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E. Comparini, M. Ughi

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On the Existence of Shock Propagation in a Flow Through Deformable Porous Media (*).

E. Comparini - M. Ughi

Sunto. – *Consideriamo il flusso unidimensionale di un fluido incomprimibile in un mezzo poroso deformabile, in cui la porosità e la conduttività idraulica dipendono dall'intensità del flusso. Trascurando fenomeni di capillarità, una frontiera regolare penetra nella zona asciutta (inizialmente occupante l'intero mezzo) dividendola dalla zona bagnata. Assumendo che la pressione sul bordo sia una funzione convessa, studiamo il problema della continuazione della soluzione nel caso di eventuali singolarità interpretabili fisicamente come «collassi» locali del mezzo. In particolare si danno condizioni sufficienti per garantire l'esistenza di una soluzione continua fino ad un tempo assegnato e si studia il comportamento della soluzione nel caso in cui appaiano singolarità, dimostrando un teorema di esistenza locale e unicità della soluzione.*

Summary. – *We consider a one-dimensional incompressible flow through a porous medium undergoing deformations such that the porosity and the hydraulic conductivity can be considered to be functions of the flux intensity. The medium is initially dry and we neglect capillarity, so that a sharp wetting front proceeds into the medium. We consider the open problem of the continuation of the solution in the case of onset of singularities, which can be interpreted as a local collapse of the medium, in the general case of convex boundary pressure. We study the behaviour of the solution after the development of a singularity and we study the existence of the solution after the time at which the shock line reaches the surface.*

1. – Introduction.

In the recent papers [1], [3] a generalization of the classical Green-Ampt model for the penetration of a wetting front in a dry porous medium [4] has been considered. It has been assumed that the physical parameters k (hydraulic conductivity) and ε (porosity) depend on the volumetric velocity q. Thus the model takes into account the possibility of flow-induced deformations on the microscopic scale. In summary the model gives rise to the following free

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boundary problem

$$p_x = -\frac{q}{k(q)}, \qquad \text{in } \{0 < x < s(t),\ t > 0\} = D(t), \tag{1.1}$$

$$q_x + \varepsilon'(q)\, q_t = 0, \qquad \text{in } D(t), \tag{1.2}$$

$$p(0,t) = p_0(t) > 0, \qquad t > 0, \tag{1.3}$$

$$p(s(t),t) = 0, \qquad t > 0, \tag{1.4}$$

$$\dot{s} = \frac{q(s(t),t))}{\varepsilon(q(s(t),t)}, \qquad t > 0,\ s(0) = 0. \tag{1.5}$$

where p is the pressure, $p_0(t)$ is a given function, $s(t)$ is the thickness of the wet region.

In the Green-Ampt model ε and k are positive constants while here we assume that ε and k are given functions of q satisfying some physically intuitive conditions (see [3])

$$\varepsilon \in C^3, \qquad \varepsilon' < 0, \qquad \varepsilon'' > 0, \qquad \varepsilon \geqslant \varepsilon_0 > 0 \qquad \forall q > 0, \tag{1.6}$$

$$k \in C^3, \qquad k' \leqslant 0, \qquad k'' \geqslant 0, \qquad k \geqslant k_0 > 0 \qquad \forall q > 0. \tag{1.7}$$

From a mathematical point of view the problem (1.1)-(1.5) is a free boundary problem for a non-homogeneous reducible quasilinear hyperbolic system, which is strictly hyperbolic in the assumption $\varepsilon' < 0$ (see [5], [6], [11]). Let also mention that p and q are already the Riemann invariant of the problem. The domain $\{0 < x < s(t), t > 0\}$ is typical in the sense of Li Ta-Tsien (see [5]) because the characteristic starting from the origin do not enter in the domain ($\dot{s} > 0$, $\varepsilon' < 0$), but the boundary conditions (1.3), (1.4) are not typical.

However in the mentioned paper [3], local existence and uniqueness of a classical solution are proved, but the possibility of the developement of singularities of the solution is envisaged, which can be interpreted as a local collapse of the medium. In such a case it is an open question if and how the solution can be continued after the onset of the above mentioned singularities.

In the paper [1] global existence of a classical solution has been proved for $p_0(t)$ concave. Moreover an example has been studied under the assumption that $p_0(t)$ has a piecewise constant increasing derivative. In this case it has been proved that an infinite sequence of shocks exists, starting from the time at which $\dot{p}_0$ jumps. These shocks travel from the free boundary $x = s(t)$ towards the surface $x = 0$.

This type of qualitative behaviour of the solution is not unusual for this kind of hyperbolic problems, see e.g. [6], [7].

Here we want to consider more generally the case of $p_0(t)$ convex.

In Sect. 2 we give conditions on $\ddot{p}_0(t)$ which guarantee existence without shocks up to a general time T. A study of the solution after the shock develops is given is Sect. 3 up to the time T_k at which the shock line reaches the surface $x = 0$. The last two sections are devoted to the study of the local existence and uniqueness of the solution after T_k.

2. – Nonshock condition.

Let us assume that $p_0(t)$ is a given function such that

$$p_0 \in C^2, \qquad p_0(0) = 0, \qquad \dot{p}_0(0) > 0, \qquad \ddot{p}_0(t) \geqslant 0$$

In [3] the local in time existence was proved up to a sufficiently small time T_0, moreover it was shown that q is bounded

$$q^* < q < Q(t) \tag{2.1}$$

where

$$q^*\colon F(q^*) = \dot{p}_0(0), \qquad Q(t)\colon F(Q(t)) = \dot{p}_0(t), \qquad F(q) = \frac{q^2}{\varepsilon(q)\, k(q)} \tag{2.2}$$

(Remark that $Q(t)$ is increasing).

From now on we will use the notation

$$q_0(t) = q(0, t), \qquad q_s(t) = q(s(t), t).$$

Condition (2.1) implies the a priori boundeness of both q_0 and q_s.

As for $q_0'(t)$ and $q_s'(t)$ for $0 < t < T_0$ we have (see [10])

$$0 < q_0' < L_0, \qquad 0 < q_s' < L_0 \tag{2.3}$$

where

$$L_0 = \sup_{t < T_0,\, q^* < q < Q(t)} \frac{\ddot{p}_0}{H'(q)}, \tag{2.4}$$

and

$$H(q) = F(q) + G(q), \tag{2.5}$$

G given by

$$G'(q) = -\frac{1}{\varepsilon'(q)} \frac{d}{dq}\left(\frac{q}{k(q)}\right), \qquad G(0) = 0.$$

The solution can be continued after T_0 solving the first order problem (1.2) with datum $q_s(t)$ on $s(t)$ for $0 < t < T_0$.

This is possible because the characteristics starting from the free boundary are stright lines whose equation is the following:

$$t = \varepsilon'(q_s(\tau))(x - s(\tau)) + \tau, \qquad 0 < \tau < T_0. \tag{2.6}$$

Since $\varepsilon' < 0$ the above characteristics enter the domain $D(t)$ so we can prescribe on $x = s(t)$ the known function $q_s(t)$ in $0 < t < T_0$, and solve equation (1.2) in $D(t)$ locally after T_0 (remark that since equation (1.2) is homogeneous q is constant along the characteristics).

The time at which the above characteristic reaches the surface $x = 0$ is given by θ

$$\theta = -\varepsilon'(q_s(\tau))\, s(\tau) + \tau > \tau. \tag{2.7}$$

By classical theory [2] we have

$$q_0(\theta) = q_s(\tau), \tag{2.8}$$

hence

$$\begin{cases} q_0'(\theta) = q_s'(\tau)\,\dfrac{1}{\frac{d\theta}{d\tau}} \\[2ex] \dfrac{d\theta}{d\tau} = -\varepsilon'' q_s' s - \varepsilon' \dot{s} + 1 \Big|_\tau . \end{cases} \tag{2.9}$$

If there are no shocks up to time

$$T_1 = T_0 - \varepsilon'(q_s(T_0))\, s(T_0), \tag{2.10}$$

then

$$\frac{d\theta}{d\tau} \geqslant \delta > 0.$$

This gives immediately an upper bound for q_s'. In fact we have

$$q_s' \leqslant \frac{1 - \varepsilon' \dot{s} - \delta}{\varepsilon'' s}, \qquad 0 < t < T_1. \tag{2.11}$$

On the other hand if there are no shocks before T_1, see [3], we have the following relation between q_s and q_0

$$H(q_s) = \dot{p}_0 + G(q_0) \tag{2.12}$$

so that

$$H'(q_s)\, q_s' = \ddot{p}_0 + G'(q_0)\, q_0'. \tag{2.13}$$

Equation (2.12) is obtained in [3] integrating (1.1) in x between 0 and $s(t)$, taking into account the boundary values of $p(t)$ ((1.3), (1.4)) and then differentiating with respect to t the integral relation thus obtained and using the free boundary condition (1.5).

Since $q_0(t)$ is defined locally after T_0 by (2.7) and (2.8), then $q_s(t)$ is defined in the same time interval by (2.12) and so is $s(t)$ through (1.5). Therefore a classical solution is continued after T_0 as long as no shock forms inside the domain $D(t)$.

In terms of pressure, if we have no shocks up to T_1 then we must have assumed

$$\ddot{p}_0(t) \leqslant (1 - \varepsilon' \dot{s} - \delta) \frac{H'(q_s)}{\varepsilon''(q_s)\, s} - G'(q_0)\, q_0', \qquad t < T_1. \tag{2.14}$$

This can be generalized to any $T > 0$ in the sense that, if there are no shocks before T, (2.11) has to hold for $t < T$. We remark that condition (2.11) holds for small T; precisely, since the existence is guaranteed up to T_0 (see [3]), then in particular (2.11) holds for $\tau_0 < t < T_0$, where τ_0 is defined by

$$T_0 = -\varepsilon'(q_s(\tau_0))\, s(\tau_0) + \tau_0. \tag{2.15}$$

We are now able to prove a theorem which gives a necessary and sufficient condition of existence of a unique classical solution up to a time T.

THEOREM 2.1. – *Necessary condition: for any* $T > 0$, *if*

$$\ddot{p}_0 > \frac{1 - \varepsilon'(Q(T))\, Q(T)/\varepsilon(q^*)}{\inf\limits_{q^* < q < Q} \varepsilon''(q)\, \tau_0 q^*/\varepsilon(q^*)} \sup_{q^* < q < Q} H',$$

then at least one shock occurred before T.

Sufficient condition: If $\dfrac{d^2}{dz^2}\left(\dfrac{z}{k}\right) \geqslant 0$, *then* $\forall T > 0$ *there exists a constant* $C(T)$ *such that if* $\ddot{p}_0(t) < C(T)$, $0 \leqslant t \leqslant T$, *then there is no shock up to T. Here* $Q(t)$ *is given in* (2.2).

PROOF. – The proof of the necessary condition is obvious from (2.14) and (2.15).

As for the sufficient condition, from (2.3) (2.4) we have that if $\ddot{p}_0 < C$ then

$$0 < q_s' < \alpha C, \qquad \alpha = \sup_{q^* < q < Q(t)} \frac{1}{H'(q)}, \qquad 0 < t < T_0.$$

So clearly for C small enough condition (2.11) holds with $\delta = 1$. Let us denote by

$$\gamma(T) = \inf_{0<t<T,\, q^*<q<Q} -\frac{\varepsilon'\dot{s}}{\varepsilon'' s} = \frac{-\varepsilon'(Q)\, Q/\varepsilon(Q)}{\sup_{q^*<q<Q} \varepsilon''(q)\, Tq^*/\varepsilon(q^*)} . \tag{2.16}$$

We only need $C < \gamma/\alpha$. Then for $T_0 < t < T_1$, if $C < \gamma/\alpha$

$$q_0'(t) < \frac{L_0}{dt/d\tau} < L_0 \leqslant \alpha C$$

(since $dt/d\tau \geqslant 1$ by the choice of C).

Then from (2.13) we have

$$q_s'(t) < C(\alpha + K\alpha), \qquad T_0 < t < T_1, \tag{2.17}$$

where $K = \sup \dfrac{G'(q_0)}{H'(q_s)}$.

If $C\alpha(1+K) < \gamma$ then we have again that $\dfrac{d\theta}{d\tau} \geqslant 1$ for $T_0 < \tau < T_1$, hence

$$q_0'(t) \leqslant C\alpha(1+K)$$

for $T_1 < t < T_2$, where $T_2 = -\varepsilon'(q_s(T_1))\, s(T_1) + T_1$.

Let us define the sequence

$$T_{i+1} = -\varepsilon'(q_s(T_i))\, s(T_i) + T_i, \qquad i = 0, 1, \ldots \tag{2.18}$$

T_0 given in [3].

Repeating the previous argument after n steps we have

$$q_0'(t) < C\alpha\left(\sum_{j=0}^{n-2} K^j + K^{n-1}\right), \qquad T_{n-1} < t < T_n, \tag{2.19}$$

so that

$$q_s'(t) < C\alpha\left(\sum_{j=0}^{n-1} K^j + K^n\right) = C\alpha\left(\frac{1-K^n}{1-K} + K^n\right), \qquad T_{n-1} < t < T_n. \tag{2.20}$$

In Prop. 2.1 below we will prove that in our assumptions $K < 1$.

Then the generic inequalities which guarantees that there is no shock is satisfied if

$$q_s' < C\alpha\left(\frac{1-K^n}{1-K} + K^n\right) < C\alpha\left(\frac{1}{1-K} + 1\right) < \gamma . \tag{2.21}$$

So to prove the sufficient condition it is sufficient to take

$$C < \frac{\gamma}{\alpha\left(1 + \frac{1}{1-K}\right)} . \qquad \blacksquare \tag{2.22}$$

PROPOSITION 2.1. – *If* $\frac{d^2}{dz^2}\left(\frac{z}{k}\right) \geqslant 0$ *then*

$$K < \frac{1}{1+e_1 \sigma} < 1,$$

where $e_1 = |\varepsilon'(Q(T))|$, $\sigma = \frac{q^*}{\varepsilon(q^*)}$.

PROOF. – From (2.2), (2.5) we have

$$\frac{G'(q_0)}{H'(q_s)} = \frac{-\frac{1}{\varepsilon'(q_0)} \frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{q_0}}{-\frac{1}{\varepsilon'(q_s)}\left[\frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{q_s} - \frac{\varepsilon'(q_s)\,\dot{s}(t)}{k(q_s)}\right](1-\varepsilon'(q_s)\,\dot{s}(t))}. \tag{2.23}$$

We have that

$$\frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{q_s} = \frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{q_0} + \frac{d^2}{dz^2}\left(\frac{z}{k}\right)\Big|_{\hat{q}}(q_s - q_0), \tag{2.24}$$

with $\hat{q} \in (q_0, q_s)$.

So, assuming $\frac{d^2}{dz^2}\left(\frac{z}{k}\right) \geqslant 0$, and noting that $\frac{\varepsilon'(q_s)}{\varepsilon'(q_0)} < 1$, being $q_0 < q_s$, we obtain

$$\frac{G'(q_0)}{H'(q_s)} \leqslant \frac{1}{1-\varepsilon'(q_s)\,\dot{s}} \leqslant \frac{1}{1+e_1\sigma}. \qquad \blacksquare \tag{2.25}$$

REMARK 2.1. – Suppose that there has been no shocks up to time $\widehat{T}$. If there are not to be shocks for $\widehat{T} < t < T$, then (2.14) has to hold for $\widehat{T} < t < T$. Since $s(t)$ is increasing, this allows to improve the limiting constant in the necessary condition of Theorem 2.1.

REMARK 2.2. – The condition $\frac{d^2}{dz^2}\left(\frac{z}{k}\right) \geqslant 0$ is satisfied for slowly varying k, which is in agreement with fisical intuition. Infact

$$\frac{d^2}{dz^2}\left(\frac{z}{k}\right) = \frac{-zkk'' - 2kk' + 2zk'^2}{k^3}.$$

So it is nonnegative for $z \in [q^*, Q(T)]$ if k'' is small enough.

If $Q(T) \to \infty$ an example is $k = k_0 + cq^{-m}$ with $m < 1$.

Also the assumption on the «smallness» of the data is not unusual for this kind of hyperbolic problem (see e.g. [6]).

3. – Qualitative description of the solution after a shock, «Entropy» solution.

If $\dot{p}_0$ grows fast enough (see Sec. 2) then a shock starts at some time T_c.

In Appendix 1 we will give a qualitative description of the possible shocks.

According to Theorem 2.1 it is possible to have data allowing many shocks at different times. However, for the sake of simplicity, we assume that $p_0(t)$ is such that only one shock is formed in

$$\Sigma_c = (x_c, T_c), \qquad 0 < x_c < s(T_c), \qquad T_c > T_0.$$

This means that $q(x, T_c)$ is such that only one shock line, which we will denote by $x = \Sigma(t)$, exists locally after T_c. According to the classical theory of scalar conservation laws in one space variable (see e.g. [2])we have now to solve equation (1.2) in the class of piecewise continuous and smooth functions (see [6], [7]) which jump across the shock line. Such a solution $q(x, t)$ which we will call «Classical Entropy Solution» (C.E.S. for brevity hereafter) is well defined and unique provided that $\Sigma(t)$ is defined through the Rankine-Hugoniot jump condition:

$$\dot{\Sigma}(t) = \frac{q_l(t) - q_r(t)}{\varepsilon(q_l(t)) - \varepsilon(q_r(t))}, \qquad \Sigma(T_c) = x_c, \tag{3.1}$$

where the subscripts l and r denote the limit from the left and from the right respectively of $q(x, t)$ across $x = \Sigma(t)$.

Moreover the entropy condition must be satisfied, i.e. in our case the following

$$\varepsilon'(q_l) < \frac{1}{\dot{\Sigma}(t)} < \varepsilon'(q_r). \tag{3.2}$$

Therefore we will define locally after T_c a solution of the free boundary problem (1.1)-(1.5) in the following way

DEFINITION 3.1. – Given $h > 0$, a pair $(q(x, t), s(t))$ of piecewise C^1 functions is a solution of Problem (1.1)-(1.5) in $D_h^+ = \{0 < x < s(t),\ T_c < t < T_c + h\}$ if

i) $q(x, t)$ is a C.E.S. in D_h^+,

ii) $p_0(t) = \int_0^{s(t)} \frac{q(x, t)}{k(q(x, t))} dx$,

iii) (1.5) holds for $T_c < t < T_c + h$.

REMARK 3.1. – Of course a C.E.S. is also an Integral Entropy Solution in a more usual sense (see again [2]), obtained multiplying by a suitable test function and integrating. We think however that the previous definition (more in the style of [5], [7]) gives a better insight in the qualitative behaviour of the solution of (1.1)-(1.5) after T_c, which is our goal. We refer also to the work of [11], App.B, for a definition of weak or integral entropy solution of a free boundary problem. However let us remark that in our case we know a priori that the characteristics starting on the free boundary enter the domain D_h^+, so there are no problems in prescribing data there (which is a main question for general bounded domains as in [11]).

We can state the following

THEOREM 3.1. – *Let $p_0(t)$ be such that only one shock is formed in Σ_c. Then a unique solution in the sense of Def.* 3.1 *exists up to time T_k when $\Sigma(t)$ reaches $x = 0$.*

PROOF. – We remark that up to T_c there exists a unique classical solution of our problem, so that $s(t)$ and q_s are known up to T_c, $q_0(t)$ is known almost up to a time, say θ_c, that is greater than T_c, and both q_0 and q_s are increasing with t. As a consequence the functions $q_l(t)$ and $q_r(t)$ in (3.1), given by

$$\begin{cases} q_l(t) = q_s(\tau_l), \ t = -\varepsilon'(q_s(\tau_l))(\Sigma(t) - s(\tau_l)) + \tau_l, \\ q_r(t) = q_s(\tau_r), \ t = -\varepsilon'(q_s(\tau_r))(\Sigma(t) - s(\tau_r)) + \tau_r, \end{cases} \tag{3.3}$$

are known up to time $t_c > T_c$, given by $t_c = -\varepsilon'(q_s(T_c))(\Sigma(t_c) - s(T_c)) + T_c$, i.e. the time in which the characteristic starting from $s(T_c)$, T_c) reaches Σ. $\Sigma(t)$ on its turn is given by (3.1) and is a decreasing function because $q_l < q_r$. Moreover the entropy condition (3.2) is satisfied for the same reason.

Of course after T_c we have to change the relation between q_0 and q_s. Proceeding in the same way as for (2.12) and looking for a solution with only one shock we arrive at the following:

$$H(q_s) = \dot{p}_0(t) + G(q_r) - G(q_l) + G(q_0) - \dot{\Sigma}(t)\left[\left(\frac{q}{k}\right)_l - \left(\frac{q}{k}\right)_r\right]. \tag{3.4}$$

Although more complicated than (2.12), (3.3) defines uniquely $q_s(t)$ in term of $q_0(t)$ and $\dot{p}_0(t)$ locally after T_c, since everything is known there but $q_s(t)$. The free boundary $x = s(t)$ is then defined by (1.5) and it is C^1 in T_c since $q(x, t)$ is continuous in $(s(T_c), T_c)$ (see (3.3)). Moreover, since ε is a convex C^2 function (see [2] Thm 3 Sect. 3.4) equation (1.2) can be solved classically along the characteristics starting on $x = s(t)$ locally after T_c, provided no new shock lines start in the domain of influence of the free boundary data.

The situation does not change as long as $\Sigma(t) > 0$, but due to the bounds on q (see (2.1),(2.2)) and to the entropy condition (3.2) the shock line will reach the fixed boundary $x = 0$ in a finite time T_k, which can be estimated. ■

REMARK 3.2. – As we showed in [1] through an example, no new shock line starts if $\dot{p}_0(t)$ is slowly varying after T_c. One could precise the condition on the pressure needed to ensure that no new shocks appear after T_c, following a method similar to the one of Section 2 (with (2.12) substituted by (3.4)), but it would be too technical.

We have seen that between T_c and T_k we have a solution which is classical for $0 < x < \Sigma(t)$ and $\Sigma(t) < x < s(t)$ and has a jump on Σ. Also the porosity has a jump at $\Sigma(t)$ which implies a local collapse of the medium there.

At time T_k the discontinuity of q along Σ becomes a discontinuity of q_0. Then istantaneously we must have a discontinuity of q_s. In fact if we assume that q_s is continuous in T_k, then (2.12) should hold for $t > T_k$. Then

$$H(q_s(T_k)) = \dot{p}_0(T_k) + G(q_0(T_k^+)).$$

On the other hand from (3.4) (for $t < T_k$) we have

$$H(q_s(T_k)) = \dot{p}_0(T_k) + G(q_0(T_k^+)) - \dot{\Sigma}(T_k)\left[\frac{q_0(T_k^-)}{K(q_0(T_k^-))} - \frac{q_0(T_k^+)}{K(q_0(T_k^+))}\right]$$

and this gives a contradiction since the last term in the above equality is different from zero $\left(\text{since } \dot{\Sigma}(T_k) = \dfrac{q_0(T_k^-) - q_0(T_k^+)}{\varepsilon(q_0(T_k^-)) - \varepsilon(q_0(T_k^+))},\ q_0(T_k^+) > q_0(T_k^-)\right)$.

Since q_s has to be discontinuous in T_k, a new shock line (which we will denote again with $x = \Sigma(t)$) has to form in $(s(T_k), T_k))$. Even if $\dot{p}_0$ is slowly varying so to have a solution with only one shock after T_k, we cannot repeat now the argument used in Theorem 3.1 because $q_r(t)$ is no more given by $q_s(\tau)$ for $\tau < T_k$. A similar problem appeared in the case of $\dot{p}_0$ piecewise constant (see [1]). It was shown there that after T_k a solution with one shock exists up to the time when the new shock line reaches the fixed boundary and then one has a new jump in q_s. Hence in that case the unique jump in $\dot{p}_0$ trigges an infinite sequence of solutions with one shock. A similar behaviour happens in the present more general case; a unique shock in T_c generates a global in time solution with a sequence of shocks (see fig. 1). The main point is to prove the local existence after T_k of a solution in the sense of Def. 3.1.

To give a better idea of the proof in the next section we will simulate a similar situation imposing a jump of $\dot{p}_0$ in T_k. Infact this gives an instantaneous jump in q_s (with $q_s(T_k^+) > q_s(T_k^-)$) and a fixed point argument is needed to show that the solution can be continued after T_k.

This case has its own interest since it considers the case of a discontinuous $\dot{p}_0$, thus generalizing the «explicit» example given in [1]. In Section 5 we will return to the case of a smooth $\dot{p}_0$ and prove local existence after T_k with a similar method as the one of Sect. 4 and with suitable assumptions on the data. Let us remark here that the free boundary $x = s(t)$ will be globally piecewise C^1.

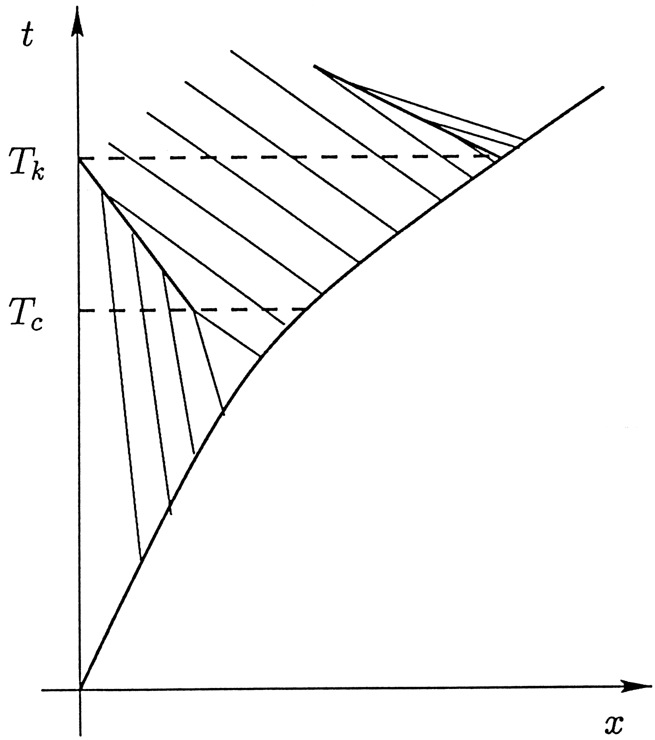

Figure 1. – Propagation of the shock.

4. – Fixed point.

We assume in this section that k is slowly varying so that $\frac{d^2}{dz^2}\left(\frac{z}{k}\right) \geqslant 0$ (see Remark 2.2).

As for $p_0(t)$ for the sake of simplicity we assume

$$\dot{p}_0(t) = \begin{cases} \dot{p}_0^-, & t < T_k, \\ \dot{p}_0^+(t), & t > T_k, \end{cases} \tag{4.1}$$

where $\dot{p}_0^+(t)$ is increasing, $\dot{p}_0^-$ is a given constant. and $\dot{p}_0^+(T_k) > \dot{p}_0^-(T_k)$.

This implies that for $t < T_k$ q is the solution of the classical Green-Ampt model, so that

$$q = F^{-1}(\dot{p}_0^-) = \overline{q}_0 = const$$

(F defined in (2.2)).

Then we have a shock line starting from $(s(T_k), T_k)$, infact we can determine $q_s(T_k^+) = q_s^+$ from (3.4) which in this case becomes

$$\dot{p}_0^+(T_k) = F(q_s^+) + f(q_s^+) \tag{4.2}$$

where

$$f(z) = \frac{\overline{q}_0 - z}{\varepsilon(\overline{q}_0) - \varepsilon(z)}\left(\frac{\overline{q}_0}{k(\overline{q}_0)} - \frac{z}{k(z)}\right). \tag{4.3}$$

From Lemma 2.1 of [1] we have that there exists a unique q_s^+ satisfying (4.2) and

$$\overline{q}_0 < q_s^+ < F^{-1}(\dot{p}_0^+(T_k)) = Q(T_k). \tag{4.4}$$

We will prove the following

THEOREM 4.1. – *Under assumptions* (1.6), (1.7) *and* (4.1) *there exists a unique solution in the sense of Def.* 3.1 *for the problem* (1.1)-(1.5) *for t in a small interval after* T_k.

PROOF. – We use a fixed point argument. We proceed in a different way than in [3], since we assume $q_s(t)$ instead of $q_0(t)$ given. This seems to us more close to the physical situation.

For a fixed $T > 0$ we introduce the set

$$X(T, L_1) = \Big\{ q_s(t) \in C(T_k, T_k + T),\ q_s(T_k) = q_s^+,$$

$$0 < \overline{q}_0 < q_s < Q(t),\ 0 < \frac{q_s(t_2) - q_s(t_1)}{t_2 - t_1} \leqslant L_1 \ \text{ for } \ t_2 > t_1 \Big\}$$

where T, L_1 are to be determined later and

$$\overline{q}_0 = F^{-1}(\dot{p}_0^-), \qquad Q(t) = F^{-1}(\dot{p}_0^+(t)), \tag{4.5}$$

and q_s^+ defined in (4.2).

For a q_s selected in $X(T, L_1)$ we find $s(t)$ as

$$s(t) = s(T_k) + \int_{T_k}^{t} \frac{q_s(\tau)}{\varepsilon(q_s(\tau))}\, d\tau, \qquad T_k < t < T_k + T. \tag{4.6}$$

Then we solve the first order equation in the domain $0 < x < s(t)$, $t > T_k$ with data q_s on $x = s(t)$ and $\overline{q}_0$ on $t = T_k$, $0 < x < s(T_k)$. This problem has a unique entropy solution (see previous section)

$$(4.7) \qquad q = \begin{cases} \bar{q}_0, & 0 \leqslant x < \Sigma(t), \\ q_s(\tau), & \Sigma(t) < x < s(t), \qquad T_k < t < T + T_k, \end{cases}$$

where τ is given by

$$t = \varepsilon'(q_s(\tau))(x - s(\tau)) + \tau ,$$

and $\Sigma(t)$ is solution of the following equation:

$$(4.8) \qquad \dot{\Sigma}(t) = \frac{\bar{q}_0 - q_r(t)}{\varepsilon(\bar{q}_0) - \varepsilon(q_r(t))}, \qquad \Sigma(T_k) = s(T_k),$$

$$(4.9) \qquad q_r(t) = q_s(\tau)$$

where

$$(4.10) \qquad t = \varepsilon'(q_s(\tau))(\Sigma(t) - s(\tau)) + \tau .$$

Now we define $\tilde{q}_s(t)$ by solving the equation

$$(4.11) \qquad H(\tilde{q}_s(t)) = \dot{p}_0(t) + G(q_r(t)) - f(q_r(t))$$

with G, H defined in (2.5), f in (4.3). The relation (4.11) is simply relation (3.4) in the present case ($\bar{q}_0 = q_l$).

Then (4.11) defines the mapping

$$\mathcal{C}q_s = \tilde{q}_s, \qquad \mathcal{C} : X \to C(T_k, T_k + T).$$

We want to prove that for suitable T and L_1, $\mathcal{C}$ is a contraction.

Then we we have first to prove the claim

$$(4.12) \qquad \bar{q}_0 < \tilde{q}_s < Q(t).$$

From (4.11) and from (4.5) we have

$$F(\tilde{q}_s) + G(\tilde{q}_s) \leqslant F(Q(t)) + G(q_r(t)) < F(Q(t)) + G(Q(t))$$

hence $\tilde{q}_s < Q(t)$.

The lower estimate comes from the fact that, from (4.11) at $t = T_k$, we have $\tilde{q}_s(T_k) = q_s^+$ and from the estimate $\tilde{q}_s' \geqslant 0$, that we are going to prove.

Let us prove that

$$(4.13) \qquad 0 \leqslant \tilde{q}_s' \leqslant L_1,$$

for suitable L_1, T To get the lower estimate formally we differentiate (4.11) obtaining

$$H'(\tilde{q}_s)\,\tilde{q}_s' = \ddot{p}_0(t) + (G' - f')(q_r(t))\,q_r' \tag{4.14}$$

From (2.5) and (4.3) we obtain

$$(G'-f')(z) = \left(1 - \frac{\varepsilon'(z)}{\varepsilon'(\bar{z})}\right)\left(-\frac{1}{\varepsilon'(z)}\frac{d}{dz}\left(\frac{z}{k}\right)\right)\left[1 - \frac{\varepsilon'(z)}{\varepsilon'(\bar{z})}\frac{\left.\frac{d}{dz}\left(\frac{z}{k}\right)\right|_{\hat{z}}}{\frac{d}{dz}\left(\frac{z}{k}\right)}\right], \tag{4.15}$$

with $\bar{z}, \hat{z}$ suitable values in $(\bar{q}_0, z)$, $z > \bar{q}_0$.

Since $\bar{z} < \hat{z}$ and ε' increasing, we have

$$0 < \frac{\varepsilon'(z)}{\varepsilon'(\bar{z})} < 1.$$

Moreover, since $\frac{d^2}{dz^2}\left(\frac{z}{k}\right) > 0$ and $\hat{z} < z$

$$0 < \frac{\left.\frac{d}{dz}\left(\frac{z}{k}\right)\right|_{\hat{z}}}{\frac{d}{dz}\left(\frac{z}{k}\right)} < 1.$$

Then

$$0 < (G' - f')(z) < -\frac{1}{\varepsilon'(z)}\frac{d}{dz}\left(\frac{z}{k}\right) = G'(z). \tag{4.16}$$

Recalling that $q_r(t) = q_s(\tau)$ (see (4.9) (4.10)) we have

$$q_r'(t) = \frac{q_s'(\tau)}{\frac{dt}{d\tau}} \tag{4.17}$$

where

$$\frac{dt}{d\tau} = \frac{1 - \varepsilon'(q_s(\tau)\,\dot{s}(\tau) + \varepsilon''(q_s(\tau))\,q_s'(\tau)(\Sigma(t) - s(\tau))}{1 - \varepsilon'(q_s(\tau))\,\dot{\Sigma}(t)}. \tag{4.18}$$

We remark that

$$0 < 1 - \varepsilon'(q_s(\tau))\,\dot{\Sigma}(t) = 1 - \frac{\varepsilon'(q_s(\tau)}{\varepsilon'(\bar{q})} < 1, \tag{4.19}$$

with $\bar{q} \in (\bar{q}_0, q_s(\tau))$, and that

$$-\varepsilon'(q_s(\tau))\,\dot{s}(\tau) + \varepsilon''(q_s(\tau))\,q_s'(\tau)(\Sigma(t) - s(\tau)) > 0,$$

for T sufficiently small such that

$$T < \frac{|\varepsilon'(\overline{Q})|\,\bar{q}_0/\varepsilon(\bar{q}_0)}{E_2\left(\frac{\overline{Q}}{\varepsilon(\overline{Q})} + \frac{1}{|\varepsilon'(\overline{Q})|}\right)L_1}, \tag{4.20}$$

where $E_2 = \sup_q \varepsilon''(q)$, and $\overline{Q} = \sup_{T_k < t < 2T_k} Q(t)$.

For T satisfaying (4.20) we have that (see (4.18) and (4.19))

$$\frac{dt}{d\tau} \geqslant 1\,. \tag{4.21}$$

Therefore $q_r' \geqslant 0$ and consequently from (4.14) (remarking that $H' \geqslant 0$) we have $\tilde{q}_s' \geqslant 0$.

Let us prove the upper estimate of the claim, that is $\tilde{q}_s' \leqslant L_1$. From (4.21) and (4.17) we have that

$$q_r' \leqslant \frac{L_1}{\frac{dt}{d\tau}} < L_1.$$

Taking into account (4.16), in order to estimate $\tilde{q}_s'$ we need to estimate the following

$$0 < \frac{(G' - f')(q_r)}{H'(\tilde{q}_s)}\,q_r' < \frac{-1/\varepsilon'(q_r)\,\frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{q_r}}{-\frac{(1-\varepsilon'(\tilde{q}_s)\,\dot{s})}{\varepsilon'(\tilde{q}_s)}\left[-\frac{\varepsilon'(\tilde{q}_s)\,\dot{s}}{k} + \frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{\tilde{q}_s}\right]}\,L_1. \tag{4.22}$$

We can express the quantities in (4.22) as

$$\varepsilon'(\tilde{q}_s) = \varepsilon'(q_r) + \varepsilon''(\hat{q})(\tilde{q}_s - q_r).$$

We have that, if $\tilde{q}_s > q_r$, then $|\varepsilon'(\tilde{q}_s)| < |\varepsilon'(q_r)|$; if $\tilde{q}_s < q_r$, then $|\tilde{q}_s - q_r| < (q_r - \bar{q}_0) \leqslant L_1 T$.

Henceforth in any case

$$|\varepsilon'(\tilde{q}_s)| \leqslant |\varepsilon'(q_r)| + E_2 L_1 T. \tag{4.23}$$

Moreover

$$\frac{d}{dz}\left(\frac{z}{k}\right)\bigg|_{q_r} = \frac{d}{dz}\left(\frac{z}{k}\right)\bigg|_{\tilde{q}_s} + \frac{d^2}{dz^2}\left(\frac{z}{k}\right)\bigg|_{\hat{q}}(q_r - \tilde{q}_s),$$

where $\hat{q}$ stays between q_r and $\tilde{q}_s$.

Proceeding as above we have

$$\frac{d}{dz}\left(\frac{z}{k}\right)\bigg|_{q_r} \leqslant \frac{d}{dz}\left(\frac{z}{k}\right)\bigg|_{\tilde{q}_s} + D_2 L_1 T \tag{4.24}$$

with $D_2 = \sup\limits_q \frac{d^2}{dq^2}\left(\frac{q}{k(q)}\right)$.

Substituting in (4.22) we have

$$\frac{(G' - f')(q_r)\, q_r'}{H'(\tilde{q}_s)} \leqslant C_2 L_1 (1 + C_3 L_1 T)(1 + C_4 L_1 T) \tag{4.25}$$

where

$$C_2 = \frac{1}{1 - \varepsilon'(\overline{Q})\, \bar{q}_0/\varepsilon(\bar{q}_0)} < 1, \quad C_3 = \frac{E_2}{|\varepsilon'(\overline{Q})|}, \quad C_4 = \frac{D_2}{\frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{\bar{q}_0}}. \tag{4.26}$$

For T sufficiently small, since $C_2 < 1$, we have still

$$C_2(1 + C_3 L_1 T)(1 + C_4 L_1 T) < 1. \tag{4.27}$$

Eventually the claim is proved choosing L_1 so that

$$0 < \sup \frac{\ddot{p}_0}{H'} \leqslant L_1[1 - C_2(1 + C_3 L_1 T)(1 + C_4 L_1 T)] \tag{4.28}$$

which is possible for T satisfying (4.27).

The two Claims prove that $\mathcal{T}$ maps the closed, convex and compact set X into itself.

In order to prove the continuity of $\mathcal{T}$ in the chosen topology, we consider two elements q_s^1, q_s^2 in X, and we estimate, for any $t < T$ (shifting the origin of time to T_k), the difference

$$\tilde{q}_s^1 - \tilde{q}_s^2 = H^{-1}(\dot{p}_0 + G(q_r^1) - f(q_r^1)) - H^{-1}(\dot{p}_0 + G(q_r^2) - f(q_r^2)), \tag{4.29}$$

which is in turn estimated in terms of the difference

$$q_r^1 - q_r^2 = q_s^1(\tau_1) - q_s^2(\tau_2), \tag{4.30}$$

with

$$\tau_i = t - \varepsilon'(q_r^i)(\Sigma_i(t) - S_i(t)) \tag{4.31}$$

and $S_i(t) = s_i(\tau_i(t))$, $i = 1, 2$.

From (4.31) we have

$$\tau_1 - \tau_2 = \varepsilon'(q_r^2)(\Sigma_2 - S_2) - \varepsilon'(q_r^1)(\Sigma_1 - S_1) = \tag{4.32}$$

$$= \varepsilon''(\hat{q})(q_r^2 - q_r^1)(\Sigma_2 - S_2) + \varepsilon'(q_r^1)[(\Sigma_2 - \Sigma_1) + (S_1 - S_2)],$$

where

$$(\Sigma_2 - S_2)(t) = (\dot{\Sigma}_2 - \dot{S}_2)(\xi)\, t < 0, \tag{4.33}$$

$$(\Sigma_2 - \Sigma_1)(t) = (\dot{\Sigma}_2 - \dot{\Sigma}_1)(\overline{\xi})\, t, \tag{4.34}$$

$$(S_1 - S_2)(t) = \dot{s}_1(\overline{\tau})(\tau_1 - \tau_2) + \int_0^{\tau_2} (\dot{s}_1 - \dot{s}_2)(\tau)\, d\tau, \tag{4.35}$$

with

$$\frac{\overline{q}_0}{\varepsilon(\overline{q}_0)} < \dot{S}_i < \frac{\overline{Q}}{\varepsilon(\overline{Q})}, \qquad \frac{1}{\varepsilon'(\overline{Q})} < \dot{\Sigma}_i < \frac{1}{\varepsilon'(\overline{q}_0)}, \quad i = 1, 2.$$

Now, recalling the expression of $\dot{\Sigma}_i(t)$, we can estimate the difference $\dot{\Sigma}_2 - \dot{\Sigma}_1$ in (4.34) in terms of $q_r^1 - q_r^2$, obtaining

$$\dot{\Sigma}_1 - \dot{\Sigma}_2 = \frac{(\overline{q}_0 - q_r^1)\, \varepsilon'(\overline{q}) - (\varepsilon(\overline{q}_0) - \varepsilon(q_r^1))}{(\varepsilon(\overline{q}_0) - \varepsilon(q_r^1))(\varepsilon(\overline{q}_0) - \varepsilon(q_r^2))} (q_r^1 - q_r^2). \tag{4.36}$$

From the above estimates, (4.32) gives

$$|\tau_1 - \tau_2| \leqslant \frac{1}{1 - \varepsilon'(q_r^1)\, \dot{s}_1(\overline{\tau})} [C_5 T \| q_r^1 - q_r^2 \| + C_6 T \| q_s^1 - q_s^2 \|] \tag{4.37}$$

with

$$C_5 = \sup_q \varepsilon''(q) \left[\frac{\overline{Q}}{\varepsilon(\overline{Q})} + \frac{1}{|\varepsilon'(\overline{Q})|} + \frac{\overline{Q} - \overline{q}_0}{|\varepsilon'(\overline{Q})|^2 (q_s^+ - \overline{q}_0)} \right],$$

and

$$C_6 = |\varepsilon'(\overline{q}_0)| \frac{|\varepsilon'(\overline{q}_0)|\, \overline{Q} + \varepsilon(\overline{q}_0)}{[\varepsilon(\overline{Q})]^2}.$$

Remarking that (4.30) can be espressed as

$$q_r^1 - q_r^2 = (q_s^1)'(\overline{\tau})(\tau_1 - \tau_2) + (q_s^1 - q_s^2)(\tau_2),$$

we have from (4.37)

$$\|q_r^1 - q_r^2\| \leqslant \frac{1 + C_7 T}{1 - C_8 T} \|q_s^1 - q_s^2\| \tag{4.38}$$

where

$$C_7 = \frac{L_1 C_6}{1 + |\varepsilon'(\overline{Q})| \dfrac{\overline{Q}}{\varepsilon(\overline{Q})}}, \qquad C_8 = \frac{L_1 C_5}{1 + |\varepsilon'(\overline{Q})| \dfrac{\overline{Q}}{\varepsilon(\overline{Q})}},$$

under the assumption

$$T < \frac{1}{C_8}, \tag{4.39}$$

that guarantees the positivity of the constant in (4.38).

Coming back to (4.29) we have

$$\|\tilde{q}_s^1 - \tilde{q}_s^2\| \leqslant \sup_q \frac{G' - f'}{H'} \|q_r^1 - q_r^2\|, \tag{4.40}$$

which, recalling (4.25) and (4.38), finally gives

$$\|\tilde{q}_s^1 - \tilde{q}_s^2\| \leqslant C_9 \frac{1 + C_7 T}{1 - C_8 T} \|q_s^1 - q_s^2\|, \tag{4.41}$$

with $C_9 < 1$ given by (4.29).

From (4.41) we have that the operator $\mathcal{C}$ is a contraction if

$$T < \frac{1 - C_9}{C_9(C_7 + C_8)}. \qquad \blacksquare \tag{4.42}$$

5. – Generalization.

In this section we want to extend the results of existence and uniqueness of the solution proved in previous section (see Thm 4.1) to the case of a shock line starting at $(s(T_k), T_k))$, due to a discontinuity of $q_0(T_k)$.

We assume that $p_0(t)$ is continuous and such that only one shock is formed in Σ_c and that the shock line reaches $x = 0$ at $t = T_k$, so that the discontinuity of q becomes a discontinuity of $q_0(T_k)$ and instantaneously a discontinuity of $q_s(T_k)$.

Let us assume moreover that

$$0 < q_x(x, T_k) < L_0 \tag{5.1}$$

with L_0 sufficiently small.

We remark that this hypothesis is consistent with the assumption we have done that only one shock is present between T_c and T_k, see Sect. 2.

We start determining q_s^+ that in this case is the unique solution of

$$\begin{cases} \dot{p}_0(T_k) = F(q_s^+) - G(q_0^+) + G(q_s^-) + \mathcal{F}(q_s^-, q_s^+), \\ \dot{p}_0(T_k) = H(q_s^-) - G(q_0^+) + \mathcal{F}(q_0^-, q_0^+), \end{cases} \tag{5.2}$$

with

$$\mathcal{F}(q^-, q^+) = \frac{q^- - q^+}{\varepsilon(q^-) - \varepsilon(q^+)}\left[\left(\frac{q}{k}\right)^- - \left(\frac{q}{k}\right)^+\right]. \tag{5.3}$$

Equations (5.2) have been obtained from (1.5) at T_k^+ and T_k^- respectively.

The proof of the theorem of existence and uniqueness of the solution of problem (1.1)-(1.5) can be obtained following a fixed point argument as in previous section remarking that the mapping $\mathcal{C} q_s = \tilde{q}_s$ is defined now by

$$H(\tilde{q}_s(t)) = \dot{p}_0(t) + G(q_0) - G(q_l) + G(q_r) - \mathcal{F}(q_l, q_r). \tag{5.4}$$

Remarking that $G(q_0) - G(q_l) < 0$ because of the assumption (5.1) and $G' > 0$, and that $\mathcal{F}(q_l, q_r) > 0$, from (5.4) we immediately obtain

$$\tilde{q}_s < Q(t) = F^{-1}(\dot{p}_0(t)).$$

Differentiating (5.4) we have

$$\tilde{q}_s' = \frac{1}{H'(\tilde{q}_s)}\left[\ddot{p}_0 + G'(q_0)\, q_0' + \left(G'(q_r) - \frac{\partial \mathcal{F}}{\partial q_r}\right) q_r' - \left(G'(q_l) - \frac{\partial \mathcal{F}}{\partial q_l}\right) q_l'\right]. \tag{5.5}$$

Then we need an estimate of the last term in (5.5). We have

$$\left(G' + \frac{\partial \mathcal{F}}{\partial q_l}\right)(q_l) = -\frac{\varepsilon'(q_l)}{[\varepsilon'(\overline{q})]^2}\, \frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{\hat{q}} \left(1 - \frac{\varepsilon'(\overline{q})}{\varepsilon'(q_l)}\right)\left(1 - \frac{\varepsilon'(\overline{q})}{\varepsilon'(q_l)}\, \frac{\frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{q_l}}{\frac{d}{dz}\left(\frac{z}{k}\right)\Big|_{\hat{q}}}\right), \tag{5.6}$$

with $\overline{q}, \hat{q}$ suitable values in (q_l, q_r).

From the hypotheses on ε and on k we have, recalling the expression of $H'(\tilde{q}_s)$

$$(5.7)\qquad 0<\frac{\left(G'+\dfrac{\partial\mathcal{F}}{\partial q_l}\right)(q_l)}{H'(\tilde{q}_s)}<\frac{-\dfrac{\varepsilon'(q_l)}{[\varepsilon'(\overline{q})]^2}\dfrac{d}{dz}\left(\dfrac{z}{k}\right)\Big|_{\overline{\overline{q}}}}{-\dfrac{1-\varepsilon'(\tilde{q}_s)\,\dot{s}}{\varepsilon'(\tilde{q}_s)}\left[\dfrac{d}{dz}\left(\dfrac{z}{k}\right)\Big|_{\tilde{q}_s}-\dfrac{\varepsilon'(\tilde{q}_s)\,\dot{s}}{k(\tilde{q}_s)}\right]}\leqslant$$

$$[1+C_3(\overline{Q}-q_0)](1+C_3L_1T)\,C_2(1+C_4L_1T)=C_9[1+C_3(\overline{Q}-q_0)],$$

with C_2, C_3, C_4 defined in (4.26), q_0 denoting $q(0, T_k^+)$, and $C_9<1$ (see 4.27).

Let us remark that the coefficients of q_0' and q_l' are bounded and q_0' and q_l' are bounded by L_0, so in order to ensure $q_s'\geqslant 0$, we assume L_0 to be such that

$$L_0\leqslant\inf_t\frac{\ddot{p}_0}{C_9[1+C_3(\overline{Q}-q_0)]}\,.$$

Then we can choose L_1 such that

$$(5.8)\quad 0<\frac{1}{H'(\tilde{q}_s)}\left[\ddot{p}_0+G'(q_0)\,q_0'-\left(G'(q_l)-\frac{\partial\mathcal{F}}{\partial q_l}\right)q_l'\right]\leqslant L_1(1-C_9),$$

and consequently, remarking that the term $\tilde{q}_s\dfrac{1}{H'(\tilde{q}_s)}\left(G'(q_r)-\dfrac{\partial\mathcal{F}}{\partial q_r}\right)q_r'$ satisfies again (4.25), we have for T sufficiently small

$$0\leqslant\tilde{q}_s'\leqslant L_1.$$

The proof of the continuity of the operator $\mathcal{C}$ follows as before estimating the difference $\tilde{q}_s^1-\tilde{q}_s^2$, that now implies, besides the term $G(q_r^1)-G(q_r^2)$, the estimates of $G(q_l^1)-G(q_l^2)$ and $\mathcal{F}(q_l^1, q_r^1)-\mathcal{F}(q_l^2, q_r^2)$.

Proceeding as in the previous section we obtain

$$(5.9)\qquad q_l^1-q_l^2=q(\eta_1,0)-q(\eta_2,0),$$

with η_i given by

$$(5.10)\qquad t=\varepsilon'(q(\eta_i,0))(\Sigma_i(t)-\eta_i).$$

Using the above definition of η_i we have

$$(5.11)\qquad \eta_2-\eta_1=\frac{\varepsilon'(q(\eta_1,0)}{\varepsilon''(\overline{q})(\Sigma_2-\eta_2)-\varepsilon'(q(\eta_1,0))}(\dot{\Sigma}_2-\dot{\Sigma}_1)\,t,$$

where the difference $\dot{\Sigma}_2 - \dot{\Sigma}_1$ can be estimated as in (4.36)

$$\dot{\Sigma}_1 - \dot{\Sigma}_2 = \frac{(q_l^1 - q_l^2)[(\varepsilon(q_l^2) - \varepsilon(q_r^2)) - (\varepsilon'(\overline{q}_l)(q_l^2 - q_r^2)]}{(\varepsilon(q_l^1) - \varepsilon(q_r^1))(\varepsilon(q_l^2) - \varepsilon(q_r^2))} - \frac{(q_r^1 - q_r^2)[(\varepsilon(q_l^2) - \varepsilon(q_r^2)) - (\varepsilon'(\overline{q}_r)(q_l^2 - q_r^2)]}{(\varepsilon(q_l^1) - \varepsilon(q_r^1))(\varepsilon(q_l^2) - \varepsilon(q_r^2))}. \tag{5.12}$$

From (5.9) we obtain then

$$\|q_l^1 - q_l^2\| \leqslant \frac{C_{10}}{1 - C_{11} T} \|q_r^1 - q_r^2\| \, T, \tag{5.13}$$

where C_{10} and C_{11} can be computed using (5.11), (5.12).

We remark that C_{11} is multiplied by T, so that estimate (6.13) makes sense assumed

$$T < \frac{1}{C_{11}}. \tag{5.14}$$

Now let us consider the difference

$$\mathcal{F}(q_l^1, q_r^1) - \mathcal{F}(q_l^2, q_r^2) = (\dot{\Sigma}_1 - \dot{\Sigma}_2)\left[\frac{q_l^1}{k(q_l^1)} - \frac{q_r^1}{k(q_r^1)}\right] + \dot{\Sigma}_2\left[\frac{d}{dz}\left(\frac{q}{k}\right)\bigg|_{\overline{q}_l}(q_l^1 - q_l^2) - \frac{d}{dz}\left(\frac{q}{k}\right)\bigg|_{\overline{q}_r}(q_r^1 - q_r^2)\right], \tag{5.15}$$

where again $\dot{\Sigma}_1 - \dot{\Sigma}_2$ is given by (5.12), so that

$$|\mathcal{F}(q_l^1, q_r^1) - \mathcal{F}(q_l^2, q_r^2)| \leqslant C_{12}|q_l^1 - q_l^2| + C_{13}|q_r^1 - q_r^2|, \tag{5.16}$$

where C_{12} and C_{13} are obtained from (5.12) and (5.15).

We remark also that the difference $q_r^1 - q_r^2$ can be expressed in terms of $q_s^1 - q_s^2$ as we did in Sect. 4, recalling that the difference $\tau_1 - \tau_2$ now depends on $q_l^1 - q_l^2$ too, that in its turn is estimated by $\|q_r^1 - q_r^2\|$ multiplied by T, so that the constants C_7, C_9 in (4.38) have to be modified.

Finally we have

$$\|\tilde{q}_s^1 - \tilde{q}_s^2\| \leqslant \overline{C}_9 \frac{1 + \overline{C}_7 T}{1 - \overline{C}_8 T}\left(1 + \frac{C_{10} T}{1 - C_{11} T}\right)\|q_s^1 - q_s^2\|, \tag{5.17}$$

where

$$\overline{C}_9 = C_9\left[1+\frac{1+C_3(\overline{Q}-q_0)}{1-C_{11}T}C_{10}T\right]<1,$$

and $\overline{Q}$ defined as in the previous section.

This proves the contractivity of the operator $\mathcal{T}$ for T sufficiently small and hence the existence of a solution in the sense of Def. 3.1 locally after T_k.

A.1. – Qualitative description of the possible shocks.

If $\dot{p}_0$ grows fast enough (see Sect. 2) then a shock starts at the point Σ_c given by

$$\Sigma_c = (x_c, T_c),\ 0 < x_c < s(T_c),\ T_c > 0, \tag{A.1}$$

where $x = x_c$, $t = T_c$ is the cusp of the envelope of the characteristics obtained assuming given $q_s(t)$ on $s(t)$, for $t = \tau$.

The parametric equations of the envelope are (see classical theory [2])

$$\begin{cases} x = s(\tau) + \left.\dfrac{\varepsilon' q_s - \varepsilon}{\varepsilon\varepsilon'' q_s'}\right|_\tau \\ t = \varepsilon' \left.\dfrac{\varepsilon' q_s - \varepsilon}{\varepsilon\varepsilon'' q_s'}\right|_\tau + \tau \end{cases} \tag{A.2}$$

Differentiating (A.2) we have

$$\begin{cases} \dfrac{dx}{d\tau} = \left.\dfrac{2q_s}{\varepsilon}\right|_\tau + \dfrac{\varepsilon - \varepsilon' q_s}{(\varepsilon\varepsilon'' q_s')^2}(\varepsilon\varepsilon'' q_s')'\,|_\tau, \\ \dfrac{dt}{d\tau} = \varepsilon' \left.\dfrac{dx}{d\tau}\right|_\tau. \end{cases} \tag{A.3}$$

From (A.3) we have that the envelope has either two branches, if $\frac{dx}{d\tau}$ and consequently $\frac{dt}{d\tau}$ becomes zero at the cusp, or one branch if $\frac{dx}{d\tau}$ does not vanish. Moreover being $\frac{dt}{dx} = \varepsilon' < 0$ the branches are decreasing and, since

$$\frac{d^2 t}{dx^2} = \varepsilon'' q_s' \frac{d\tau}{dx}, \tag{A.4}$$

the concavity depends on the sign of $\frac{dx}{d\tau}$.

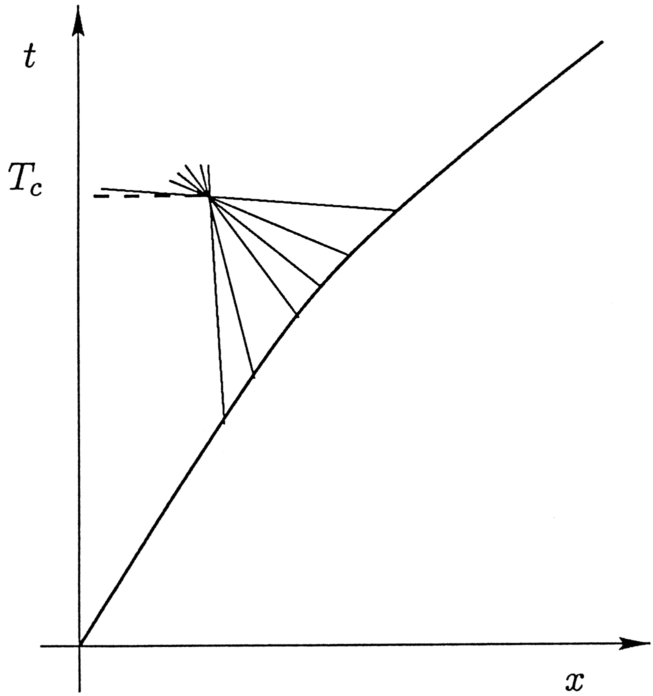

Figure 2. – Instantaneous shock.

REMARK 5.1. – Let us remark that the condition (2.11) given in Sect. 2 is equivalent to have $x < 0$ in (A.2) so that the envelope lies outside our domain. From (A.2) we have that anyhow $x < s(\tau)$ and $t > \tau$.

Let us consider a finite interval (τ_1, τ_2) with $\tau_1 > \tau_0$, defined in Sect. 2, and assume that the characteristics in this interval form a shock.

We can divide the possible shocks in four different classes, according with [9]:

1) instantaneous shock: all characteristics meet in Σ_c,

2) «fast» shock: $\frac{dx}{d\tau} < 0$, Σ_c belongs to the characteristic starting from $(s(\tau_1), \tau_1)$,

3) «slow» shock: $\frac{dx}{d\tau} > 0$, Σ_c belongs to the characteristic starting from $(s(\tau_2), \tau_2)$,

4) shock with two branches: x_c is the maximum of $x(\tau)$ and T_c is the minimum of $t(\tau)$ (see A.2).

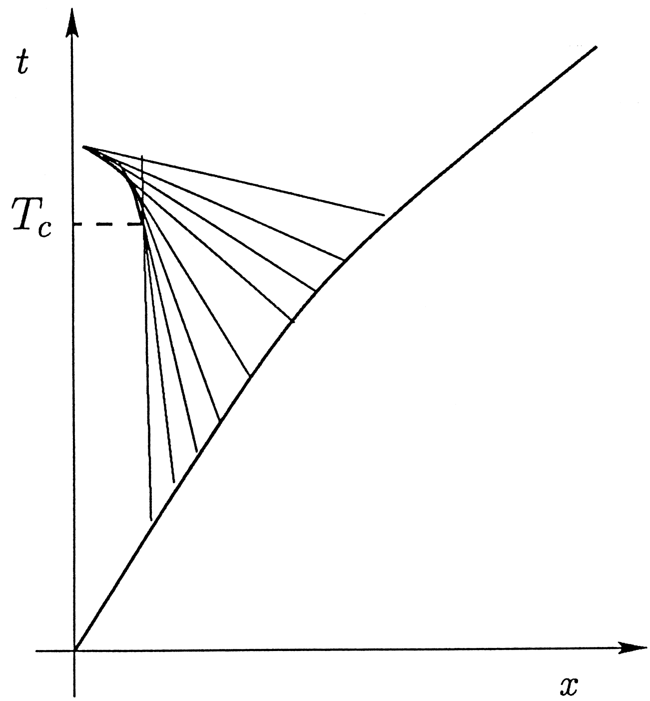

Figure 3. – «Fast» shock.

1. Instantaneous shock

The generic characteristic passing through Σ_c is

$$\frac{T_c - \tau}{x_c - s(\tau)} = \varepsilon'(q_s(\tau)), \qquad \tau_1 < \tau < \tau_2, \tag{A.5}$$

with

$$x_c = s(\tau_1) + \frac{T_c - \tau_1}{\varepsilon'(q_s(\tau_1))}.$$

Let us define the «base» function E_b (see [9]) as follows

$$E_b(\tau) = \varepsilon'(q_s(\tau)). \tag{A.6}$$

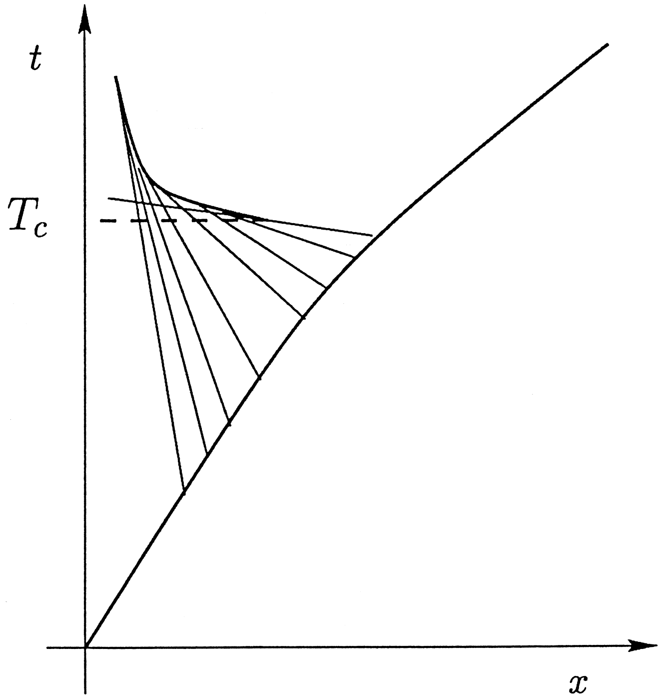

Figure 4. – «Slow» shock.

We have that

$$E_b(\tau) < 0,$$

$$E_b'(\tau) = \frac{1}{x_c - s}\left(-1 + \dot{s}\,\frac{T_c - \tau}{x_c - s}\right) = -\frac{\ddot{s}\,\varepsilon}{(x_c - s)\,q_s'} > 0,$$

(A.7) $$E_b''(\tau) = \frac{1}{x_c - s}\left[\ddot{s}\,\frac{T_c - \tau}{x_c - s} + 2\,\dot{s}\,\frac{\ddot{s}\,\varepsilon}{(x_c - s)\,q_s'}\right] = - \frac{q_s'\,E_b'}{\varepsilon\,\ddot{s}}(\ddot{s}E_b + 2\,\dot{s}E_b') > 0.$$

The function E_b can be used as reference function in order to characterize the other classes of shocks.

2. «fast» shock

Let us define again the function

$$E(\tau) = \varepsilon'(q_s(\tau)). \tag{A.8}$$

The condition $\frac{dx}{d\tau} < 0$ characterizing this case through (A.3) can be expressed in the form

$$E'' < -\frac{q_s' E'}{\varepsilon \ddot{s}}(\ddot{s}E + 2\dot{s}E'). \tag{A.9}$$

Let us remark that if we consider the equality in (A.9) then E_b is a solution of the differential equation, while E, with $0 < x_c < s(T_c)$, $T_c > 0$, is a subsolution.

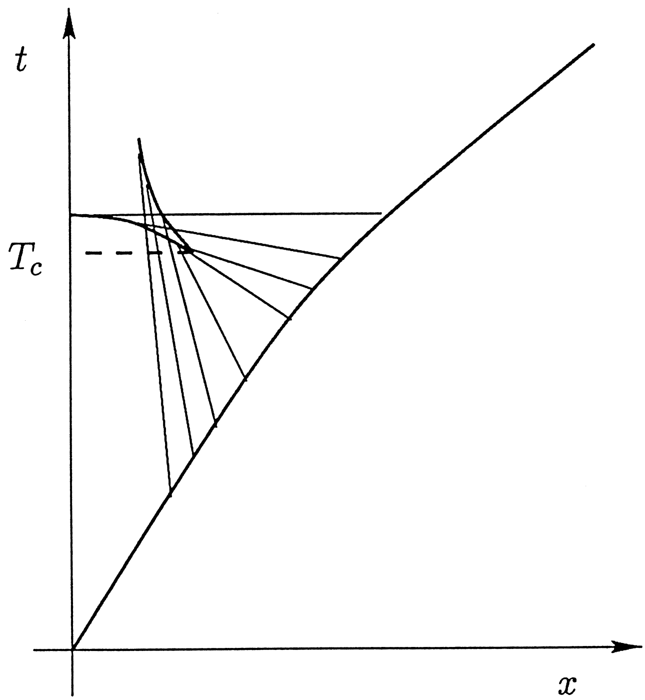

Figure 5. – Two branches shock.

3. «slow» shock

Defining again $E(\tau)$ through (A.8) the condition $\frac{dx}{d\tau} > 0$ is expressed by

$$E'' > -\frac{q_s' E'}{\varepsilon \ddot{s}}(\ddot{s}E + 2\dot{s}E'). \tag{A.10}$$

4. two branches shock

In this case we have a maximum for the function $x(\tau)$ obtained imposing $\frac{dx}{d\tau} = 0$; this gives the condition

$$E''(T_c) = -\frac{q_s' E'}{\varepsilon \ddot{s}} (\ddot{s} E + 2\,\dot{s} E')\,|_{T_c}. \tag{A.11}$$

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E. Comparini: Dipartimento di Matematica «U. Dini», Università di Firenze
Viale Morgagni 67/A, 50134 Firenze (Italy)

M. Ughi: Dipartimento di Scienze Matematiche, Università di Trieste
Piazzale Europa 1, 34127 Trieste (Italy)

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