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On the Analytic Approximation
of Differentiable Functions from Above.

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Summary. – We determine conditions in order that a differentiable function be approximable from above by analytic functions, being left invariate on a fixed analytic subset which is a locally complete intersection.

1. – Introduction.

Let $X$ be a closed analytic subset of an open domain $\Omega$ of $\mathbb{R}^n$ and let $f$ be a $C^\infty$ differentiable function on $\Omega$. If we want to approximate $f$ by analytic functions, in strong Whitney’s topology, without changing its values on $X$, obviously, a necessary condition is that $f$ is analytic on $X$. Another necessary condition is that $X$ is coherent, otherwise it could not exist any analytic function on $\Omega$ that coincides with $f$ on $X$. In [6] it is proved that for every continuous positive function $\eta : \Omega \to \mathbb{R}$ there exists an analytic function $h$ on $\Omega$ such that $f|_X = h|_X$ and

$$|D^\alpha f(x) - D^\alpha h(x)| < \eta(x) \text{ for } |\alpha| \leq \frac{1}{\eta(x)},$$

where $\alpha \in \mathbb{N}^n$.

In this short note we deal with the conditions that allow us to make such an approximation from above, i.e. we want $h(x) \geq f(x)$ for every $x \in \Omega$. When $X$ is a locally complete intersection subset we find a necessary and sufficient condition in order to obtain the approximation. This allows us to say a little more about the problem, posed by C. Andradas, of the extension of a nonnegative analytic function off $X$.

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2. – Results.

Let \( \mathcal{O}_{\mathbb{R}^n} \) and \( \mathcal{E}_{\mathbb{R}^n} \) be the sheaves of real analytic and, respectively, differentiable \( (C^\infty) \) functions on \( \mathbb{R}^n \).

If \( \Omega \) is an open domain in \( \mathbb{R}^n \), we denote by \( \mathcal{O}_\Omega \) (resp. \( \mathcal{E}_\Omega \)) the sheaf \( \mathcal{O}_{\mathbb{R}^n} \mid_\Omega \) (resp. \( \mathcal{E}_{\mathbb{R}^n} \mid_\Omega \)).

Let \( X \) be a closed analytic subset of \( \Omega \); we denote by \( \mathcal{I}_X \) the ideal sheaf of real analytic functions vanishing on \( X \) and by \( \mathcal{J}_X \) the ideal sheaf of differentiable functions vanishing on \( X \).

As usual, we denote by \( \mathcal{O}_X \) the sheaf \( (\mathcal{O}_\Omega / \mathcal{I}_X) \mid_X \) of the analytic functions on \( X \).

In the following, by an analytic subset \( X \) we always mean a closed locally complete intersection coherent analytic subset so that, by a result of S. Coen ([3]), the ideal \( \mathcal{I}_X \) has finitely many global sections that generate its fiber at each point.

**Lemma 1.** – Let \( \Omega \) be an open domain in \( \mathbb{R}^n \), \( \eta \) a continuous positive function on \( \Omega \), and \( \theta, \theta_1, \ldots, \theta_s \in \mathcal{E}_\Omega(\Omega) \). Then there exists a positive function \( \mu \in \mathcal{E}_\Omega(\Omega) \) such that, for \( |\alpha| \leq 1/\eta(x) \),

\[
\begin{align*}
&i) \quad |D^\alpha(\mu\theta)(x)| < \frac{\eta(x)}{2}; \\
&ii) \quad \mu(x) \sum_{i=1}^s \sum_{\beta=0}^\alpha \left(\begin{array}{c}
\alpha \\
\beta
\end{array}\right) |D^{\alpha-\beta} \theta_i(x)| \leq \frac{\eta(x)}{2}.
\end{align*}
\]

**Proof.** – Let \( (K_v)_{v \in \mathbb{N}} \) be a sequence of compact sets such that \( \Omega = \bigcup_{v} K_v, \)

\( K_0 = \emptyset, K_v \subset K_{v+1}, \) and let \( A_v = K_{v+2} - K_v \). Then \( (A_v)_{v \in \mathbb{N}} \) is a locally finite open covering of \( \Omega \) and every \( x \in \Omega \) has a neighbourhood \( U \) such that \( U \cap A_v = \emptyset \) if \( v \neq p, p+1 \) for one and only one \( p \in \mathbb{N} \). Let \( (\phi_v)_{v \in \mathbb{N}} \) be a differentiable partition of unity such that \( \text{supp}(\phi_v) \subset A_v \).

For every \( v \in \mathbb{N} \), let \( \delta_v \in \mathbb{R}_+ \) be such that \( \delta_v < \inf_{x \in K_{v+2}} \eta(x) \) and let \( q_v \geq 1 \) be a real number strictly bigger than

\[
\sup_{|\alpha| \leq (1/\delta_v)} \|D^\alpha \phi_v\|_{K_{v+3}}, \quad \sup_{|\alpha| \leq (1/\delta_v)} \|D^\alpha \theta\|_{K_{v+3}}, \quad \sup_{|\alpha| \leq (1/\delta_v)} \sum_{i=1}^s \|D^\alpha \theta_i\|_{K_{v+3}}.
\]

We can suppose that \( q_{v+1} \geq q_v \) and we can find a sequence of positive real numbers \( (\varepsilon_v)_{v \in \mathbb{N}} \) such that \( \varepsilon_{v+1} \leq \varepsilon_v \) and

\[
\varepsilon_v q_v^2 \sup_{|\alpha| \leq (1/\delta_v)} \sum_{\beta=0}^\alpha \left(\begin{array}{c}
\alpha \\
\beta
\end{array}\right) < \frac{\delta_v}{4}.
\]

Now, let us consider the differentiable function \( \mu = \sum_{v \in \mathbb{N}} \varepsilon_v \phi_v \), since
we can suppose that $\mu$ is locally equal to $\varepsilon_p \phi_p + \varepsilon_{p+1} \phi_{p+1}$, it is straightforward to check that $\mu$ satisfies the required conditions.

**Theorem 1.** — Let $\Omega$ be an open domain in $\mathbb{R}^n$ and $f \in \mathcal{C}(\Omega)$ a differentiable function. For every continuous positive function $\eta$ on $\Omega$ there exists an analytic function $h \in \mathcal{C}_\Omega(\Omega)$ such that

i) $|D^\alpha f(x) - D^\alpha h(x)| < \eta(x)$, for $|\alpha| \leq \frac{1}{\eta(x)}$;

ii) $h(x) > f(x)$ for every $x \in \Omega$.

**Proof.** — It follows from Lemma 1 that there exists a positive differentiable function $\mu \in \mathcal{C}(\Omega)$ such that $|D^\alpha(\mu)(x)| < \frac{\eta(x)}{2}$, for $|\alpha| \leq \frac{1}{\eta(x)}$. By Whitney’s approximation theorem (see [5]) there exists an analytic function $h \in \mathcal{C}_\Omega(\Omega)$ such that

$$|D^\alpha(f + \mu)(x) - D^\alpha h(x)| < \frac{\mu(x)}{2} \quad \text{for} \quad |\alpha| \leq \frac{2}{\mu(x)}.$$ 

It is easy to check that $h$ satisfies the required conditions.

**Lemma 2.** — Let $\Omega$ be an open domain in $\mathbb{R}^n$, $\mathfrak{J}$ a coherent ideal of $\mathcal{C}_\Omega$ generated by finitely many global sections $\theta_1, \ldots, \theta_s$, $\mathfrak{J}$ the ideal generated by $\mathfrak{J}$ in $\mathcal{C}_\Omega$ and $f \in \Gamma(\Omega, \mathfrak{J})$. For every continuous positive function $\eta$ on $\Omega$ there exists an analytic function $h \in \Gamma(\Omega, \mathfrak{J})$ such that

$$|D^\alpha f(x) - D^\alpha h(x)| < \eta(x), \quad \text{for} \quad |\alpha| \leq \frac{1}{\eta(x)}.$$ 

Moreover, if $f \in \Gamma(\Omega, \mathfrak{J}^2)$ it is possible to find $h \in \Gamma(\Omega, \mathfrak{J}^2)$ such that $h(x) \geq f(x)$ for every $x \in \Omega$.

**Proof.** — There exist differentiable functions $f_1, \ldots, f_s$ on $\Omega$ such that $f = \sum_{i=1}^s f_i \theta_i$. As in Lemma 1, let $\mu$ be a differentiable function such that $\mu(x) < \eta(x)$ for every $x \in \Omega$ and $\mu(x) \sum_{i=1}^s \sum_{\beta=0}^\alpha \left(\frac{\alpha}{\beta}\right) |D^{\alpha-\beta} \theta_i(x)| \leq \frac{1}{\eta(x)}$. By Theorem 1, for every $i = 1, \ldots, s$, there exist analytic functions $h_i \in \mathcal{C}_\Omega(\Omega)$ such that $|D^\alpha f_i(x) - D^\alpha h_i(x)| < \mu(x)$, for $|\alpha| \leq \frac{1}{\mu(x)}$ and $h_i(x) > f_i(x)$ for every $x \in \Omega$. It is easy to see that the analytic function $h = \sum_{i=1}^s h_i \theta_i$ satisfies the first condition.

Moreover, if $f$ is a section of $\mathfrak{J}^2$, then, by replacing the functions $\theta_i$ by the functions $(\theta_i + \theta_j)^2$, $i, j = 1, \ldots, s$, we can suppose that they are nonnegative on $\Omega$, and so the second condition is satisfied too.
THEOREM 2. – Let $\Omega$ be an open domain in $\mathbb{R}^n$, $X$ an analytic subset of $\Omega$ and $f \in \mathcal{E}_\Omega(\Omega)$ a differentiable function such that $f_x \in \mathcal{E}_{X,x}$ for all $x \in X$. For every continuous positive function $\eta$ on $\Omega$ there exists an analytic function $h \in \mathcal{O}_\Omega(\Omega)$ such that

i) $|D^\alpha f(x) - D^\alpha h(x)| < \eta(x)$, for $|\alpha| \leq \frac{1}{\eta(x)}$;

ii) $h(x) \geq f(x)$ for every $x \in \Omega$;

iii) $h|_X = 0$.

Moreover, if $f$ is nonnegative on $\Omega$, $h$ can be chosen such that $X = h^{-1}(0)$.

PROOF. – Since $f$ is in $\Gamma(\Omega, \mathcal{J}^2)$, by Lemma 2 we only need to prove that, if $f$ is nonnegative, it is possible to find $h$ such that $X = h^{-1}(0)$.

Since $X$ is coherent, there exists a nonnegative function $\theta \in \mathcal{O}_\Omega(\Omega)$ such that $X = \theta^{-1}(0)$, and, by Lemma 1, there exists a positive differentiable function $\mu \in \mathcal{E}_\Omega(\Omega)$ such that $|D^\alpha(\mu\theta)(x)| < \frac{\eta(x)}{2}$, for $|\alpha| \leq \frac{1}{\eta(x)}$. On the other hand, as in the proof of Lemma 2, applied to the ideal generated by $\theta$, there exists an analytic function $\delta \in \mathcal{O}_\Omega(\Omega)$ such that $|D^\alpha(\mu\theta)(x) - D^\alpha(\delta\theta)(x)| < \frac{\eta(x)}{4}$, for $|\alpha| \leq \frac{1}{\eta(x)}$. It follows that $|D^\alpha(\delta\theta)(x)| < \frac{\eta(x)}{2}$ for $|\alpha| \leq \frac{1}{\eta(x)}$.

Now let us consider the analytic function $g = h + \delta\theta$; it follows immediately that $X = g^{-1}(0)$ and that $g(x) \geq h(x) \geq f(x)$ for every $x \in \Omega$. Moreover, for $|\alpha| \leq \frac{1}{\eta(x)}$, we have $|D^\alpha f(x) - D^\alpha g(x)| < \eta(x)$ and then, by replacing $h$ by $g$, we get the conclusion.

LEMMA 3. – Let $X$ be an analytic subset of an open domain $\Omega$ of $\mathbb{R}^n$:

i) $\{f_a \in \mathcal{J}_{X,a} | f_a \geq 0\} \subset \mathcal{J}^2_{X,a}$;

ii) $\{f_a \in \mathcal{J}_{X,a} | f_a \geq 0\} \subset \mathcal{J}^2_{X,a}$.

PROOF. – There exist an open neighbourhood $U$ of $a$ and functions $\theta_1, \ldots, \theta_s \in \mathcal{J}_X(U)$ that generate $\mathcal{J}_{X,x}$ for every $x \in U$, with $s = n - \dim X_a$. By a well known result of B. Malgrange (see [4]), they generate $\mathcal{J}_{X,x}$ as $\mathcal{E}_{X,x}$-module. If $f_a \in \mathcal{J}_{X,a}$ (resp. $\mathcal{J}_{X,a}$), we can suppose, by further shrinking $U$, that there exist functions $g_i \in \mathcal{O}_\Omega(U)$ (resp. $g_i \in \mathcal{E}_\Omega(U)$) such that $f = \sum_{i=1}^{s} g_i \theta_i$ and $f(x) \geq 0$ for every $x \in U$. It follows that $0 = d_x f = \sum_{i=1}^{s} g_i(x) d_x \theta_i$ for every $x \in U \cap X$, hence $g_i(x) = 0$ for every regular point of dimension $n - s$. Since such points are dense, $g_i \in \mathcal{J}_\Omega(U)$ (resp. $g_i \in \mathcal{J}_\Omega(U)$) and the conclusion follows.

DEFINITION 1. – Let $X$ be an analytic subset of an open domain $\Omega$ of $\mathbb{R}^n$ and $a \in X$. We say that a differentiable function $f \in \mathcal{E}_\Omega(\Omega)$ is strongly analytic at $a$.
if there exists a germ of analytic function $g_a \in \mathcal{O}_X, a$ such that $f_a - g_a \in \mathfrak{j}^2_{X, a}$. Of course, if $f_a \in \mathcal{O}_X, a$, then $f$ is strongly analytic at $a$.

We say that a differentiable function $f \in \mathcal{E}_X(\Omega)$ is strongly analytic on $X$ if it is strongly analytic at every point of $X$.

It is not hard to exhibit an example of an analytic function on an analytic subset $X$ that is not strongly analytic. Let us consider an analytic subset $X$ of $\mathbb{R}^n$ such that the ideal $\mathfrak{j}_X$ is generated by an analytic function $\theta$ at a point $a \in X$ and the differentiable function defined by $\phi(x) = \exp(-1/\|x - a\|^2)$, for $x \neq a$, and $\phi(a) = 0$. For every analytic function $h$ on a neighborhood of $a$, the differentiable function $\phi h + h$ is analytic but not strongly analytic at the point $a$.

**Lemma 4.** Let $X$ be an analytic subset of an open domain $\Omega$ of $\mathbb{R}^n$. For every strongly analytic function $f$ on $X$ there exists an analytic function $g \in \mathcal{O}_X(\Omega)$ such that $f - g \in \Gamma(\Omega, \mathfrak{j}^2_X)$.

**Proof.** For every $x \in \Omega$ the canonical inclusion $\mathcal{O}_{X, x} \to \mathcal{E}_{X, x}$ is faithfully flat and then, by the cited result of B. Malgrange, $\mathcal{O}_{X, x} \cap \mathcal{O}_X = \mathfrak{j}^2_{X, x}$. It follows that the sheaf $\mathcal{O}_X/\mathfrak{j}^2_X$ identifies with a subsheaf of $\mathcal{E}_X/\mathfrak{j}^2_X$. Let $\pi : \mathcal{E}_X \to \mathcal{O}_X/\mathfrak{j}^2_X$ and $\tau : \mathcal{E}_X \to \mathcal{E}_X/\mathfrak{j}^2_X$ be the canonical morphisms and let us consider the section $\tau(f) \in \Gamma(\Omega, \mathcal{O}_X/\mathfrak{j}^2_X)$; since $f$ is strongly analytic on $X$ and $\tau_x(f_x) = 0$ for every $x \in \Omega - X$, it follows that $\tau_x(f_x) \in \mathcal{O}_{X, x}/\mathfrak{j}^2_{X, x}$ for every $x \in \Omega$ and so $\tau(f) \in \Gamma(\Omega, \mathcal{O}_X/\mathfrak{j}^2_X)$. By Cartan’s Theorem B (see [2]) there exists an analytic function $g \in \mathcal{O}_X(\Omega)$ such that $\pi_x(g_x) = \tau_x(f_x)$ for every $x \in \Omega$ and so $f - g \in \Gamma(\Omega, \mathfrak{j}^2_X)$.

**Theorem 3.** Let $\Omega$ be an open domain in $\mathbb{R}^n$, $X$ an analytic subset of $\Omega$ and $f \in \mathcal{E}_X(\Omega)$ a strongly analytic function on $X.$ For every continuous positive function $\eta$ on $\Omega$ there exists an analytic function $h \in \mathcal{O}_X(\Omega)$ such that

i) $|D^\alpha f(x) - D^\alpha h(x)| < \eta(x), \text{ for } |\alpha| \leq 1/\eta(x);$  
ii) $h(x) \geq f(x)$ for every $x \in \Omega$;  
iii) $h|_X = f|_X$.

**Proof.** By Lemma 4 there exists $g \in \mathcal{O}_X(\Omega)$ such that $f - g \in \Gamma(\Omega, \mathfrak{j}^2_X)$. By Theorem 2 there exists $\delta \in \mathcal{O}_X(\Omega)$ such that $|D^\alpha(f - g)(x) - D^\alpha \delta(x)| < \eta(x)$, for $|\alpha| \leq 1/\eta(x)$, $\delta(x) \geq f(x) - g(x)$ for every $x \in \Omega$ and $\delta|_X = 0$. The analytic function $h = \delta + g$ satisfies the required conditions.

**Corollary 1.** A differentiable function $f \in \mathcal{E}_X(\Omega)$ is approximable, in the strong Whitney’s topology, by analytic functions $h \in \mathcal{O}_X(\Omega)$ such
that \( h|_X = f|_X \) and \( h(x) \geq f(x) \) for every \( x \in \Omega \) if and only if it is strongly analytic on \( X \).

**Proof.** – If there exists such an \( h \), then by Lemma 3 the function \( f \) is strongly analytic on \( X \). The other implication follows from Theorem 3.

The previous results allow us to say something about the following problem: let \( X \) be a closed coherent analytic subset of an open domain \( \Omega \) of \( \mathbb{R}^n \) and let \( \lambda \in \mathcal{O}_X(X) \) be a nonnegative function; does there exist a nonnegative function \( h \in \mathcal{O}_\Omega(\Omega) \) such that \( h|_X = \lambda \)? As it is shown in [1], the answer is in general negative: the function \( \lambda = x_1 \) on the subset \( X = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 - x_2^2 = 0\} \) does not even admit any local differential extension on a neighbourhood of \( O = (0, 0) \). Indeed, for such an extension \( f \), the germ of \( f - x_1 \) at \( O \) would be a multiple of the generator of \( \mathcal{O}_{X, O} \) and \( f \) could not be nonnegative in a neighbourhood of \( O \). Moreover it is proved in [1] that there exists a local nonnegative analytic extension of \( \lambda \) if and only if there exists a local nonnegative differentiable extension. In the following corollary we give a necessary and sufficient condition for the global extension of functions defined on locally complete intersection coherent analytic subsets.

**Corollary 2.** – A nonnegative analytic function \( \lambda \in \mathcal{O}_X(X) \) extends to a nonnegative analytic function \( h \in \mathcal{O}_\Omega(\Omega) \) if and only if it extends to a differentiable function \( f \) on \( \Omega \) which is a strongly analytic function on \( X \).

**Proof.** – By Theorem 3 there exists \( h \in \mathcal{O}_\Omega(\Omega) \) such that \( h|_X = f|_X = \lambda \) and \( h(x) \geq f(x) \) for every \( x \in \Omega \).

**Remark 1.** – If there exists a nonnegative analytic extension \( g \) of \( \lambda \) to an open neighbourhood \( U \) of \( X \) in \( \Omega \), then there exists a nonnegative analytic extension \( h \in \mathcal{O}_\Omega(\Omega) \). Indeed, let us consider an open neighbourhood \( V \) of \( X \) in \( U \) such that \( V \subseteq U \) and a differentiable function \( \phi \in \mathcal{E}(\Omega) \) such that \( \phi|_\Omega = 1 \), \( \phi(\Omega) \subseteq [0, 1] \) and \( \text{supp} (\phi) \subseteq U \). By Lemma 3 the differentiable function \( f = \phi g \in \mathcal{E}_\Omega(\Omega) \) is strongly analytic on \( X \) and extends \( \lambda \); it follows from Corollary 2 that \( \lambda \) extends to a nonnegative analytic function.

Such an extension \( g \) can be determined when it is possible to find a family \( (h_i)_{i \in I} \) of nonnegative analytic local extensions of \( \lambda \) to open sets \( U_i \), which there exist by the cited result of [1], such that the functions \( h_i/h_j \) define an analytic cocycle on \( U = \bigcup_i U_i \). In this situation there exist an analytic line bundle \( E \) on \( U \), trivial on \( X \), with an analytic section \( s \in \mathcal{I}(U, E) \) such that \( s|_X = \lambda \). Since \( E \) is trivial on \( X \), by Cartan’s Theorem B (see [2]), the constant section 1 on \( X \) can be extended, by shrinking \( U \) if necessary, to an analytic section \( t \in \mathcal{I}(U, E) \) such that \( t(x) > 0 \) for every \( x \in U \). The function \( g = s/t \) gives the required extension of \( \lambda \).
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