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On the Suspension Homomorphism.

S. DRAGOTTI - G. MAGRO - L. PARLATO

Sunto. – *In questa nota vengono studiate le condizioni affinché l'omomorfismo di sospensione*

$$s : \Theta^{\mathcal{F}}_{r-1}(S^{n-1}, x_0) \to \Theta^{\mathcal{F}}_{r}(S^{n}, x_0)$$

sia un epimorfismo o un isomorfismo.

Summary. – *In this paper we investigate the conditions for the suspension homomorphism*

$$s : \Theta^{\mathcal{F}}_{r-1}(S^{n-1}, x_0) \to \Theta^{\mathcal{F}}_{r}(S^{n}, x_0)$$

is onto or an isomorphism.

Introduction.

The functor $\Theta^{\mathcal{F}}$ associated to a manifold class $\mathcal{F}$ satisfies all the Eilenberg-Steenrod axioms except excision. The last axiom holds provided that $\mathcal{F}$ satisfies the geometric excision property (see [4]). In this case $\Theta^{\mathcal{F}}$ coincides with the classical homology functor.

In this paper we introduce a weaker geometric property: the «dimension-controled» excision, or (m, n) excision, and we investigate how it is related to the suspension homomorphism

$$s : \Theta^{\mathcal{F}}_{r-1}(S^{n-1}, x_0) \to \Theta^{\mathcal{F}}_{r}(S^{n}, x_0)\,.$$

Already in a recent note ([6]) we have proved that s is an isomorphism if $r = n$. By making power to the, purely geometric, custom there employed, we arrive to devoid a definition wich allows us to obtain a larger result for the homomorphism s (Theor. 3.1).

The intuitive idea of «dimension-controled» excision is that of eliminating a part X of a geometrical space S without shifting another part Y, provided that the dimensions of X, Y, S have selected values.

Elementary techniques of PL topology (regular neighbourhoods, general position theorems) state conditions for the manifold class $\mathcal{PL}$ of the standard

PL-spheres satisfies the new definition (Prop. 2.3). In this case the functor $\Theta^{\mathcal{F}}$ agrees with the classical homotopy functor π, and our main theorem, joint to the recalled result of [6], provides, in greater geometric simplicity and in a broadened point of view, a proof of the well- known Freudenthal's suspension theorem.

The original construction of the functor $\Theta^{\mathcal{F}}$ is developed in [3]. The papers [4], [5], [7] contain more closely investigations about their basic properties, and their behaviour in interesting special cases.

In order to make the current paper self-contained enough that the main theorem can be understood we include a section that provides the definitions of manifold class and functor associated.

1. – Manifold classes and associated functors.

A manifold class is a graded collection $\mathcal{F} = \{\mathcal{F}_h\}_{h \geq 0}$ of compact polyhedra, defined up to a PL-isomorphism, closed under link and join, and such that $S^0 \in \mathcal{F}_0$ ($S^n =$ standard PL-sphere).

The collection $\mathcal{C}$ of the geometric cycles without boundary is so, and for each $\mathcal{F}$ such that $\mathcal{F}_0 = \{S^0\}$ we have $\mathcal{F} \subseteq \mathcal{C}$.

The collection $\mathcal{PL}$ of the standard PL-spheres is so, and $\mathcal{PL} \subseteq \mathcal{F}$ for each manifold class $\mathcal{F}$.

A polyhedron $\Sigma \in \mathcal{F}_h$ is called $\mathcal{F}_h$-sphere, a polyhedron P of the form $\Sigma - \overset{\circ}{st}(x, \Sigma)$ is called $\mathcal{F}_h$-pseudodisc. $\mathcal{F}_h$-spheres and $\mathcal{F}_h$-pseudodiscs are allowable links for a theory of generalized manifolds, the $\mathcal{F}$-manifolds, and a subsequent cobordism theory, the $\mathcal{F}$-cobordism.

A manifold class $\mathcal{F}$ is said to be connected if the polyhedron obtained attaching two $\mathcal{F}_h$-pseudodiscs, by a PL-homeomorphism between their boundaries (if there exists), is an $\mathcal{F}_h$-sphere.

Let $\mathcal{F}$ be a connected manifold class such that $\mathcal{F}_0 = \{S^0\}$. The last hypothesis implies that any $\mathcal{F}$-manifold M is a geometric cycle, so it makes sense to define M to be orientable if M is orientable as geometric cycle. If M denotes an oriented $\mathcal{F}$-manifold, then $-$M will denote the same manifold with the opposite orientation.

An $\mathcal{F}$-cobordism between two oriented $\mathcal{F}_h$-spheres Σ_1 and Σ_2 is an oriented $\mathcal{F}$-manifold W such that:

a) ∂W is the disjoint union of Σ_1 and $-\Sigma_2$;

b) $W \cup c_1 * \Sigma_1 \cup c_2 * \Sigma_2$ is an $\mathcal{F}_{h+1}$-sphere.

An $\mathcal{F}$-cobordism between two oriented $\mathcal{F}_h$-pseudodiscs P_1 and P_2 is an oriented $\mathcal{F}$-manifold W such that:

a′) $\partial W = P_1 \cup P_2 \cup W_0$, where W_0 is a cobordism between ∂P_1 and ∂P_2;

b′) $W \cup c_1 * P_1 \cup c_2 * P_2$ is an $\mathcal{T}_{h+1}$-pseudodisc.

Let (X, x_0) be a pointed topological space. A singular $\mathcal{T}_h$-sphere of (X, x_0) is a triple (Σ, Δ, f), where Σ is an oriented $\mathcal{T}_h$-sphere, $\Delta \subseteq \Sigma$ is a top dimensional simplex and $f:(\Sigma, \Delta) \to (X, x_0)$ a continuous map.

Two singular $\mathcal{T}_h$-spheres $(\Sigma_1, \Delta_1, f_1)$ and $(\Sigma_2, \Delta_2, f_2)$ are said $\mathcal{T}$-cobordant if there exists a triple (W, W', g), called $\mathcal{T}$-cobordism, where W is an $\mathcal{T}$-cobordism between Σ_1 and Σ_2, $W' \subseteq W$ is a PL-disc such that $W' \cap \Sigma_i = \Delta_i$, $i = 1, 2$, and $g:(W, W') \to (X, x_0)$ is a continuous map, such that: $g/\Sigma_i = f_i$, $i = 1, 2$.

The $\mathcal{T}$-cobordism between singular $\mathcal{T}$-spheres is an equivalence relation.

Let $\Theta_h^{\mathcal{T}}(X, x_0)$ denote the set of the $\mathcal{T}$-cobordism classes of singular $\mathcal{T}_h$-spheres of (X, x_0).

As in the case of the homotopy theory we can geometrically define an addition in $\Theta_h^{\mathcal{T}}(X, x_0)$ $(h \geq 1)$ which give a group structure. The zero element is the class of the $\mathcal{T}_h$-spheres cobordant to zero, and a cobordism to zero of (Σ, Δ, f) is a triple (P, D, g), where P is an oriented $(h+1)$-pseudodisc, $D \subset P$ is a top-dimensional simplex, and $g:(P, D) \to (X, x_0)$ is a continuous map such that $\partial P = \Sigma$, $D \cap \Sigma = \Delta$, and $g/\Sigma = f$.

Let (X, A) be a pair of topological spaces and let x_0 be a point of A. By relative $\mathcal{T}_h$-sphere of (X, A, x_0) we mean a triple (P, Δ, f), where P is an oriented $\mathcal{T}_h$-pseudodisc, $\Delta \subseteq P$ is a top-dimensional simplex meeting ∂P in a top-dimensional simplex, and $f:(P, \Delta) \to (X, x_0)$ is a map which carries ∂P to A.

Given a relative $\mathcal{T}_h$-sphere (P, Δ, f) of (X, A, x_0), $(\partial P, \Delta \cap \partial P, f/)$ is a singular $\mathcal{T}_{h-1}$-sphere of (A, x_0) which will be denoted by $\partial(P, \Delta, f)$.

Two relative $\mathcal{T}$-spheres (P_i, Δ_i, g_i), $i = 1, 2$, of (X, A, x_0) are called $\mathcal{T}$-cobordant if there exists a triple (V, V', G) where V is an $\mathcal{T}$-cobordism between P_1 and P_2, $V' \subseteq V$ a $\mathcal{PL}$-cobordism between Δ_1 and Δ_2, and $G:(V, V') \to (X, x_0)$ is a continuous map such that the following conditions hold:

(1) $V' \cap P_i = \Delta_i$, $i = 1, 2$

(2) $G/P_i = g_i$, $i = 1, 2$

(3) Let $W = \partial V - (\mathring{P}_1 \cup \mathring{P}_2)$ (1) and $W' = W \cap V'$. Then $(W, W', G/)$ is an $\mathcal{T}$-cobordism between $\partial(P_1, \Delta_1, g_1)$ and $\partial(P_2, \Delta_2, g_2)$ with $G(W) \subseteq A$.

The $\mathcal{T}$-cobordism between relative $\mathcal{T}$-spheres is an equivalence relation.

Let $\Theta_h^{\mathcal{T}}(X, A, x_0)$ denote the set of the $\mathcal{T}$-cobordism classes of relative $\mathcal{T}_h$-spheres of (X, A, x_0). As before, we can introduce in $\Theta_h^{\mathcal{T}}(X, A, x_0)$ $(h \geq 2)$ a group structure. The zero element is the class of the relative $\mathcal{T}_h$-spheres cobor-

(1) $\mathring{P}$ stands for $P - \partial P$.

dant to zero, and a cobordism to zero of (P, Δ, f) is a triple (Q, D, F) where Q is an $\mathcal{F}$-pseudodisc of dimension $h+1$, $D \subseteq Q$ is an $h+1$ simplex of Q, $F:(Q, D) \to (X, x_0)$ is a continuous map such that

1) $P \subset \partial Q$, $D \cap P = \Delta$

2) $F/P = f$

3) Let $Q' = \partial Q - \overset{\circ}{P}$ and $D' = D \cap (\partial Q - \overset{\circ}{P})$. Then $(Q', D', F/$ is a cobordism to zero of $\partial(P, \Delta, f)$ with $F(Q') \subset A$.

Given a continuous map $f:(X, A, x_0) \to (Y, B, y_0)$, we can define, for each $h \geq 2$, a homomorphism $\Theta^{\mathcal{F}}(f)$: $\Theta_h^{\mathcal{F}}(X, A, x_0) \to \Theta_h^{\mathcal{F}}(Y, B, y_0)$ by setting

$$\Theta^{\mathcal{F}}(f)([(P, \Delta, g)]) = [(P, \Delta, f \circ g)].$$

As proved in [3], the definitions above recalled allow us to build a covariant functor $\Theta^{\mathcal{F}}$ which assigns to every pointed pair of topological spaces (X, A, x_0) a graded group $\Theta^{\mathcal{F}}(X, A, x_0)$, just as $\mathcal{PL}$ determines the classical functor π using $\mathcal{F}$-spheres and $\mathcal{F}$-pseudodiscs instead of PL-spheres and discs.

Every $\Theta^{\mathcal{F}}$ satisfies the first six axioms of Eilenberg and Steenrod (excision is excluded).

If $\mathcal{F}' \subseteq \mathcal{F}$ there exists a canonical homomorphism (forgetful) of graded groups $\psi_{\mathcal{F}', \mathcal{F}}$: $\Theta^{\mathcal{F}'}(X, A, x_0) \to \Theta^{\mathcal{F}}(X, A, x_0)$ which allows to factorize the classical Hurewicz homomorphism

$$\begin{array}{ccc} \pi(X, x_0) & \xrightarrow{\psi_{\mathcal{PL}, \mathcal{C}}} & H(X, x_0) \\ \psi_{\mathcal{PL}, \mathcal{F}} \searrow & & \nearrow \psi_{\mathcal{F}, \mathcal{C}} \\ & \Theta^{\mathcal{F}}(X, x_0) & \end{array}$$

A manifold class $\mathcal{F}$ is said to be coconnected if for every $\mathcal{F}_h$-sphere Σ and $\mathcal{F}_h$-pseudodisc $P \subset \Sigma$, the polyhedron $\Sigma - \overset{\circ}{P}$ is an $\mathcal{F}_h$-pseudodisc.

PROPOSITION 1.1. – *Let $\mathcal{F}$ be a connected, coconnected manifold class. All the $\mathcal{F}$-spheres and $\mathcal{F}$-pseudodiscs of positive dimension are connected. The only $\mathcal{F}_0$-sphere is S^0.*

PROPOSITION 1.2. – *Let $\mathcal{F}$ be a connected, coconnected manifold class, and let $P' \subseteq P$ be $\mathcal{F}_h$-pseudodiscs such that $\partial P \cap \partial P' = P''$ is an $\mathcal{F}_{h-1}$-pseudodisc, then the polyhedron $P - (\overset{\circ}{P} \cup P'')$ is an $\mathcal{F}_h$-pseudodisc.*

PROPOSITION 1.3. – *Let $\mathcal{F}$ be a connected manifold class. The cylinder $P \times I$ and the cone cP on an $\mathcal{F}_{h-1}$-pseudodisc P are $\mathcal{F}_h$-pseudodiscs.*

PROPOSITION 1.4. – *Let $\mathcal{F}$ be a connected, coconnected manifold class. The polyhedron obtained by gluing two $\mathcal{F}_h$-pseudodiscs belong an $\mathcal{F}_{h-1}$-pseudodisc of their boundary is an $\mathcal{F}_h$-pseudodisc.*

2. – The (m, n) excision property.

In this section we do the definition of «dimension-controled» excision and some observations about this. Moreover we study the behaviour with respect to this property of the manifold class $\mathcal{PL}$ of the standard PL-spheres.

The intuitive idea of dimension-controled excision, described in the Introduction, in case the objects are polyhedra and pseudodiscs of a manifold class can be well formulated in the following fashion.

DEFINITION 2.1. – *Let $m > n \geq 0$ integers. A manifold class $\mathcal{F}$ is said (m, n)-excisive if for each pair of disjoint polyhedra X_1, X_2 of codimension n contained in an m-pseudodisc P and such that $X_i' = X_i \cap \partial P$ has codimension $\geq n$ in ∂P, $i = 1, 2$, for every $(m-1)$-pseudodisc P_i' of ∂P containing X_i' and not meeting X_j, if there is one, there exists an m-pseudodisc P_i such that*

i) $X_i \subset \text{int } P_i$

ii) $P_i \cap X_j = \emptyset,\ i \neq j$

iii) $P_i \cap \partial P$ *is an* $(m-1)$*-pseudodisc contained in int* P_i'.

Roughly speaking, if $\mathcal{F}$ is (m, n)-excisive, it is possible to eliminate a polyhedron X contained in a pseudodisc P without moving another polyhedron Y provided $\dim P = m$, $\dim X = \dim Y = m - n$, and provided that it is possible to eliminate on the boundary ∂P the polyhedron $X \cap \partial P$ without moving $Y \cap \partial P$.

REMARK 2.2. – *At first glance the Definition* 2.1 *may seem strange because considers only polyhedra X_i of P which meet ∂P. Newertheless if $X_i \cap \partial P = \emptyset$, we can replace, without loss of the generality, X_i by $Y_i = X_i \cup x_i$, where x_i is a point of $\partial P - X_j$. Indeeed the hypothesis $m > n$ implies that the codimension $m - 1$ of $Y_i \cap \partial P$ in ∂P results greater or equal to n.*

Since a polyhedron of codimension n is contained in a polyhedron of codimension $n' > n$, it is evident that a manifold class $\mathcal{F}(m, n)$-excisive is also (m, n')-excisive for each n' such that $m > n' > n$.

The manifold class $\mathcal{C}$ of the geometric cycles is (m, n)-excisive for each m and n, $m > n$. This follows from the excision property of the geometric cycles.

Finally we consider the class $\mathcal{PL}$ of the standard PL-spheres.

PROPOSITION 2.3. – *The manifold class $\mathcal{PL}$ is (m, n)-excisive provided $n \geqslant 2$ and $m < 2n - 1$.*

PROOF. – Let X_i, $i = 1, 2$, be disjoint polyhedra of dimension $m - n$ contained in a PL-disc m-dimensional D, such that $\dim X_i \cap \partial D \leqslant m - 1 - n$. Suppose that there is a top dimensional PL-disc D_i' of ∂D such that $X_i \cap \partial D \subset \overset{\circ}{D}{}_i'$ and $D_i' \cap X_j = \emptyset$, $i \neq j$.

Being $m < 2n - 1$, we have $2(m - n - 1) + 1 < m - 1$. Hence there is a point c_i' of $\overset{\circ}{D}{}_i'$ in general position with respect to $X_i' = X_i \cap \partial D_i'$, so c_i' and X_i' are joinable, and the cone $C_i' = c_i' * X_i'$ is contained in $\overset{\circ}{D}{}_i'$.

Now consider a point $c_i \in \overset{\circ}{D}$ in general position with respect to $C_i' \cup X_i \cup X_j$ (this is possible because $2(m - n) + 1 < m$), hence the cone $C_i = c_i * (C_i' \cup X_i)$ is disjoint from X_j. An ε-neighbourhood of C_i is a PL-disc D_i (C_i is collapsible), and there exists $\varepsilon > 0$ sufficiently small so that such a disc not meet X_j.

Moreover $D_i \cap \partial D$ is a regular neighbourhood, in ∂D, of the cone C_i', so it is a PL-disc contained in $\overset{\circ}{D}{}_i'$. ■

3. – The main theorem.

THEOREM 3.1. – *Let $\mathcal{F}$ be a connected, coconnected manifold class.*

If $\mathcal{F}$ is (r, n) excisive for each $r \leqslant m$ $(r > n \geqslant 2)$ then the suspension homomorphism:

$$s : \Theta^{\mathcal{F}}_{r-1}(S^{n-1}, x_0) \to \Theta^{\mathcal{F}}_r(S^n, x_0)$$

is an isomorphism for each $r \leqslant m - 1$.

If $r = m$, s is onto.

PROOF. – The suspension homomorphism s is obtained by composition

$$\Theta^{\mathcal{F}}_{r-1}(S^{n-1}, x_0) \underset{\sim}{\overset{\partial}{\leftarrow}} \Theta^{\mathcal{F}}_r(D^n_+, S^{n-1}, x_0) \overset{i}{\rightarrow} \Theta^{\mathcal{F}}_r(S^n, D^n_-, x_0) \underset{\sim}{\overset{j}{\leftarrow}} \Theta^{\mathcal{F}}_r(S^n, x_0)$$

$$s = j^{-1} \circ i \circ \partial^{-1}$$

where D^n_+ and D^n_- are the northern and southern hemispheres of S^n, and x_0 is a fixed point of $S^{n-1} = \dot{D}^n_+ \cap \dot{D}^n_-$.

The boundary homomorphisms ∂ and j also are isomorphisms by standard properties of the involved spaces, and by homotopy and exactness axioms of the functor $\Theta^{\mathcal{F}}$.

Hence s is an isomorphism or s is onto if, and only if, the homomorphism i is so.

Let $p_1 \in \overset{\circ}{D}{}^n_+$ the north pole, $p_2 \in \overset{\circ}{D}{}^n_-$ the south pole of S^n, consider the diagram $(r \geq 2)$

$$\begin{array}{ccc} \Theta_r^{\mathcal{T}}(D_+^n, S^{n-1}, x_0) & \xrightarrow{\partial_1} & \Theta_{r-1}^{\mathcal{T}}(S^{n-1}, x_0) \\ \downarrow i_1 & & \downarrow j_1 \\ \Theta_r^{\mathcal{T}}(S^n - p_2, S^n - (p_1 \cup p_2), x_0) & \xrightarrow{\partial_2} & \Theta_{r-1}^{\mathcal{T}}(S^n - (p_1 \cup p_2), x_0) \end{array}$$

where the vertical maps are induced by inclusion maps. Because D_+^n and $S^n - p_2$ are contractible, the homotopy and exactness axioms of the functor $\Theta^{\mathcal{T}}$ assure that the connecting homomorphisms ∂_1, ∂_2 are isomorphisms. Moreover S^{n-1} is a strong deformation retract of $S^n - (p_1 \cup p_2)$, and hence j_1 is an isomorphism.

From the trivial commutativity of the above diagram it follows that i_1 is an isomorphism.

A similar argument shows that if $r \geq 2$ also the homomorphism

$$i_2 \colon \Theta_r^{\mathcal{T}}(S^n, D_-^n, x_0) \to \Theta_r^{\mathcal{T}}(S^n, S^n - p_1, x_0)$$

is an isomorphism.

Finally consider the commutative diagram

$$\begin{array}{ccc} \Theta_r^{\mathcal{T}}(D_+^n, S^{n-1}, x_0) & \xrightarrow{i_1} & \Theta_r^{\mathcal{T}}(S^n - p_2, S^n - (p_1 \cup p_2), x_0) \\ \downarrow i & & \downarrow h \\ \Theta_r^{\mathcal{T}}(S^n, D_-^n, x_0) & \xrightarrow{i_2} & \Theta_r^{\mathcal{T}}(S^n, S^n - p_1, x_0) \end{array}$$

where h is induced by inclusion.

Being i_1 and i_2 isomorphisms, to prove that i is onto (or injective) is equivalent to prove that h is onto (or injective).

Then we now show that, if $r \leq m$, the homomorphism

$$h : \Theta_r^{\mathcal{T}}(S^n - p_2, S^n - (p_1 \cup p_2), x_0) \to \Theta_r^{\mathcal{T}}(S^n, S^n - p_1, x_0)$$

is onto.

Let (P, Δ, f) be a representative triple of an element α of $\Theta_r^{\mathcal{T}}(S^n, S^n - p_1, x_0)$ that is $f \colon P \to S^n$, $f(\partial P) \subseteq S^n - p_1$, $f(\Delta) = x_0$.

We need a representative triple (P', Δ', f') of α such that $f'(P') \subseteq S^n - p_2$, $f'(\partial P') \subseteq S^n - (p_1 \cup p_2)$, $f'(\Delta') = x_0$.

Up to a homotopy we can suppose that f is a simplicial map, p_1 is the barycentre of a top dimensional simplex of D_+^n and p_2 is the barycentre of a top dimensional simplex of D_-^n.

If $f^{-1}(p_2) = \emptyset$ it suffices to take

$$P' = P, \qquad \Delta' = \Delta, \qquad f' = f.$$

If not, being f a simplicial map, $f^{-1}(p_2)$ is a polyhedron X_2 of dimension $r - n$ contained in the r-pseudodisc P, and analogously $f^{-1}(p_1)$, if not empty, is a polyhedron Y_1 of dimension $r - n$ contained in $\mathring{P}$.

Let $X_1 = Y_1 \cup b(\Delta \cap \partial P)$ ($b(\Delta \cap \partial P)$ is the barycentre of the face $\Delta_0 = \Delta \cap \partial P$ of Δ).

We have

$$\dim X_1 = \dim Y_1 = r - n, \qquad \dim X_1 \cap \partial P = 0 \leqslant r - 1 - n$$

$$X_1 \cap X_2 = \emptyset \text{ because } f(b) = x_0 \neq p_2$$

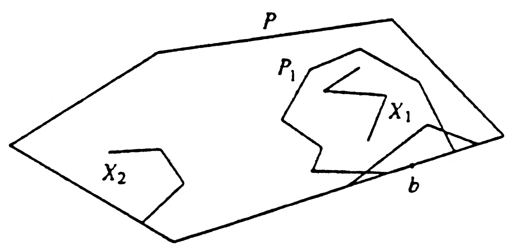
$$X_1 \cap \partial P = \{b(\Delta \cap \partial P)\} \subset \Delta_0$$

$$X_2 \cap \Delta_0 = \emptyset \text{ because } f(b) = x_0 \neq p_2 .$$

Being $\mathcal{F}(r, n)$ excisive, there exists an r-pseudodisc P_1 such that

(a) $X_1 \subset \operatorname{int} P_1$, $P_1 \cap X_2 = \emptyset$

(b) $P_1 \cap \partial P$ is an $(r-1)$-pseudodisc contained in Δ_0



Let Δ_1 be an r-simplex contained in $\Delta \cap P_1$ such that $\Delta_1 \cap \partial P_1 \subseteq \Delta_1 \cap \partial P$ is an $r-1$ simplex containing $b(\Delta \cap \partial P)$.

The triple (P, Δ, f) is $\mathcal{F}$-cobordant to (P, Δ_1, f) (cfr. [3] theor. 2.10).

Now if we take $P' = P_1$, $\Delta' = \Delta_1$, $f' = f/P_1$ we have

(1) $f'(P') = f(P_1) \subseteq S^n - p_2$ because $P_1 \cap X_2 = \emptyset$

(2) $f'(\partial P') = f(\partial P_1) \subseteq S^n - (p_1 \cup p_2)$ because $f^{-1}(p_1) \subset X_1 \subset \mathring{P}_1$ by (a)

(3) $f'(\Delta') = f(\Delta') \subseteq f(\Delta) = x_0$.

Hence the triple (P', Δ', f') determines an element α' of $\Theta_r^{\mathcal{F}}(S^n - p_2, S^n - (p_1 \cup p_2), x_0)$.

In order to prove that $h(\alpha') = \alpha$ it remains to construct an $\mathcal{F}$-cobordism (W, W', G) between (P, Δ', f) $(\sim (P, \Delta, f))$ and (P', Δ', f').

Let $W = P \times I$, $W' = \Delta' \times I$, $G = f \times \mathrm{id}$.

We have

$$W' \cap P \times \{0\} = \Delta' \times \{0\}; \qquad W' \cap P' \times \{1\} = \Delta' \times \{1\}$$

$$G(\partial W - (\mathring{P} \times \{0\} \cup \mathring{P}' \times \{1\})) \subseteq S^n - p_1$$

Then, by using the Propositions 1.2, 1.3, 1.4, it is easy to verify that the triple (W, W', G) satisfies all the conditions which assure that it is the required cobordism between (P, Δ', f) and (P', Δ', f'), where P and P' are respectively identified with $P \times \{0\}$ and $P' \times \{1\}$.

Now we prove that h is injective for $r \leqslant m-1$.

Let $\alpha = [(P, \Delta, f)]$ be an element of $\Theta_r^{\mathcal{F}}(S^n - p_2, S^n - (p_1 \cup p_2), x_0)$ such that $h(\alpha) = 0$. As above, we can suppose, up to a homotopy, that f is a simplicial map, p_1 is the barycentre of a top dimensional simplex of D_+^n and p_2 is the barycentre of a top dimensional simplex of D_-^n.

Let (Q, D, F) be a cobordism to zero of (P, Δ, f) in $(S^n, S^n - p_1, x_0)$, that is:

Q is an $\mathcal{F}$-pseudodisc of dimension $r+1$, $D \subseteq Q$ is an $r+1$ simplex of Q, $F: (Q, D) \to (S^n, x_0)$ is a simplicial map such that

1) $P \subset \partial Q$, $D \cap P = \Delta$

2) $F/P = f$

3) $(\partial Q - \mathring{P}, D \cap (\partial Q - \mathring{P}), F)$ is a cobordism to zero of $\partial(P, \Delta, f)$ such that $F(\partial Q - \mathring{P}) \subset S^n - p_1$.

In order to prove the assert we need a cobordism to zero of (P, Δ, f) in $(S^n - p_2, S^n - (p_1 \cup p_2), x_0)$.

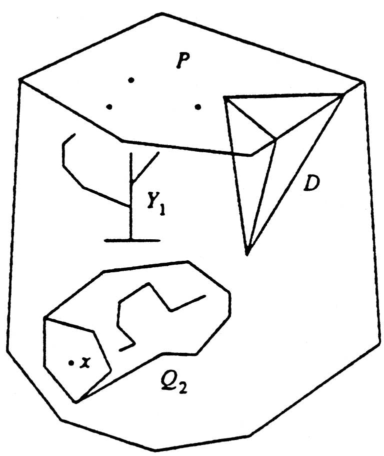
If $F^{-1}(p_2) = \emptyset$, it suffices to take (Q, D, F) itself.

If not, let $Y_1 = F^{-1}(p_1)$, $Y_2 = F^{-1}(p_2) \cup x$, where x is a point of $\partial Q - P$. Y_1 and Y_2 are disjoint polyhedra of dimension $r+1-n$ contained in $Q = Q - (\mathring{D} \cup \mathring{\Delta})$, which is an $(r+1)$-pseudodisc by Prop. 1.2 again.

Being $r+1 \leqslant m$, $\mathcal{F}$ is $(r+1, n)$-excisive by hypothesis, and being $\partial Q - \mathring{P}$ an r-pseudodisc containing $Y_2 \cap \partial\overline{Q}$ and not meeting Y_1, there exists an $(r+1)$-pseudodisc Q_2 of $\overline{Q}$ such that

(a′) $Y_2 \subset \operatorname{int} Q_2$, $Y_1 \cap Q_2 = \emptyset$

(b′) $Q_2 \cap \partial\overline{Q}$ is an r-pseudodisc Q_2' contained in $\operatorname{int}(\partial Q - \mathring{P})$



Let D' be an $(r+1)$ simplex contained in int$D$ such that $D' \cap \partial Q = \Delta$.

Being $\mathcal{F}$ a coconnected manifold class, by Prop. 1.2, the polyhedron $Q' = Q - (\overset{\circ}{Q}_2 \cup \overset{\circ}{Q}{}'_2)$ is an $(r+1)$-pseudodisc.

We have

$$P \subset \partial Q' \quad \text{because} \quad Q_2' \subseteq \operatorname{int}(\partial Q - \overset{\circ}{P})$$

$$F(Q') \subseteq S^n - p_2 \quad \text{because} \quad F^{-1}(p_2) \subset \operatorname{int} Q_2 \quad \text{and} \quad Q' \cap \operatorname{int} Q_2 = \emptyset$$

$$F(\partial Q' - \overset{\circ}{P}) \subseteq S^n - (p_1 \cup p_2).$$

At this point it is straightforward to see that the triple $(Q', D', F/)$ is the required cobordism to zero of (P, Δ, f) in $(S^n - p_2, S^n - (p_1 \cup p_2), x_0)$. ■

COROLLARY 3.2 (Freudenthal's Suspension Theorem). – *If* $r < 2n - 2$, *then*

$$s : \pi_{r-1}(S^{n-1}) \approx \pi_r(S^n)$$

$$s(\pi_{2n-3}(S^{n-1})) = \pi_{2n-2}(S^n)$$

PROOF. – Being $\Theta^{\mathcal{PL}} = \pi$, the assert follows

– if $r \neq n$, from the Prop. 2.3 and Theorem 3.1;

– if $r = n$, as particular case of the result of [6] recalled in Introduction.

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