(x_j x_k + x_{p+j}x_{p+k}) + \\
x_j x_{p+k}(c_{ij}^\alpha c_{ip+k}^\alpha - c_{ip+j}^\alpha c_{ik}^\alpha)]\Big\}\,.
\end{aligned} \tag{8.10}$$

PROOF. – From (8.2), (8.5) and (8.8) we get

$$b(\xi) = \frac{1}{8}\{n\|\xi_{\mathfrak{z}}\|^2 + m\|\xi_{\mathfrak{v}}\|^2 + \operatorname{tr}(U \mapsto J_{\mathrm{ad}_\xi U}\xi_{\mathfrak{v}})\}\,.$$

Since

$$\operatorname{tr}(U \mapsto J_{\mathrm{ad}_\xi U}\xi_{\mathfrak{v}}) = \sum_{i=1}^{p}(\|[U_i,\, \xi]\|^2 + \|[J_{X_1}U_i,\, \xi]\|^2)$$

and, by using (8.9),

$$\begin{aligned}
[U_i,\, \xi] &= x_{p+i}X_1 + \sum_{\alpha=2}^{m}\Big(\sum_{j=1,\, j\neq i}^{p}(x_j c_{ij}^\alpha + x_{p+j}c_{ip+j}^\alpha)\Big)X_\alpha\,,\\
[J_{X_1}U_i,\, \xi] &= -x_i X_1 + \sum_{\alpha=2}^{m}\Big(\sum_{j=1,\, j\neq i}^{p}(x_j c_{ip+j}^\alpha - x_{p+j}c_{ij}^\alpha)\Big)X_\alpha\,,
\end{aligned}$$

a straightforward computation yields (8.10). ■

When dim $\mathfrak{z} = 1$, $\mathfrak{n}$ is isomorphic to a classical Heisenberg algebra. The corresponding Heisenberg group is isomorphic to the group of matrices $H(p, 1)$, with $n = 2p$, of the form

$$a = \begin{pmatrix} 1 & A & c \\ 0 & I_p & B^t \\ 0 & 0 & 1 \end{pmatrix}$$

where I_p denotes the identity matrix of type $p \times p$ and where $A = (a_1, \ldots, a_p) \in \mathbb{R}^p$, $B = (b_1, \ldots, b_p) \in \mathbb{R}^p$ and $c \in \mathbb{R}$. In this case, (8.10) reduces to

$$b(\xi) = \frac{1}{8}\left(n x_{n+1}^2 + 2 \sum_{l=1}^{n} x_l^2 \right)$$

and then, using Proposition 3.2, ξ is a harmonic unit vector if and only if

$$(n-2)\, x_{n+1} x_k = 0, \qquad k = 1, \ldots, n\,.$$

Then, putting $Z = X_1$ and applying Proposition 8.1, we have

PROPOSITION 8.4. – *Let $N \cong H(p, 1)$ be the classical Heisenberg group of dimension $2p+1$. For $p = 1$, every left-invariant unit vector field is harmonic. For $p \geq 2$, the set of left-invariant harmonic unit vector fields is given by $\{\pm Z\} \cup (\mathcal{S} \cap \mathfrak{v})$ and they all determine harmonic maps into the unit tangent bundle.*

In principle, we can also determine by using Proposition 3.2, all left-invariant harmonic unit vector fields on N when dim $\mathfrak{z} \geq 2$ and by considering case by case. Here, we restrict to the case $n = 4$. Then the corresponding non-isomorphic and non-classical generalized Heisenberg algebras have dim $\mathfrak{z} = 2$ or 3 [2]. (8.10) then takes the form

$$b(\xi) = \frac{1}{8}\left\{ 4 \sum_{\alpha=1}^{m} x_{n+\alpha}^2 + c_m \sum_{k=1}^{4} x_k^2 \right\}$$

where $c_m = 1 + m + \sum_{\alpha=2}^{m} \{(c_{12}^{\alpha})^2 + (c_{14}^{\alpha})^2\}$ for $m = 2$ or 3. So, ξ is harmonic if and only if

$$(4 - c_m)\, x_k x_{n+\alpha} = 0\,, \qquad k = 1, \ldots, 4 \quad \text{and} \quad \alpha = 1, \ldots, m\,.$$

Hence, we have the following two propositions.

PROPOSITION 8.5. – *Let N^6 be the six-dimensional generalized Heisenberg group and let $\{U_i, J_{X_1} U_i, X_1, X_2;\, i = 1, 2\}$ be an orthonormal basis of its Lie algebra satisfying* (8.9). *Then we have*:

(i) *every left-invariant unit vector field is harmonic if and only if* $(c_{12}^2)^2 + (c_{14}^2)^2 = 1$;

(ii) *in the other cases, the set of left-invariant harmonic unit vector fields is* $\mathcal{S} \cap (\mathfrak{v} \cup \mathfrak{z})$.

PROPOSITION 8.6. – *Let* N^7 *be the seven-dimensional non-classical generalized Heisenberg group and let* $\{U_i, J_{X_1} U_i, X_1, X_2, X_3; i = 1, 2\}$ *be an orthonormal basis of its Lie algebra satisfying* (8.9). *Then we have*:

(i) *every left-invariant unit vector field is harmonic if and only if* $c_{12}^2 = c_{14}^2 = c_{12}^3 = c_{14}^3 = 0$;

(ii) *in the other cases, the set of left-invariant harmonic unit vector fields is* $\mathcal{S} \cap (\mathfrak{v} \cup \mathfrak{z})$.

REMARK 8.7. – We refer to [7], [13] for some results concerning minimal unit fields on generalized Heisenberg groups of both types.

9. – Damek-Ricci spaces.

The Damek-Ricci spaces are a particular class of solvable extensions of generalized Heisenberg groups. They have a remarkable geometry and play an important role in several topics of study. We refer to [2] for more information. Here we study again the harmonicity of invariant unit vector fields and start with some needed basic material.

Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a generalized Heisenberg algebra, $\mathfrak{a}$ a one-dimensional real vector space and A a non-zero element of it. On the direct sum $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ we define an inner product $\langle , \rangle$ and a Lie bracket $[,]$ by

$$\langle U + X + rA, V + Y + sA \rangle = \langle U + X, V + Y \rangle_{\mathfrak{n}} + rs ,$$

$$[U + X + rA, V + Y + sA] = [U, V]_{\mathfrak{n}} + \frac{1}{2} rV - \frac{1}{2} sU + rY - sX ,$$

where the index $\mathfrak{n}$ denotes the corresponding product and bracket on $\mathfrak{n}$. As before, $U, V \in \mathfrak{v}$, $X, Y, Z \in \mathfrak{z}$ and r, s are real numbers. The corresponding connected, simply connected Lie group S with Lie algebra $\mathfrak{s}$ and with the induced left-invariant metric g is called a Damek-Ricci space.

We have

$$\mathrm{ad}_A(U + X + rA) = \frac{1}{2} U + X , \qquad \mathrm{tr}\, \mathrm{ad}_A = \frac{n}{2} + m .$$

S is non-unimodular, $\mathfrak{n}$ is its unimodular kernel and $\mathrm{ad}_{A|\mathfrak{n}}$ is a symmetric endomorphism of $\mathfrak{n}$ with respect to $\langle , \rangle$.

The Levi Civita connection ∇ of (S, g) is given by

$$\nabla_{V+Y+sA}(U+X+rA) = -\frac{1}{2}J_X V - \frac{1}{2}J_Y U - \frac{1}{2}rV - \frac{1}{2}[U, V] - rY + \frac{1}{2}\langle U, V\rangle A + \langle X, Y\rangle A\,. \tag{9.1}$$

Now, let $\xi = \xi_{\mathfrak{v}} + \xi_{\mathfrak{z}} + \lambda A \in \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ be a unit vector of $\mathfrak{s}$. Using (9.1), we then obtain

$$A_\xi(U+X+rA) = \frac{1}{2}(J_{\xi_{\mathfrak{z}}}U + J_X \xi_{\mathfrak{v}} + \lambda U + [\xi_{\mathfrak{v}}, U] - \langle U, \xi\rangle A) + \lambda X - \langle X, \xi\rangle A\,. \tag{9.2}$$

First, we prove a non-existence result.

PROPOSITION 9.1. – *On a Damek-Ricci space there do not exist left-invariant unit vector fields such that the corresponding map into its unit tangent bundle $(T_1 S, g_S)$ is harmonic.*

PROOF. – We show that $\tilde{\nu}_\xi(A) \neq 0$ for all $\xi \in \mathcal{S}$. To do this, we use the following formula for the curvature tensor R:

$$R(U+X+rA, V+Y+sA)A = \frac{1}{4}(J_Y U - J_X V - sU + rV) + \frac{1}{2}[U, V] - sX + rY\,. \tag{9.3}$$

Then, by using (8.1), (8.5) and (9.2) we get

$$\tilde{\nu}_\xi(A) = -\frac{1}{8}\{(n+8)\|\xi_{\mathfrak{z}}\|^2 + (2m+1)\|\xi_{\mathfrak{v}}\|^2 + (8m+n)\lambda^2 + \mathrm{tr}(U \mapsto J_{\mathrm{ad}_{\xi_{\mathfrak{v}}}U}\xi_{\mathfrak{v}})\}\,.$$

Since the last term in the brackets is positive, the result follows at once. ■

Next, we have

PROPOSITION 9.2. – *Any left-invariant unit vector field on* (S, g) *belonging to the set* $\mathfrak{v} \cup \mathfrak{z} \cup \{\pm A\}$ *is harmonic.*

PROOF. – From (9.2) and for $\xi \in \mathfrak{v}$ we have

$$A^t_\xi(U + X + rA) = \frac{1}{2}(J_X \xi - r\xi + [\xi, U]).$$

Now, it easy to check with (9.1) that $\nu_\xi(X + rA) = 0$. So ξ is harmonic.

For $\xi \in \mathfrak{z}$ we have

$$A^t_\xi(U + X + rA) = -\left(\frac{1}{2}J_\xi U + r\xi\right)$$

and again $\nu_\xi(U + rA) = 0$.

In a similar way we get that $\pm A$ are harmonic vectors. ■

When dim $\mathfrak{z} = 1$, the Damek-Ricci spaces are isometric to complex hyperbolic spaces. Denote by Z a unit vector of $\mathfrak{z}$. Then we have

PROPOSITION 9.3. – *The set of left-invariant harmonic unit vector fields on a Damek-Ricci space with one-dimensional center* $\mathfrak{z}$ *is given by* $(\{A, Z\}_{\mathbb{R}} \cup \mathfrak{v}) \cap \mathcal{S}$.

PROOF. – From (9.2) we have

$$b(\xi) = \frac{1}{8}\{(n+4)\|\xi_{\mathfrak{z}}\|^2 + 2\|\xi_{\mathfrak{v}}\|^2 + (n+4)\lambda^2 + \mathrm{tr}(U \mapsto J_{\mathrm{ad}_{\xi_{\mathfrak{v}}} U}\xi_{\mathfrak{v}})\}.$$

On $\mathfrak{s}$ we consider the orthonormal basis adapted to the decomposition $\mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ given by $\{U_1, \dots, U_p, J_Z U_1, \dots, J_Z U_p, Z, A\}$ where $n = 2p$, and we put $\xi = \sum_{i=1}^{p} (x_i U_i + x_{p+i} J_Z U_i) + x_{n+1} Z + x_{n+2} A$. Then, from (8.9) we obtain

$$b(\xi) = \frac{1}{8}\left\{3 \sum_{l=1}^{n} x_l^2 + (n+4)(x_{n+1}^2 + x_{n+2}^2)\right\}.$$

Hence, Proposition 3.4 then yields that ξ is harmonic if and only if

$$x_k x_{n+1} = 0, \qquad x_k x_{n+2} = 0, \qquad k = 1, \dots, n.$$

This shows that the required result holds. ■

REMARK 9.4. – In [7] it has been shown that the left-invariant vector field A on a Damek-Ricci space is always minimal.

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