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On the Analytic Approximation of Differentiable Functions from Above.

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Sunto. – *Si determinano condizioni affinché una funzione differenziabile sia approssimabile dall'alto con funzioni analitiche, restando immutata su un fissato sottoinsieme analitico che sia localmente intersezione completa.*

Summary. – *We determine conditions in order that a differentiable function be approximable from above by analytic functions, being left invariate on a fixed analytic subset which is a locally complete intersection.*

1. – Introduction.

Let X be a closed analytic subset of an open domain Ω of $\mathbb{R}^n$ and let f be a $\mathcal{C}^\infty$ differentiable function on Ω. If we want to approximate f by analytic functions, in strong Whitney's topology, without changing its values on X, obviously, a necessary condition is that f is analytic on X. Another necessary condition is that X is *coherent*, otherwise it could not exist any analytic function on Ω that coincides with f on X. In [6] it is proved that for every continuous positive function $\eta : \Omega \to \mathbb{R}$ there exists an analytic function h on Ω such that $f|_X = h|_X$ and

$$|D^\alpha f(x) - D^\alpha h(x)| < \eta(x) \text{ for } |\alpha| \leq \frac{1}{\eta(x)}, \qquad \alpha \in \mathbb{N}^n.$$

In this short note we deal with the conditions that allow us to make such an approximation from above, i.e. we want $h(x) \geq f(x)$ for every $x \in \Omega$. When X is a locally complete intersection subset we find a necessary and sufficient condition in order to obtain the approximation. This allows us to say a little more about the problem, posed by C. Andradas, of the extension of a nonnegative analytic function off X.

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2. – Results.

Let $\mathcal{O}_{\mathbb{R}^n}$ and $\mathcal{E}_{\mathbb{R}^n}$ be the sheaves of real analytic and, respectively, differentiable ($\mathcal{C}^\infty$) functions on $\mathbb{R}^n$.

If Ω is an open domain in $\mathbb{R}^n$, we denote by $\mathcal{O}_\Omega$ (resp. $\mathcal{E}_\Omega$) the sheaf $\mathcal{O}_{\mathbb{R}^n}|_\Omega$ (resp. $\mathcal{E}_{\mathbb{R}^n}|_\Omega$).

Let X be a closed analytic subset of Ω; we denote by $\mathfrak{I}_X \subset \mathcal{O}_\Omega$ the ideal sheaf of real analytic functions vanishing on X and by $\mathfrak{J}_X \subset \mathcal{E}_\Omega$ the ideal sheaf of differentiable functions vanishing on X.

As usual, we denote by $\mathcal{O}_X$ the sheaf $(\mathcal{O}_\Omega/\mathfrak{I}_X)|_X$ of the analytic functions on X.

In the following, by an analytic subset X we always mean a *closed locally complete intersection coherent analytic subset* so that, by a result of S. Coen ([3]), the ideal $\mathfrak{I}_X$ has finitely many global sections that generate its fiber at each point.

LEMMA 1. – *Let Ω be an open domain in $\mathbb{R}^n$, η a continuous positive function on Ω, and $\theta, \theta_1, \ldots, \theta_s \in \mathcal{E}_\Omega(\Omega)$. Then there exists a positive function $\mu \in \mathcal{E}_\Omega(\Omega)$ such that, for $|\alpha| \leqslant \dfrac{1}{\eta(x)}$,*

i) $|D^\alpha(\mu\theta)(x)| < \dfrac{\eta(x)}{2}$;

ii) $\mu(x) \sum_{i=1}^{s} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta}\theta_i(x)| \leqslant \dfrac{\eta(x)}{2}$.

PROOF. – Let $(K_\nu)_{\nu\in\mathbb{N}}$ be a sequence of compact sets such that $\Omega = \bigcup_\nu K_\nu$, $K_0 = \emptyset$, $K_\nu \subset \overset{\circ}{K}_{\nu+1}$, and let $A_\nu = \overset{\circ}{K}_{\nu+2} - K_\nu$. Then $(A_\nu)_{\nu\in\mathbb{N}}$ is a locally finite open covering of Ω and every $x \in \Omega$ has a neighbourhood U such that $U \cap A_\nu = \emptyset$ if $\nu \neq p, p+1$ for one and only one $p \in \mathbb{N}$. Let $(\phi_\nu)_{\nu\in\mathbb{N}}$ be a differentiable partition of unity such that $\operatorname{supp}(\phi_\nu) \subset A_\nu$.

For every $\nu \in \mathbb{N}$, let $\delta_\nu \in \mathbb{R}_+$ be such that $\delta_\nu < \inf_{x\in K_{\nu+3}} \eta(x)$ and let $\varrho_\nu \geqslant 1$ be a real number strictly bigger than

$$\sup_{|\alpha|\leqslant(1/\delta_\nu)} \|D^\alpha\phi_\nu\|_{K_{\nu+3}}, \qquad \sup_{|\alpha|\leqslant(1/\delta_\nu)} \|D^\alpha\theta\|_{K_{\nu+3}}, \qquad \sup_{|\alpha|\leqslant(1/\delta_\nu)} \sum_{i=1}^{s} \|D^\alpha\theta_i\|_{K_{\nu+3}}.$$

We can suppose that $\varrho_{\nu+1} \geqslant \varrho_\nu$ and we can find a sequence of positive real numbers $(\varepsilon_\nu)_{\nu\in\mathbb{N}}$ such that $\varepsilon_{\nu+1} \leqslant \varepsilon_\nu$ and

$$\varepsilon_\nu \varrho_\nu^2 \sup_{|\alpha|\leqslant(1/\delta_\nu)} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} < \frac{\delta_\nu}{4}.$$

Now, let us consider the differentiable function $\mu = \sum_{\nu\in\mathbb{N}} \varepsilon_\nu \phi_\nu$; since we can suppose that μ is locally equal to $\varepsilon_p \phi_p + \varepsilon_{p+1} \phi_{p+1}$, it is straightforward to check that μ satisfies the required conditions.

THEOREM 1. – *Let Ω be an open domain in $\mathbb{R}^n$ and $f \in \mathcal{E}_\Omega(\Omega)$ a differentiable function. For every continuous positive function η on Ω there exists an analytic function $h \in \mathcal{O}_\Omega(\Omega)$ such that*

i) $|D^\alpha f(x) - D^\alpha h(x)| < \eta(x)$, for $|\alpha| \leqslant \dfrac{1}{\eta(x)}$;

ii) *$h(x) > f(x)$ for every $x \in \Omega$.*

PROOF. – It follows from Lemma 1 that there exists a positive differentiable function $\mu \in \mathcal{E}(\Omega)$ such that $|D^\alpha(\mu)(x)| < \dfrac{\eta(x)}{2}$, for $|\alpha| \leqslant \dfrac{1}{\eta(x)}$. By Whitney's approximation theorem (see [5]) there exists an analytic function $h \in \mathcal{O}_\Omega(\Omega)$ such that

$$|D^\alpha(f+\mu)(x) - D^\alpha h(x)| < \frac{\mu(x)}{2} \qquad \text{for } |\alpha| \leqslant \frac{2}{\mu(x)}.$$

It is easy to check that h satisfies the required conditions.

LEMMA 2. – *Let Ω be an open domain in $\mathbb{R}^n$, $\mathfrak{I}$ a coherent ideal of $\mathcal{O}_\Omega$ generated by finitely many global sections $\theta_1, \ldots, \theta_s$, $\mathfrak{J}$ the ideal generated by $\mathfrak{I}$ in $\mathcal{E}_\Omega$ and $f \in \Gamma(\Omega, \mathfrak{J})$. For every continuous positive function η on Ω there exists an analytic function $h \in \Gamma(\Omega, \mathfrak{I})$ such that*

$$|D^\alpha f(x) - D^\alpha h(x)| < \eta(x), \qquad \text{for } |\alpha| \leqslant \frac{1}{\eta(x)}.$$

Moreover, if $f \in \Gamma(\Omega, \mathfrak{J}^2)$ it is possible to find $h \in \Gamma(\Omega, \mathfrak{I}^2)$ such that $h(x) \geqslant f(x)$ for every $x \in \Omega$.

PROOF. – There exist differentiable functions $f_1, \ldots, f_s$ on Ω such that $f = \sum_{i=1}^{s} f_i \theta_i$. As in Lemma 1, let μ be a differentiable function such that $\mu(x) < \eta(x)$ for every $x \in \Omega$ and $\mu(x) \sum_{i=1}^{s} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta}\theta_i(x)| \leqslant \dfrac{1}{\eta(x)}$ for $|\alpha| \leqslant \dfrac{1}{\eta(x)}$. By Theorem 1, for every $i = 1, \ldots, s$, there exist analytic functions $h_i \in \mathcal{O}_\Omega(\Omega)$ such that $|D^\alpha f_i(x) - D^\alpha h_i(x)| < \mu(x)$, for $|\alpha| \leqslant \dfrac{1}{\mu(x)}$ and $h_i(x) > f_i(x)$ for every $x \in \Omega$. It is easy to see that the analytic function $h = \sum_{i=1}^{s} h_i \theta_i$ satisfies the first condition.

Moreover, if f is a section of $\mathfrak{J}^2$, then, by replacing the functions θ_i by the functions $(\theta_i + \theta_j)^2$, $i, j = 1, \ldots, s$, we can suppose that they are nonnegative on Ω, and so the second condition is satisfied too.

THEOREM 2. – *Let Ω be an open domain in $\mathbb{R}^n$, X an analytic subset of Ω and $f \in \mathcal{E}_\Omega(\Omega)$ a differentiable function such that $f_x \in \mathfrak{J}^2_{X,x}$ for all $x \in X$. For every continuous positive function η on Ω there exists an analytic function $h \in \mathcal{O}_\Omega(\Omega)$ such that*

i) $|D^\alpha f(x) - D^\alpha h(x)| < \eta(x)$, for $|\alpha| \leq \dfrac{1}{\eta(x)}$;

ii) $h(x) \geq f(x)$ *for every* $x \in \Omega$;

iii) $h|_X = 0$.

Moreover, if f is nonnegative on Ω, h can be chosen such that $X = h^{-1}(0)$.

PROOF. – Since f is in $\Gamma(\Omega, \mathfrak{J}^2)$, by Lemma 2 we only need to prove that, if f is nonnegative, it is possible to find h such that $X = h^{-1}(0)$.

Since X is coherent, there exists a nonnegative function $\theta \in \mathcal{O}_\Omega(\Omega)$ such that $X = \theta^{-1}(0)$, and, by Lemma 1, there exists a positive differentiable function $\mu \in \mathcal{E}_\Omega(\Omega)$ such that $|D^\alpha(\mu\theta)(x)| < \dfrac{\eta(x)}{4}$, for $|\alpha| \leq \dfrac{1}{\eta(x)}$. On the other hand, as in the proof of Lemma 2, applied to the ideal generated by θ, there exists an analytic function $\delta \in \mathcal{O}_\Omega(\Omega)$ such that $|D^\alpha(\mu\theta)(x) - D^\alpha(\delta\theta)(x)| < \dfrac{\eta(x)}{4}$, for $|\alpha| \leq \dfrac{1}{\eta(x)}$. It follows that $|D^\alpha(\delta\theta)(x)| < \dfrac{\eta(x)}{2}$ for $|\alpha| \leq \dfrac{1}{\eta(x)}$.

Now let us consider the analytic function $g = h + \delta\theta$; it follows immediately that $X = g^{-1}(0)$ and that $g(x) \geq h(x) \geq f(x)$ for every $x \in \Omega$. Moreover, for $|\alpha| \leq \dfrac{1}{\eta(x)}$, we have $|D^\alpha f(x) - D^\alpha g(x)| < \eta(x)$ and then, by replacing h by g, we get the conclusion.

LEMMA 3. – *Let X be an analytic subset of an open domain Ω of $\mathbb{R}^n$:*

i) $\{f_a \in \mathfrak{I}_{X,a} \,|f_a \geq 0\} \subset \mathfrak{I}^2_{X,a}$;

ii) $\{f_a \in \mathfrak{J}_{X,a} \,|f_a \geq 0\} \subset \mathfrak{J}^2_{X,a}$.

PROOF. – There exist an open neighbourhood U of a and functions $\theta_1, \ldots, \theta_s \in \mathfrak{I}_X(U)$ that generate $\mathfrak{I}_{X,x}$ for every $x \in U$, with $s = n - \dim X_a$. By a well known result of B. Malgrange (see [4]), they generate $\mathfrak{J}_{X,x}$ as $\mathcal{E}_{X,x}$-module. If $f_a \in \mathfrak{I}_{X,a}$ (resp. $\mathfrak{J}_{X,a}$), we can suppose, by further shrinking U, that there exist functions $g_i \in \mathcal{O}_\Omega(U)$ (resp. $g_i \in \mathcal{E}_\Omega(U)$) such that $f = \sum_{i=1}^{s} g_i \theta_i$ and $f(x) \geq 0$ for every $x \in U$. It follows that $0 = d_x f = \sum_{i=1}^{s} g_i(x)\, d_x \theta_i$ for every $x \in U \cap X$, hence $g_i(x) = 0$ for every regular point of dimension $n - s$. Since such points are dense, $g_i \in \mathfrak{I}_\Omega(U)$ (resp. $g_i \in \mathfrak{J}_\Omega(U)$) and the conclusion follows.

DEFINITION 1. – Let X be an analytic subset of an open domain Ω of $\mathbb{R}^n$ and $a \in X$. We say that a differentiable function $f \in \mathcal{E}_\Omega(\Omega)$ is *strongly analytic at a*

if there exists a germ of analytic function $g_a \in \mathcal{O}_{\Omega, a}$ such that $f_a - g_a \in \mathfrak{J}^2_{X, a}$. Of course, if $f_a \in \mathcal{O}_{\Omega, a}$, then f is strongly analytic at a.

We say that a differentiable function $f \in \mathcal{E}_\Omega(\Omega)$ is *strongly analytic on* X if it is strongly analytic at every point of X.

It is not hard to exhibit an example of an analytic function on an analytic subset X that is not strongly analytic. Let us consider an analytic subset X of $\mathbb{R}^n$ such that the ideal $\mathfrak{I}_X$ is generated by an analytic function θ at a point $a \in X$ and the differentiable function defined by $\phi(x) = \exp(-1/\|x-a\|^2)$, for $x \neq a$, and $\phi(a) = 0$. For every analytic function h on a neighbourhood of a, the differentiable function $\phi\theta + h$ is analytic but not strongly analytic at the point a.

LEMMA 4. – *Let X be an analytic subset of an open domain Ω of $\mathbb{R}^n$. For every strongly analytic function f on X there exists an analytic function $g \in \mathcal{O}_\Omega(\Omega)$ such that $f - g \in \Gamma(\Omega, \mathfrak{J}^2_X)$.*

PROOF. – For every $x \in \Omega$ the canonical inclusion $\mathcal{O}_{\Omega, x} \to \mathcal{E}_{\Omega, x}$ is faithfully flat and then, by the cited result of B. Malgrange, $\mathfrak{J}^2_{X, x} \cap \mathcal{O}_{\Omega, x} = \mathfrak{I}^2_{X, x}$. It follows that the sheaf $\mathcal{O}_\Omega / \mathfrak{I}^2_X$ identifies with a subsheaf of $\mathcal{E}_\Omega / \mathfrak{J}^2_X$. Let $\pi : \mathcal{O}_\Omega \to \mathcal{O}_\Omega / \mathfrak{I}^2_X$ and $\tau : \mathcal{E}_\Omega \to \mathcal{E}_\Omega / \mathfrak{J}^2_X$ be the canonical morphisms and let us consider the section $\tau_\Omega(f) \in \Gamma(\Omega, \mathcal{E}_\Omega / \mathfrak{J}^2_X)$; since f is strongly analytic on X and $\tau_x(f_x) = 0$ for every $x \in \Omega - X$, it follows that $\tau_x(f_x) \in \mathcal{O}_{\Omega, x} / \mathfrak{I}^2_{X, x}$ for every $x \in \Omega$ and so $\tau_\Omega(f) \in \Gamma(\Omega, \mathcal{O}_\Omega / \mathfrak{I}^2_X)$. By Cartan's Theorem B (see [2]) there exists an analytic function $g \in \mathcal{O}_\Omega(\Omega)$ such that $\pi_x(g_x) = \tau_x(f_x)$ for every $x \in \Omega$ and so $f - g \in \Gamma(\Omega, \mathfrak{J}^2_X)$.

THEOREM 3. – *Let Ω be an open domain in $\mathbb{R}^n$, X an analytic subset of Ω and $f \in \mathcal{E}_\Omega(\Omega)$ a strongly analytic function on X. For every continuous positive function η on Ω there exists an analytic function $h \in \mathcal{O}_\Omega(\Omega)$ such that*

i) $|D^\alpha f(x) - D^\alpha h(x)| < \eta(x)$, *for* $|\alpha| \leq \dfrac{1}{\eta(x)}$;

ii) $h(x) \geq f(x)$ *for every* $x \in \Omega$;

iii) $h|_X = f|_X$.

PROOF. – By Lemma 4 there exists $g \in \mathcal{O}_\Omega(\Omega)$ such that $f - g \in \Gamma(\Omega, \mathfrak{J}^2_X)$. By Theorem 2 there exists $\delta \in \mathcal{O}_\Omega(\Omega)$ such that $|D^\alpha (f-g)(x) - D^\alpha \delta(x)| < \eta(x)$, for $|\alpha| \leq \dfrac{1}{\eta(x)}$, $\delta(x) \geq f(x) - g(x)$ for every $x \in \Omega$ and $\delta|_X = 0$. The analytic function $h = \delta + g$ satisfies the required conditions.

COROLLARY 1. – *A differentiable function $f \in \mathcal{E}_\Omega(\Omega)$ is approximable, in the strong Whitney's topology, by analytic functions $h \in \mathcal{O}_\Omega(\Omega)$ such* *that* $h|_X = f|_X$ *and* $h(x) \geqslant f(x)$ *for every* $x \in \Omega$ *if and only if it is strongly analytic on X.*

PROOF. – If there exists such an h, then by Lemma 3 the function f is strongly analytic on X. The other implication follows from Theorem 3.

The previous results allow us to say something about the following problem: let X be a closed coherent analytic subset of an open domain Ω of $\mathbb{R}^n$ and let $\lambda \in \mathcal{O}_X(X)$ be a nonnegative function; does there exist a nonnegative function $h \in \mathcal{O}_\Omega(\Omega)$ such that $h|_X = \lambda$? As it is shown in [1], the answer is in general negative: the function $\lambda = x_1$ on the subset $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^3 - x_2^2 = 0\}$ does not even admit any local differential extension on a neighbourhood of $O = (0, 0)$. Indeed, for such an extension f, the germ of $f - x_1$ at O would be a multiple of the generator of $\mathcal{J}_{X,O}$ and f could not be nonnegative in a neighbourhood of O. Moreover it is proved in [1] that there exists a local nonnegative analytic extension of λ if and only if there exists a local nonnegative differentiable extension. In the following corollary we give a necessary and sufficient condition for the global extension of functions defined on locally complete intersection coherent analytic subsets.

COROLLARY 2. – *A nonnegative analytic function* $\lambda \in \mathcal{O}_X(X)$ *extends to a nonnegative analytic function* $h \in \mathcal{O}_\Omega(\Omega)$ *if and only if it extends to a differentiable function f on* Ω *which is a strongly analytic function on X.*

PROOF. – By Theorem 3 there exists $h \in \mathcal{O}_\Omega(\Omega)$ such that $h|_X = f|_X = \lambda$ and $h(x) \geqslant f(x)$ for every $x \in \Omega$.

REMARK 1. – If there exists a nonnegative analytic extension g of λ to an open neighbourhood U of X in Ω, then there exists a nonnegative analytic extension $h \in \mathcal{O}_\Omega(\Omega)$. Indeed, let us consider an open neighbourhood V of X in U such that $\overline{V} \subset U$ and a differentiable function $\phi \in \mathcal{E}(\Omega)$ such that $\phi|_{\overline{V}} = 1$, $\phi(\Omega) \subset [0, 1]$ and $\mathrm{supp}(\phi) \subset U$. By Lemma 3 the differentiable function $f = \phi g \in \mathcal{E}_\Omega(\Omega)$ is strongly analytic on X and extends λ; it follows from Corollary 2 that λ extends to a nonnegative analytic function.

Such an extension g can be determined when it is possible to find a family $(h_i)_{i \in I}$ of nonnegative analytic local extensions of λ to open sets U_i, which there exist by the cited result of [1], such that the functions h_i/h_j define an analytic cocycle on $U = \bigcup_i U_i$. In this situation there exist an analytic line bundle E on U, trivial on X, with an analytic section $s \in \Gamma(U, E)$ such that $s|_X = \lambda$. Since E is trivial on X, by Cartan's Theorem B (see [2]), the constant section 1 on X can be extended, by shrinking U if necessary, to an analytic section $t \in \Gamma(U, E)$ such that $t(x) > 0$ for every $x \in U$. The function $g = s/t$ gives the required extension of λ.

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