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Geometric probabilities for non convex lattices. II


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Geometric Probabilities for non Convex Lattices (II).

Andrei Duma - Marius Stoka (*)

Sunto. – Si risolvono problemi di tipo Buffon per un reticolo avente per cellula fondamentale un poligono non convesso, utilizzando come corpo test un segmento ed un cerchio.

Summary. – We solve problems of Buffon type for a lattice with elementary tile a non-convex polygon, using as test bodies a line segment and a circle.

In a previous work [3] we dealt with a lattice having as elementary tile a non–convex polygon. In this paper we examine another lattice.

Let be given, in the Euclidean plane $E_2$, a lattice $\mathcal{R}$ whose elementary tile is a concave polygon, formed by five squares of side $a$, as in figure 1.

At first we want to determine the probability $p$ that a segment of constant length $l$ and random position in $E_2$ intersects one of the sides of a fundamental cell of the lattice $\mathcal{R}$. We assume that the segment is uniformly distributed in a bounded region of the plane.

We denote by $\mathcal{R}$ the family of segments $s$, of length $l$, whose midpoints lie inside a fixed tile $C_0$ of the lattice $\mathcal{R}$ and by $\mathcal{N}$ the set of segments $s$, of length $l$, which are completely contained in $C_0$. With these notations we have [6], p. 53

\[ p_l = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{R})} \]

where $\mu$ is the Lebesgue measure.

The measures $\mu(\mathcal{N})$ and $\mu(\mathcal{R})$ are computed by means of the kinematic measure in the Euclidean plane [4], p. 126

\[ dK = dx \wedge dy \wedge d\varphi , \]

where $x$ and $y$ are the coordinates of the midpoint of the segment $s$ and $\varphi$ is an angle of rotation.

Assuming the segment $s$ «small» compared to the lattice $\mathcal{R}$, i.e. $l < a$, the

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The notion of «non-small» segment and «non-small» circle with respect to a planar lattice with elementary tile a convex polygon was first defined in the works [1] and [2]. Subsequently it was generalized to a convex test body $\mathcal{K}$ and a lattice having as fundamental cell a concave polygon (we refer to the quoted work [3] for more details). By applying this latter definition to the lattice $\mathcal{R}$ in figure 1, we have that the segment $s$ is «non–small» with respect to the lattice in the following cases:

1) $a < l < a\sqrt{2}$;  
2) $a\sqrt{2} < l < 2a$;  
3) $2a < l < \sqrt{5}a$;  
4) $\sqrt{5}a < l < 2\sqrt{2}a$;  
5) $2\sqrt{2}a < l < 3a$;  
6) $3a < l < \sqrt{10}a$.

Taking into account the symmetries of the polygon $C_0$ we can confine ourselves to consider $\varphi \in [0, \pi/4]$. Thus

$$\mu(\mathcal{K}) = \int_0^{\pi/4} d\varphi \int_{\{(x, y) \in C_0\}} dx \, dy = \int_0^{\pi/4} \text{area}(C_0) \, d\varphi = \frac{5\pi}{4} a^2.$$  

Figure 1

The probability of intersection is

$$p_l = \frac{12}{5\pi} \frac{l}{a} - \frac{2}{5\pi} \left( \frac{l}{a} \right)^2. \tag{2}$$

This result was found by S. Rizzo [5], p. 14.
For any fixed value of the angle $\varphi$, we denote by $C_0(\varphi)$ the (convex or concave) domain determined by the midpoints of the segments $s$ entirely contained in $C_0$ in each one of the limit positions, so that

$$
\mu(N) = \int_0^{\pi/4} d\varphi \int_{\{(x, y) \in C_0(\varphi)\}} dx \, dy = \int_0^{\pi/4} \text{area}(C_0(\varphi)) \, d\varphi .
$$

By formulas (1), (3) and (4) we get

$$
p_l = 1 - \frac{4}{5\pi a^2} \int_0^{\pi/4} \text{area}(C_0(\varphi)) \, d\varphi .
$$

**Theorem 1.** If $a < l < a\sqrt{2}$, the probability that a segment $s$, of length $l$, intersects a side of one of the tiles of the lattice $R$ is

$$
p_l = \frac{4}{5\pi} \left( 2 \arccos \frac{a}{l} + 1 \right) + \frac{4l}{5\pi a} - \frac{8(l^2 - a^2)}{5\pi a} + \frac{2l^2}{5\pi a^2} .
$$

**Proof.** Let $\varphi_0 := \arccos \frac{a}{l}$. We have to consider the cases $0 < \varphi < \varphi_0$ and $\varphi_0 < \varphi < \frac{\pi}{4}$.

If $0 < \varphi < \varphi_0$, we get (see figure 2)

$$
\text{area } C_0(\varphi) = 2 \frac{(a - a \tan \varphi) + (a - l \sin \varphi + a \tan \varphi)}{2} (2a - l \cos \varphi)
$$

$$
+ (a - l \sin \varphi)(l \cos \varphi - a)
$$

$$
= 3a^2 - al(\sin \varphi + \cos \varphi) .
$$

Whereas when $\varphi_0 < \varphi < \frac{\pi}{4}$ we find (see figure 3)

$$
\text{area } C_0(\varphi) = 2a(a - l \sin \varphi) + 2a(a - l \cos \varphi)
$$

$$
+ (a - l \sin \varphi)(a - l \cos \varphi) + 2 \frac{l \sin \varphi \cdot l \cos \varphi}{2}
$$

$$
= 5a^2 - 3al(\sin \varphi + \cos \varphi) + l^2 \sin 2\varphi .
$$
Formula (4) then yields

\[ \mu(N) = \int_{\varphi_0}^{\varphi_0} \left[ 3a^2 - al(\sin \varphi + \cos \varphi) \right] d\varphi \]

\[ + \int_{\varphi_0}^{\pi/4} [5a^2 - 3al(\sin \varphi + \cos \varphi) + l^2 \sin 2\varphi] d\varphi \]

\[ = \left( \frac{5}{4} \pi - 2 \varphi_0 - 1 \right) a^2 - al + 2a \sqrt{l^2 - a^2} - \frac{1}{2} l^2 \]

and formula (5) gives the probability (6). ■

Remark 1. – When \( l = a \) formulas (2) and (7) give the probability
\[ p_t = \frac{2}{\pi}. \]
THEOREM 2. – If $a \sqrt{2} < l < 2a$, the probability that a segment $s$, of length $l$, intersects a side of one of the tiles of the lattice $R$ is

$$p_i = \frac{1}{5} + \frac{2}{5\pi} \left( 2 \arcsin \frac{a}{l} - 1 \right) + \frac{4\sqrt{l^2 - a^2}}{5\pi a} + \frac{4l}{5\pi a} - \frac{l^2}{5\pi a^2}.$$  \hspace{1cm}(7)$$

PROOF. – Let $\varphi_1 := \arcsin \frac{a}{l}$. We have to examine the cases $0 < \varphi < \varphi_1$ and $\varphi_1 < \varphi < \frac{\pi}{4}$.

If $0 < \varphi < \varphi_1$, we have (see figure 2)

$$\text{area } C_0(\varphi) = 3a^2 - al(\sin \varphi + \cos \varphi).$$

If $\varphi_1 < \varphi < \frac{\pi}{4}$, we get

$$\text{area } C_0(\varphi) = 2 \frac{(a - a \tan \varphi) + (a - l \sin \varphi + a \tan \varphi)}{2} \left(2a - l \cos \varphi \right)$$

$$= 4a^2 - 2al(\sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi.$$
Hence formula (4) yields

\[
\mu(N) = \int_{0}^{\varphi_1} [3a^2 - al(\sin \varphi + \cos \varphi)] d\varphi
\]

\[
+ \int_{\varphi_1}^{\pi/4} [4a^2 - 2al(\sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi] d\varphi
\]

\[
= a^2(\pi - \varphi_1 + \frac{1}{2}) - a\sqrt{l^2 - a^2} - al + \frac{1}{4}l^2
\]

and formula (5) gives the probability (7). ■

Remark 2. – When \( l = a\sqrt{2} \) formulas (6) and (7) coincide, having the same value \( p_l = \frac{2}{5} + \frac{4\sqrt{2}}{5\pi} \).

Theorem 3. – If \( 2a < l < \sqrt{5}a \), the probability that a segment \( s \), of length \( l \), intersects a side of one of the tiles of the lattice \( \mathcal{R} \) is

\[
p_l = \frac{1}{5} - \frac{2}{\pi} + \frac{4}{5\pi} \arcsin \frac{a}{l} + \frac{4\sqrt{l^2 - a^2}}{5\pi a} + \frac{12l}{5\pi a} - \frac{3l^2}{5\pi a^2}.
\]

Proof. – Let \( \varphi_2 := \arccos \frac{2a}{l} \). As \( l < \sqrt{5}a \) we have \( \varphi_2 < \varphi_1 \), so we must consider the following three cases:

\[ 0 < \varphi < \varphi_2, \quad \varphi_2 < \varphi < \varphi_1, \quad \varphi_1 < \varphi < \frac{\pi}{4} \, . \]

When \( 0 < \varphi < \varphi_2 \) we have

area \( C_0(\varphi) = (3a - l \cos \varphi)(a - l \sin \varphi) \)

\[ = 3a^2 - al(3 \sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi \, . \]

When \( \varphi_2 < \varphi < \varphi_1 \) we find (see figure 2)

area \( C_0(\varphi) = 3a^2 - al(\sin \varphi + \cos \varphi) \).  

Finally when \( \varphi_1 < \varphi < \frac{\pi}{4} \) we get (see figure 4)

area \( C_0(\varphi) = 4a^2 - 2al(\sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi \, . \)
Hence formula (4) yields
\[
\mu(N) = \int_0^{\varphi_2} [3a^2 - al(3 \sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi] \, d\varphi
\]
\[
+ \int_{\varphi_1}^{\varphi_2} [3a^2 - al(3 \sin \varphi + \cos \varphi)] \, d\varphi
\]
\[
+ \int_{\varphi_1}^{\pi/4} [4a^2 - 2al(\sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi] \, d\varphi
\]
\[
= a^2 \left( \frac{5}{2} - \varphi_1 + \pi \right) - a \sqrt{l^2 - a^2} - 3al + \frac{3}{4} l^2.
\]
and, taking into account (1) and (3) we immediately find the stated probability (8).

Remark 3. – It is easily checked that (7) and (8) give the same probability
\[
p_l = \frac{1}{3} + \frac{2}{5\pi} + \frac{4\sqrt{3}}{5\pi} \quad \text{when } l = 2a.
\]
THEOREM 4. – If $\sqrt{5}a < l < 2\sqrt{2}a$, the probability that a segment $s$, of length $l$, intersects a side of one of the tiles of the lattice $R$ is

$$p_l = \frac{1}{5} + \frac{2}{\pi} + \frac{16}{5\pi} \arccos \frac{2a}{l} - \frac{12}{5\pi} \arcsin \frac{a}{l} + \frac{12l}{5\pi a}$$

$$- \frac{12}{5\pi a} \sqrt{l^2 - a^2} - \frac{8}{5\pi a} \sqrt{l^2 - 4a^2} + \frac{l^2}{5\pi a^2}.$$  

PROOF. – This time the angle $\varphi_1$ is less than $\varphi_2$ because we are now assuming $l > a\sqrt{5}$, thus we have to consider three possible ranges of variation for the angle $\varphi$:

$$0 < \varphi < \varphi_1, \quad \varphi_1 < \varphi < \varphi_2, \quad \varphi_2 < \varphi < \frac{\pi}{4}.$$  

In the first case, $0 < \varphi < \varphi_1$, we have (see figure 5)

$$\text{area } C_0(\varphi) = 3a^2 - al(3 \sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi.$$  

In the second case, $\varphi_1 < \varphi < \varphi_2$, we find $\text{area } C_0(\varphi) = 0$ because $C_0(\varphi) = 0$.

In the third case, $\varphi_2 < \varphi < \frac{\pi}{4}$, we get (see figure 4)

$$\text{area } C_0(\varphi) = 4a^2 - 2al(\sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi.$$  

Figure 5
By applying formula (4) to obtain the value of the measure of the set \( \mathcal{N} \) we find

\[
\mu(\mathcal{N}) = \int_{0}^{\frac{\pi}{4}} [3a^2 - al(3 \sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi] d\varphi
\]

\[
+ \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} [4a^2 - 2al(\sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi] d\varphi
\]

\[
= a^2 \left( \pi - \frac{5}{2} - 4\varphi_2 + 3\varphi_1 \right) - 3al + 3a \sqrt{l^2 - a^2} + 2a \sqrt{l^2 - 4a^2} - \frac{l^2}{4}
\]

and formula (5) gives the probability (9).

**Remark 4.** – When \( l = a\sqrt{5} \) both formula (8) and (9) become

\[
p_l = \frac{1}{5} + \frac{12\sqrt{5} - 17}{5\pi} + \frac{4}{5\pi} \arcsin \frac{1}{\sqrt{5}}.
\]

**Theorem 5.** – If \( 2\sqrt{2}a < l < 3a \), the probability that a segment \( s \), of length \( l \), intersects a side of one of the tiles of the lattice \( \mathcal{R} \) is

(10) \[
p_l = 1 - \frac{12}{5\pi} \arcsin \frac{a}{l} + \frac{2}{5\pi} - \frac{12l}{5\pi a} - \frac{12}{5\pi a} \sqrt{l^2 - a^2}.
\]

**Proof.** – If \( 0 < \varphi < \varphi_1 \), we find (see figure 5)

\[
\text{area } C_0(\varphi) = 3a^2 - al(3 \sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi.
\]

As the set \( C_0(\varphi) \) is empty when \( \varphi_1 < \varphi < \frac{\pi}{4} \) we simply have

\[
\mu(\mathcal{N}) = \int_{0}^{\varphi_1} [3a^2 - al(3 \sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi] d\varphi
\]

\[
= 3a^2 \arcsin \frac{a}{l} - 3al + 3a \sqrt{l^2 - a^2} - \frac{5a^2}{2}
\]

and this value immediately yields the probability (10).
REMARK 5. – When \( l = 2\sqrt{2}a \), formulas (9) and (10) give the same probability

\[
p_l = 1 - \frac{12}{5\pi} \arcsin \frac{1}{2\sqrt{2}} + \frac{2(1 + 12\sqrt{2} - 6\sqrt{7})}{5\pi}.
\]

THEOREM 6. – If \( 3a < l < \sqrt{10}a \), the probability that a segment \( s \), of length \( l \), intersects a side of one of the tiles of the lattice \( \mathcal{R} \) is

\[
p_l = 1 + \frac{4}{\pi} - \frac{12}{5\pi} \left( \arcsin \frac{a}{l} - \arccos \frac{3a}{l} \right)
- \frac{4(l^2 - 9a^2 + \sqrt{l^2 - a^2})}{5\pi a} + \frac{2l^2}{5\pi a^2}.
\]

PROOF. – Let \( \varphi_3 := \arccos \frac{3a}{l} \). As we are assuming \( l < \sqrt{10}a \), we have \( \varphi_3 < \varphi_1 \) and therefore the different ranges of variation of the angle \( \varphi \) which we have to consider are

\[0 < \varphi < \varphi_3, \quad \varphi_3 < \varphi < \varphi_1, \quad \varphi_1 < \varphi < \frac{\pi}{4}.\]

In the first and in the latter case the set \( C_0(\varphi) \) is empty and consequently area \( C_0(\varphi) = 0 \). When \( \varphi_3 < \varphi < \varphi_1 \) we find (see figure 5)

\[
\text{area } C_0(\varphi) = 3a^2 - al(3\sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi.
\]

Formula (4) gives

\[
\mu(\mathcal{N}) = \int_{\varphi_3}^{\varphi_1} [3a^2 - al(3\sin \varphi + \cos \varphi) + l^2 \sin \varphi \cos \varphi] \, d\varphi
= 3a^2 \left( \arcsin \frac{a}{l} - \arccos \frac{3a}{l} \right) - 5a^2
+ a(\sqrt{l^2 - 9a^2} + 3\sqrt{l^2 - a^2}) - \frac{l^2}{2}
\]

and (5) yields the probability (11). ■
Remark 6. – If we substitute \( l = 3a \) in (10) and (11) we find the same probability

\[
p_l = 1 - \frac{12}{5\pi} \arcsin \frac{1}{3} + \frac{2(19 - 12\sqrt{2})}{5\pi}.
\]

Next we choose as test body a circle \( \gamma \) of constant radius \( r \). We denote by \( C_0(r) \) the set of points \( P \in C_0 \) with the property that the circle of center \( P \) and radius \( r \) is completely contained in the elementary tile \( C_0 \). Taking into account formula (1), the probability \( p_r \) of intersection between a random circle (of constant radius \( r \)), uniformly distributed in a bounded region of the plane, and one of the sides of an elementary tile of the lattice \( R \) is

\[
p_r = 1 - \frac{\text{area } C_0(r)}{\text{area } C_0} = 1 - \frac{\text{area } C_0(r)}{5a^2}.
\]

The circle \( \gamma \) is «small» with respect to the lattice \( R \) if and only if

\[
2r < a.
\]

Theorem 7. – If \( r \) satisfies condition (13), the probability that a circle \( \gamma \) intersects one of the sides of an elementary tile of the lattice \( R \) is

\[
p_r = \frac{12r}{5a} - \frac{(8 - \pi) r^2}{5a^2}.
\]

Proof. – From figure 6 we get

\[
\text{area } C_0(r) = 2(3a - 2r)(a - 2r) - (a - 2r)^2 + 4r^2 - \pi r^2
\]

\[= 5a^2 - 12ar + (8 - \pi) r^2.
\]

Using this value of area \( C_0(r) \) in formula (12) we obtain the probability (14).

Recalling the definition of «non-small» circle with respect to the lattice \( R \), we can say that this case occurs exactly when

\[
a < 2r < \sqrt{2}a.
\]
THEOREM 8. – If \( r \) satisfies condition (15), the probability that a circle \( \gamma \) intersects one of the sides of an elementary tile of the lattice \( \mathcal{R} \) is

\[
 p_r = \frac{4}{5} - \frac{\pi r^2}{5a^2} + \frac{4r^2}{5a^2} \arcsin \frac{a}{2r} + \frac{1}{5a} \sqrt{4r^2 - a^2}.
\]
PROOF. – Using the notations of figure 7 we can write
\[ 4 \cdot \frac{\pi r^2}{4} - 4 \text{area } C_1 + \text{area } C_0(r) = a^2, \]

hence
\[ \text{(17)} \quad \text{area } C_0(r) = a^2 - \pi r^2 + 4 \text{area } C_1. \]

From figure 8 we get
\[ \text{area } C_1 = 2 \int_0^r \sqrt{r^2 - x^2} dx = r^2 \left[ \frac{\pi}{2} - \arcsin \frac{a}{2r} - \frac{4}{4r^2} \sqrt{4r^2 - a^2} \right]. \]

By substituting this value in (17) we obtain
\[ \text{area } C_0(r) = a^2 + \pi r^2 - 4r^2 \arcsin \frac{a}{2r} - a \sqrt{4r^2 - a^2} \]

and formula (12) yields the probability (16).

REMARK 7. – When \( r = \frac{a}{2} \) formulas (14) and (16) give the same probability \( p_r = \frac{4}{5} + \frac{\pi}{20}. \)

REFERENCES


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