Zindine Djadli, Antoinette Jourdain

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Nodal Solutions for Scalar Curvature Type Equations with Perturbation Terms on Compact Riemannian Manifolds.

ZINDINE DJADLI - ANTOINETTE JOURDAIN

Sunto. – *L'oggetto del presente articolo è lo studio delle soluzioni soggette a cambiamenti di segno delle equazioni di tipo curvatura scalare a perturbazione. I principali risultati in esso contenuti riguardano l'esistenza di tali soluzioni e la determinazione puntuale del loro insieme degli zeri. Da ciò deduciamo, in alcuni casi, dei risultati di molteplicità.*

Summary. – *In this paper we study the nodal solutions for scalar curvature type equations with perturbation. The main results concern the existence of such solutions and the exact description of their zero set. From this we deduce, in particular cases, some multiplicity results.*

1. – Introduction.

Let (M, g) be a compact riemannian n-manifold, $n \geqslant 3$, and denote by $Scal_g$ the scalar curvature of g. Let $\tilde{g}$ be some conformal metric to g, that is $\tilde{g} = u^{4/(n-2)} g$ for some $u \in C^\infty(M)$, $u > 0$. As well known, one has that

$$\Delta_g u + \frac{(n-2)}{4(n-1)} Scal_g u = \frac{n-2}{4(n-1)} Scal_{\tilde{g}} u^{\frac{n+2}{n-2}}$$

where Δ_g is the Laplacian of g with the minus sign convention and $Scal_{\tilde{g}}$ is the scalar curvature of $\tilde{g}$. A natural and interesting generalization of this equation is the equation

$$(E_1) \qquad \Delta_g u + au = fu^{\frac{n+2}{n-2}} + hu^q$$

where a, f and h are smooth functions, and $q \in \left(1; \frac{n+2}{n-2}\right)$. Such an equation has been studied by various authors. Among others, let us mention Brézis and Nirenberg [4] where (E_1) is studied in the Euclidean context, and Djadli [8], [9] where (E_1) is studied in the Riemannian context. The goal of this paper is to study the existence of nodal solutions to (E_1) on a compact Riemannian manifold, (M, g), with or without boundary. Given a, f and h three smooth functions on M, and $q \in \left(1; \frac{n+2}{n-2}\right)$ real, we look for changing sign solutions u to

$$(E) \qquad \Delta_g u + au = f|u|^{\frac{4}{n-2}} u + h|u|^{q-1} u$$

If M has a boundary, we look for solutions satisfying a Dirichlet condition on the boundary, that is $u \equiv 0$ on ∂M. The problem of finding nodal solutions of equation (E) has been studied by several authors (among others Atkinson-Brézis-Peletier [2], Cao-Noussair [6], Hebey-Vaugon [13], Musso-Passaseo [16] and Tarantello [17]). We also refer to the paper of Fortunato-Jannelli [10]. As far as we know, there is no result about the zero set. A main point here is that we do get, in fairly general situations, the exact description of the zero set of the solutions.

The variational method by minimization, as used by Jourdain [14] or Hebey [11], does not work for such an equation because of the Euler-Lagrange multipliers that appear in this kind of approach. To overcome this difficulty, we use instead a variational method based on the Mountain-Pass lemma of Ambrosetti and Rabinowitz [1], as developed in Brézis and Nirenberg [4] (see also Djadli [9] for an example of an application in the Riemannian context). Such an approach will be developed together with ideas taken from the isometry-concentration method as presented in Hebey [11]. This will allow us to prescribe the zero set of the nodal solutions we obtain, and hence to get multiplicity results for equation (E) in some particular cases.

Notations and remarks. In this paper, we let a, f and h be three smooth functions; we set $p = \frac{n+2}{n-2}$ and let $q \in (1; p)$ be a real number. By simplicity we take a perturbation of the form hu^q. Nevertheless, our results can be extended to more general perturbations satisfying suitable assumptions.

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2. – Terminology and general notations.

Let (M, g) be a smooth compact Riemannian manifold, with or without boundary ∂M. For seek of simplicity, we use the following notation

$$W^{1,2}(M) = \begin{cases} H_1^2(M) & \text{if } \partial M = \emptyset \\ \overset{\circ}{H}{}_1^2(M) & \text{if } \partial M \neq \emptyset \end{cases}$$

where $H_1^2(M)$ is the completion of $C^\infty(M)$ with respect to the norm

$$\|u\|_{W^{1,2}} = \sqrt{\|\nabla u\|_2^2 + \|u\|_2^2}$$

and $\overset{\circ}{H}{}_1^2(M)$ is the completion of $C_0^\infty(M)$ with respect to the same norm. Let G be a subgroup of the isometry group of (M, g), denoted by $Isom(M, g)$. Without loss of generality, up to replacing G by its closure in $Isom(M, g)$, we can assume that G is a compact. We also consider τ an involutive isometry of (M, g), that is an element of $Isom(M, g)$ such that $\tau \circ \tau = Id_M$. For x a point of M, we denote by $O_G(x)$ the orbit of x under the action of G,

$$O_G(x) = \{\sigma(x), \sigma \in G\}$$

By definition, we say that G and τ commute weakly if for all $x \in M$

$$\tau(O_G(x)) = O_G(\tau(x))$$

We also say that the set of the fixed points of τ splits M into two domains Ω_1 and Ω_2, stable under the action of G, if

(i) $M = \Omega_1 \cup \Omega_2 \cup \mathcal{F}_\tau$ with $\Omega_1 \cap \Omega_2 = \emptyset$ and *measure* $(\mathcal{F}_\tau) = 0$

(ii) $\tau(\Omega_1) = \Omega_2$ and $\forall \sigma \in G$, $\forall i = 1, 2$ $\sigma(\Omega_i) = \Omega_i$

where $\mathcal{F}_\tau$ denotes the set of the fixed points of τ, that is

$$\mathcal{F}_\tau = \{x \in M, \ \tau(x) = x\}$$

We say that a function $u \in W^{1,2}(M)$ is τ-antisymmetrical if $u \circ \tau = -u$ a.e., and G-invariant if for all $\sigma \in G$, $u \circ \sigma = u$ a.e. In what follows, we denote by *Card* the cardinality of a set (even when the set is infinite). By definition, a function $u \in C^2(M)$ is said to be a solution of (E) if u satisfies (E) pointwise and $u \equiv 0$ on ∂M (if M has a boundary). Finally, we say that an operator L defined on $W^{1,2}(M)$ is coercive on a certain subspace X of $W^{1,2}(M)$ if there exists $k \in \mathbb{R}^{+*}$ such that for all $u \in X$

$$\int_M L(u).\, u \, dv(g) \geq k \|u\|^2_{W^{1,2}(M)}$$

Note that the operator $\Delta_g + a$ is clearly coercive on $W^{1,2}(M)$ if a is a positive function.

3. – A general theorem of existence.

Let G be a compact subgroup of the isometry group of (M, g) and let τ be an involutive isometry of (M, g). We assume that G and τ commute weakly, and that for some $x_1 \in M$, $\tau(O_G(x_1)) \cap O_G(x_1) = \emptyset$. Then,

$$\mathcal{H} = \{u \in W^{1,2}(M),\ u \text{ is } G\text{-invariant and } \tau - \text{antisymmetrical}\}$$

is not a trivial set, i.e $\mathcal{H} \neq \{0\}$. See Jourdain [14] for the proof of this claim.

The purpose of this section is to prove the following existence result

THEOREM 3.1. – *Let G be a compact subgroup of the isometry group of (M, g), $n \geqslant 3$, let τ be an involutive isometry of (M, g) such that G and τ commute weakly and $\tau(O_G(x_1)) \cap O_G(x_1) = \emptyset$ for some $x_1 \in M$, and let a, f and h be three smooth G-invariant and τ-invariant functions. We assume that f is positive on M and that the operator $\Delta_g + a$ is coercive on $\mathcal{H}$ (where $\mathcal{H}$ is as above). We set $p = \frac{n+2}{n-2}$ and let $q \in (1; p)$. We consider the following functional defined on $\mathcal{H}$ by*

$$J(\varphi) = \int_M \left\{ \frac{1}{2} |\nabla \varphi|^2 + \frac{1}{2} a\varphi^2 - \frac{1}{p+1} f|\varphi|^{p+1} - \frac{1}{q+1} h|\varphi|^{q+1} \right\} dv(g)$$

We assume that for all x in M there exists $v \in \mathcal{H}$, $v \not\equiv 0$, such that

$$\sup_{t \geqslant 0} \{J(tv)\} < \frac{\mathit{Card}\, O_{\langle G, \tau \rangle}(x)}{nK(n, 2)^n (f(x))^{\frac{n-2}{2}}} \qquad (\star)$$

where $\langle G, \tau \rangle$ denotes the subgroup of $\mathrm{Isom}(M, g)$ generated by G and τ and where $K(n, 2)$ is the best constant in the Sobolev embedding of $W^{1,2}(M)$ in $L^{\frac{2n}{n-2}}(M)$. Then (E) possesses a nodal solution $u \in C^2(M)$, which is G-invariant and τ-antisymmetrical.

Moreover, if we assume that $\mathcal{F}_\tau$ splits M into two domains Ω_1 and Ω_2 stable under the action of G, we can choose u such that the zero set of u is exactly $\mathcal{F}_\tau \cup \partial M$, where $\mathcal{F}_\tau$ is the set of the fixed points of τ.

The proof of this theorem proceeds in several steps. Its first part, where u is just assumed to be G-invariant and τ-antisymmetrical, but the zero set of u is not described, could have been proved with standard arguments. Because of the second part, where the zero set of u is described, we have to be more subtle. A main tool here is the so-called deformation lemma. We recall its statement below, and refer to Brézis and Nirenberg [5] for its proof.

LEMMA 3.2. – *Let X be a Banach space, F a function of class C^1 defined on X and $c \in \mathbb{R}$. We set $F_c = \{u \in X / F(u) \leqslant c\}$. Then for any given $\delta < \frac{1}{8}$ there exists a continuous deformation $\eta : [0; 1] \times X \to X$ such that*

(1) $\eta(0, u) = u$ *for all* $u \in X$

(2) $\eta(t, .)$ *is a homeomorphism of X onto X for all* $t \in [0; 1]$

(3) $\eta(t, u) = u$ *for all* $t \in [0; 1]$ *if* $|F(u) - c| \geqslant 2\delta$ *or if* $\|F'(u)\| \leqslant \sqrt{\delta}$

(4) $0 \leqslant F(u) - F(\eta(t, u)) \leqslant 4\sqrt{\delta}$ *for all* $u \in X$, *for all* $t \in [0; 1]$

(5) $\|\eta(t, u) - u\| \leqslant 16\sqrt{\delta}$ *for all* $u \in X$, *for all* $t \in [0; 1]$

(6) *If* $u \in F_{c+\delta}$ *then either*

(*i*) $\eta(1, u) \in F_{c-\delta}$ *or*

(*ii*) *for some* $t_1 \in [0; 1]$ *we have* $\|F'(\eta(t_1, u))\| < 2\sqrt{\delta}$

Now, following in a certain sense a strategy initiated by Yamabe [19], we prove the existence of a nodal solution for (E) when p is replaced by a sub-critical exponent. First we introduce the modified energy corresponding to this change; let ε_0 be such that $0 < \varepsilon_0 < p - q$, and for $0 < \varepsilon \leqslant \varepsilon_0$, we consider the C^1 functional J_ε defined on $\mathcal{H}$ by

$$J_\varepsilon(\varphi) = \int_M \left\{ \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}a\varphi^2 - \frac{1}{p-\varepsilon+1}f|\varphi|^{p-\varepsilon+1} - \frac{1}{q+1}h|\varphi|^{q+1} \right\} dv(g)$$

LEMMA 3.3. – *Let G be a compact subgroup of the isometry group of (M, g), $n \geqslant 3$, let τ be an involutive isometry of (M, g), such that G and τ commute weakly and such that, for some $x_1 \in M$*

$$\tau(O_G(x_1)) \cap O_G(x_1) = \emptyset$$

Let also a, f and h be three smooth G-invariant and τ-invariant functions. We assume that f is positive on M and that the operator $\Delta_g + a$ is coercive on $\mathcal{H}$. We set $p = \frac{n+2}{n-2}$ and let $q \in (1; p)$. Let ε_0 be such that $0 < \varepsilon_0 < p - q$. Then for all ε such that $0 < \varepsilon \leqslant \varepsilon_0$, there exists $u_\varepsilon \in C^2(M)$, G-invariant and τ-antisymmetrical, $u_\varepsilon \not\equiv 0$ in M and $u_\varepsilon \equiv 0$ on ∂M, which is a nodal solution of

$$(E_\varepsilon) \qquad \Delta_g u_\varepsilon + a u_\varepsilon = f|u_\varepsilon|^{p-\varepsilon-1}u_\varepsilon + h|u_\varepsilon|^{q-1}u_\varepsilon$$

In addition, for all $0 < \varepsilon \leqslant \varepsilon_0$

$$J_\varepsilon(u_\varepsilon) \leqslant \inf_{u \in \mathcal{H}\, u \not\equiv 0} \sup_{t \geqslant 0} J_\varepsilon(tu) \quad \text{and} \quad J_\varepsilon(u_\varepsilon) \geqslant \varrho$$

for some positive ϱ independent of ε. Moreover, if we assume that $\mathcal{F}_\tau$ splits M into two domains Ω_1 and Ω_2 stable under the action of G, we can choose u_ε such that its zero set is exactly $\mathcal{F}_\tau \cup \partial M$, where $\mathcal{F}_\tau$ is the set of the fixed points of τ.

The proof of this lemma is divided into two parts. In part 1 we prove the existence of u_ε without the assumption that $\mathcal{F}_\tau$ splits M into two domains Ω_1 and Ω_2 stable under the action of G. In part 2, we prove that under this additional assumption, we can prescribe the zero set of u_ε.

PROOF OF LEMMA 3.3 (part 1). – Following Brézis and Nirenberg [4], one easily gets that the assumptions of the mountain-pass lemma are satisfied; that is, we can find a ball U of radius independent of ε around 0 in $\mathcal{H}$, such that $U \subset B_0(1)$, and a positive real number ϱ, independent of ε, such that

$$\forall \varphi \in \partial U,\ \forall \varepsilon / 0 < \varepsilon \leqslant \varepsilon_0, \qquad J_\varepsilon(\varphi) \geqslant \varrho > 0$$

Clearly, for all ε, $J_\varepsilon(0) = 0 < \varrho$. Notice now that given a non zero $\varphi \in \mathcal{H}$, there exists a t_0 independent on ε such that $J(t\varphi) < 0$ for all $t \geqslant t_0$. As mentionned above, the assumptions of the mountain-pass lemma are satisfied. As a consequence, there exists a sequence $(u_j) \in \mathcal{H}^{\mathbb{N}}$ such that

$$\begin{cases} J_\varepsilon(u_j) \to c_\varepsilon \\ J'_\varepsilon(u_j) \to 0 \ \text{ strongly in } \mathcal{H}' \end{cases}$$

where c_ε is defined by

$$c_\varepsilon = \inf_{\gamma \in \mathfrak{P}} \max_{w \in \gamma} J_\varepsilon(w) \geqslant \varrho$$

$\mathfrak{P}$ being the set of all continuous paths joining 0 and φ. One can easily see that the minmax level c_ε corresponding to endpoint $t_0\varphi$ is in fact independent on φ, and hence we have

$$c_\varepsilon \leqslant \inf_{u \in \mathcal{H}\ u \not\equiv 0} \sup_{t \geqslant 0} J_\varepsilon(tu)$$

Once again following Brézis and Nirenberg [4], the sequence (u_j) is bounded in $\mathcal{H}$. So we can extract a subsequence, still denoted by (u_j), such that

$$\begin{cases} u_j \rightharpoonup u_\varepsilon & \text{weakly in } \mathcal{H} \\ u_j \to u_\varepsilon & \text{strongly in } L^r(M) \text{ for all given } r < p+1 \\ u_j \to u_\varepsilon & \text{a.e. in } M \end{cases}$$

Clearly u_ε verifies weakly in $\mathcal{H}$

$$\Delta_h u_\varepsilon + a u_\varepsilon = f |u_\varepsilon|^{p-\varepsilon-1} u_\varepsilon + h |u_\varepsilon|^{q-1} u_\varepsilon$$

There is still to prove that u_ε is a weak solution of this equation in $W^{1,2}(M)$. For this we refer to Jourdain [14]. The result follows from a symmetrization argument via the Haar measure. By classical regularity $u_\varepsilon \in C^2(M)$; it follows, since $u_\varepsilon \in C^2(M) \cap W^{1,2}(M)$, that $u_\varepsilon \equiv 0$ on ∂M. Moreover, since $u_j \in \mathcal{H}$ and $u_j \to u_\varepsilon$ a.e., u_ε is G-invariant and τ-antisymmetrical. As one can easily check, $J_\varepsilon(u_\varepsilon) = c_\varepsilon \geqslant \varrho > 0$. Hence, $u_\varepsilon \not\equiv 0$. This proves the first part of lemma 3.3.

PROOF OF LEMMA 3.3 (part 2). – In order to prove the second part in lemma 3.3, we use the deformation lemma 3.2. For all $u \in \mathcal{H}$, we consider the following transformation $u \to \tilde{u}$ where $\tilde{u}$ is defined by

$$\begin{cases} \tilde{u}(x) = \ \ |u(x)| & \text{if } x \in \Omega_1 \\ \tilde{u}(x) = \ \ 0 & \text{if } x \in \mathcal{T}_\tau \\ \tilde{u}(x) = -|u(x)| & \text{if } x \in \Omega_2 \end{cases}$$

Since $M = \Omega_1 \cup \Omega_2 \cup \mathcal{T}_\tau$ with $measure(\mathcal{T}_\tau) = 0$, $\tilde{u} \in \mathcal{H}$. Let $\varphi \in \mathcal{H}$ and let $\mathfrak{P}$ be the set of all continuous paths in $\mathcal{H}$, defined on $[0; 1]$, joining 0 to $\tilde{\varphi}$. We consider once again

$$c_\varepsilon = \inf_{\Gamma \in \mathfrak{P}} \max_{s \in [0;1]} J_\varepsilon(\Gamma(s))$$

Let $\delta_j = \frac{1}{j}$ for $j > 8$. There exists $\Gamma_1^j \in \mathfrak{P}$ such that

$$\max_{s \in [0;1]} J_\varepsilon\big(\Gamma_1^j(s)\big) \leqslant c_\varepsilon + \delta_j$$

We consider

$$\Gamma_2^j : [0; 1] \to \mathcal{H}$$

$$s \to \widetilde{\Gamma_1^j(s)}$$

Clearly $\Gamma_2^j \in C([0; 1], \mathcal{H})$. As a consequence (and since the Dirichlet integral decreases along the transformation $u \to \tilde{u}$)

$$\max_{s \in [0;1]} J_\varepsilon(\Gamma_2^j(s)) \leqslant c_\varepsilon + \delta_j$$

and $\Gamma_2^j \in \mathfrak{P}$. According to lemma 3.2, there exists a continuous deformation $\eta_j : [0; 1] \times \mathcal{H} \to \mathcal{H}$ such that for all $t \in [0; 1]$ and all $s \in [0; 1]$

$$0 \leqslant J_\varepsilon\big(\Gamma_2^j(s)\big) - J_\varepsilon\big(\eta_j(t, \Gamma_2^j(s))\big) \leqslant 4\sqrt{\delta_j} \tag{3.3.1}$$

Hence

$$\forall t \in [0; 1],\ \ \forall s \in [0; 1],\ \ J_\varepsilon\big(\eta_j(t, \Gamma_2^j(s))\big) \leqslant J_\varepsilon\big(\Gamma_2^j(s)\big)$$

and it follows that

$$\forall t \in [0; 1],\ \ \max_{s \in [0;1]} J_\varepsilon\big(\eta_j(t, \Gamma_2^j(s))\big) \leqslant c_\varepsilon + \delta_j$$

Let s_1^j be such that

$$J_\varepsilon\big(\eta_j(1, \Gamma_2^j(s_1^j))\big) = \max_{s \in [0;1]} J_\varepsilon\big(\eta_j(1, \Gamma_2^j(s))\big)$$

According to lemma 3.2, either

$$\begin{aligned} &(i)\ \ J_\varepsilon\big(\eta_j(1,\,\Gamma_2^j(s_1^j))\big)\leqslant c_\varepsilon-\delta_j\\ \text{or }&(ii)\ \ \exists t_1^j\in[0;1]\ \text{ such that }\ \|J'_\varepsilon\big(\eta_j(t_1^j,\,\Gamma_2^j(s_1^j))\big)\|\leqslant 2\sqrt{\delta_j}\end{aligned}$$

while by point (3) of lemma 3.2, $\eta_j(1,\,\Gamma_2^j(1))=\tilde{\varphi}$. Then $s\to\eta_j\big(1,\,\Gamma_2^j(s)\big)$ is a continuous path joining 0 to $\tilde{\varphi}$, and it follows that

$$J_\varepsilon\big(\eta_j(1,\,\Gamma_2^j(s_1^j))\big)\geqslant c_\varepsilon$$

This means that (i) does not occur (thus, (ii) holds). Moreover, according to (3.3.1), with $t=1$,

$$J_\varepsilon\big(\eta_j(1,\,\Gamma_2^j(s))\big)\leqslant J_\varepsilon\big(\Gamma_2^j(s)\big),\ \ \forall s\in[0;\,1]$$

Taking $s=s_1^j$ in this inequality, we deduce

$$c_\varepsilon\leqslant J_\varepsilon\big(\eta_j(1,\,\Gamma_2^j(s_1^j))\big)\leqslant J_\varepsilon\big(\Gamma_2^j(s_1^j)\big)$$

Using once again (3.3.1) with $t=t_1^j$ and $s=s_1^j$, together with the previous inequality, we deduce that $J_\varepsilon(\eta(t_1^j,\,\Gamma_2(s_1^j)))$ satisfies

$$c_\varepsilon-4\sqrt{\delta_j}\leqslant J_\varepsilon(\eta(t_1^j,\,\Gamma_2^j(s_1^j)))\leqslant c_\varepsilon+\delta_j$$

This gives a sequence (u_j), namely $u_j=\eta_j(t_1^j,\,\Gamma_2^j(s_1^j))$, such that

$$\begin{cases} J_\varepsilon(u_j)\to c_\varepsilon\\ J'_\varepsilon(u_j)\to 0\ \text{ in }\ \mathcal{H}'\end{cases}$$

As done in part 1, (u_j) converges strongly in $L^{p+1-\varepsilon}(M)$ to a certain u_ε and

$$\|u_\varepsilon-\Gamma_2^j(s_1^j)\|_{p+1-\varepsilon}\leqslant\|u_\varepsilon-u_j\|_{p+1-\varepsilon}+\|u_j-\Gamma_2^j(s_1^j)\|_{p+1-\varepsilon}$$

It follows from point (5) of lemma 3.2 that

$$\lim_{j\infty}\|u_\varepsilon-\Gamma_2^j(s_1^j)\|_{p+1-\varepsilon}=0$$

Thus $\Gamma_2^j(s_1^j)\to u_\varepsilon$ a.e. Hence $u_\varepsilon\geqslant 0$ on Ω_1 and $u_\varepsilon\leqslant 0$ on Ω_2, and it follows from Hopf maximum principle that $u_\varepsilon>0$ on Ω_1 and $u_\varepsilon<0$ on Ω_2. This ends the proof of lemma 3.3. ■

The idea now is to let ε go to 0. First of all, we prove the two following lemmas

LEMMA 3.4. – *There exists* $u_\varepsilon \in W^{1,2}(M)$ *a solution of* (E_ε) *as in lemma 3.3,* $\varepsilon \ll 1$, *such that if we let*

$$c = \inf_{\gamma \in \mathfrak{P}} \max_{w \in \gamma} J(w)$$

where $\mathfrak{P}$ *is the set of all continuous paths in* $\mathcal{H}$ *joining 0 and a certain* $u \in \mathcal{H}$, *then*

$$c \geqslant \limsup_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon)$$

PROOF. – Clearly

$$J_\varepsilon(u) \leqslant J(u) + \frac{1}{p+1} \max_M f \int_M \left| |u|^{p+1} - |u|^{p+1-\varepsilon} \right| dv(g)$$

Thus

$$\inf_{\gamma \in \mathfrak{P}} \max_{u \in \gamma} J_\varepsilon(u) \leqslant \inf_{\gamma \in \mathfrak{P}} \max_{u \in \gamma} J(u) +$$

$$\frac{1}{p+1} \max_M f \times \sup_{t \in [0;\, 1]} \int_M \left| t^{p+1} |u_0|^{p+1} - t^{p+1-\varepsilon} |u_0|^{p+1-\varepsilon} \right| dv(g)$$

For u_ε as in the proof of lemma 3.3, $J_\varepsilon(u_\varepsilon) = c_\varepsilon$. Hence

$$J_\varepsilon(u_\varepsilon) = c_\varepsilon \leqslant c + \frac{\max_M f}{p+1} \times \sup_{t \in [0;\, 1]} \int_M \left| t^{p+1} |u_0|^{p+1} - t^{p+1-\varepsilon} |u_0|^{p+1-\varepsilon} \right| dv(g)$$

But

$$\limsup_{\varepsilon \to 0} \sup_{t \in [0;\, 1]} \int_M \left| t^{p+1} |u_0|^{p+1} - t^{p+1-\varepsilon} |u_0|^{p+1-\varepsilon} \right| dv(g) = 0$$

Taking the limit as $\varepsilon \to 0$,

$$\limsup_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) = \limsup_{\varepsilon \to 0} c_\varepsilon \leqslant c$$

This ends the proof of the lemma. ■

LEMMA 3.5. – *Assume that there exists a weakly convergent subsequence of* (u_ε) *which converges to* $u \not\equiv 0$. *Then* u *is G-invariant,* τ*-antisymmetrical,* $u \equiv 0$ *on* ∂M, u *is* C^2 *on* M, *and*

$$\Delta_g u + au = f|u|^{p-1}u + h|u|^{q-1}u$$

Moreover, if we assume that $\mathcal{F}_\tau$ *splits* M *into two domains* Ω_1 *and* Ω_2

stable under the action of G, we can choose u such that the zero set of u is exactly $\mathcal{F}_\tau \cup \partial M$.

PROOF. – For all $0 < \varepsilon \leq \varepsilon_0$, $J_\varepsilon(u_\varepsilon) = c_\varepsilon$ and $J_\varepsilon'(u_\varepsilon).u_\varepsilon = 0$. Moreover, the sequence (u_ε) is bounded in $W^{1,2}(M)$. Hence we can extract a subsequence, still denoted by (u_ε), so that

$$\begin{cases} (u_\varepsilon) \text{ converges to } u \text{ weakly in } W^{1,2}(M) \\ (u_\varepsilon) \text{ converges to } u \text{ strongly in } \mathrm{L}^2(\mathrm{M}) \\ (u_\varepsilon) \text{ converges to } u \text{ a.e.} \end{cases}$$

From this, we deduce that u is G-invariant and τ-antisymmetrical. Moreover,

$$|u_\varepsilon|^{p-1} u_\varepsilon \to |u|^{p-1} u \text{ a.e. in } M$$

and

$$\int_M |u_\varepsilon|^{p+1} dv(g) = \int_M (|u_\varepsilon|^p)^{\frac{p+1}{p}} dv(g) \leq C$$

where $C > 0$ is independent of ε. Then

$$\begin{cases} |u_\varepsilon|^{p-1} u_\varepsilon \to |u|^{p-1} u \text{ weakly in } (L^{p+1})' \\ |u_\varepsilon|^{q-1} u_\varepsilon \to |u|^{q-1} u \text{ weakly in } (L^{p+1})' \end{cases}$$

Since $W^{1,2}(M) \hookrightarrow L^{p+1}(M)$, we get that u is a weak solution in $W^{1,2}(M)$ of the equation

$$\Delta_g u + au = f|u|^{p-1}u + h|u|^{q-1}u$$

As in Trüdinger [18], it follows that $u \in C^2(M)$. Since $u \in C^2(M) \cap W^{1,2}(M)$, $u \equiv 0$ on ∂M. The result follows from lemma 3.3. This ends the proof of lemma 3.5. ■

We suppose now by absurdum that every weakly convergent subsequence of (u_ε) has for limit 0. We call this hypothesis (RA) (for reductio by absurdum). In the sequel, we denote by $B_x(r)$ the ball of center x and radius r. As in [13], the following holds

LEMMA 3.6. – *Suppose* (*RA*) *and that for all $x \in M$, we can find $\delta > 0$ such that*

$$(3.6.1) \qquad K(n,2)^2 (f(x))^{\frac{2}{p+1}} \limsup_{\varepsilon \to 0} \left(\int_{B_x(\delta) \backslash (B_x(\delta) \cap \partial M)} f|u_\varepsilon|^{p+1-\varepsilon} dv(g) \right)^{\frac{p-1}{p+1}} < 1$$

Then $\forall x \in M$, $\exists \delta(x) > 0$ *such that*

$$\limsup_{\varepsilon \to 0} \int_{B_x(\delta(x)) \setminus (B_x(\delta(x)) \cap \partial M)} |u_\varepsilon|^{p+1-\varepsilon} dv(g) = 0$$

Theorem 3.1 reduces to the proof that (RA) leads to a contradiction. We assume (3.6.1). Since M is compact, there exist $x_1, \dots, x_m \in M$ such that

$$M = \bigcup_{1 \leqslant i \leqslant m} B_{x_i}(\delta(x_i))$$

where $\delta(x_i)$ is given by lemma 3.6. We recall here that

$$c_\varepsilon = \frac{p-1-\varepsilon}{2(p+1-\varepsilon)} \int_M f|u_\varepsilon|^{p+1-\varepsilon} dv(g) + \frac{q-1}{2(q+1)} \int_M h|u_\varepsilon|^{q+1} dv(g)$$

According to (RA), $\lim_{\varepsilon \to 0} \int_M h|u_\varepsilon|^{q+1} dv(g) = 0$, while by lemma 3.6

$$\limsup_{\varepsilon \to 0} \int_{B_{x_i}(\delta(x_i)) \setminus (B_{x_i}(\delta(x_i)) \cap \partial M)} |u_\varepsilon|^{p+1-\varepsilon} dv(g) = 0, \quad \forall i = 1, \dots, m$$

It follows that $\limsup_{\varepsilon \to 0} c_\varepsilon = 0$, a contradiction with the fact that

$$\forall 0 < \varepsilon \leqslant \varepsilon_0, \quad c_\varepsilon \geqslant \varrho > 0$$

Hence, (3.6.1) is absurd and there exists $x_0 \in M$ such that for all $\delta \ll 1$

$$\text{(3.6.2)} \quad \limsup_{\varepsilon \to 0} \int_{B_{x_0}(\delta) \setminus (B_{x_0}(\delta) \cap \partial M)} f|u_\varepsilon|^{p+1-\varepsilon} dv(g) \geqslant \frac{1}{K(n,2)^n f(x_0)^{\frac{2}{p-1}}}$$

This implies in turn that

$$\limsup_{\varepsilon \to 0} c_\varepsilon \geqslant \frac{1}{n} \limsup_{\varepsilon \to 0} \int_{B_{x_0}(\delta) \setminus (B_{x_0}(\delta) \cap \partial M)} f|u_\varepsilon|^{p+1-\varepsilon} dv(g) \geqslant \frac{1}{nK(n,2)^n (f(x_0))^{\frac{2}{p-1}}}$$

Now, we distinguish two cases

Case 1. *We assume that Card* $O_{\langle G, \tau \rangle}(x_0) = +\infty$. Let $A > 0$ be given. Since f and $|u_\varepsilon|$ are G-invariant and τ-invariant, for all $A > 0$, we can choose $\delta > 0$ small enough such that

$$\limsup_{\varepsilon \to 0} \int_{B_{x_0}(\delta) \setminus (B_{x_0}(\delta) \cap \partial M)} f|u_\varepsilon|^{p+1-\varepsilon} dv(g) \leqslant \frac{1}{A} \limsup_{\varepsilon \to 0} c_\varepsilon$$

We choose A such that

$$A > nK(n,2)^n (f(x_0))^{\frac{2}{p-1}} \times \limsup_{\varepsilon\to 0} c_\varepsilon$$

Together with (3.6.2), this leads to a contradiction.

Case 2. *We assume that Card* $O_{\langle G,\tau\rangle}(x_0) < +\infty$. We choose δ small enough such that

$$\limsup_{\varepsilon\to 0} \int_{B_{x_0}(\delta)\setminus(B_{x_0}(\delta)\cap\partial M)} f|u_\varepsilon|^{p+1-\varepsilon} dv(g) \leqslant \frac{n \limsup_{\varepsilon\to 0} c_\varepsilon}{Card\, O_{\langle G,\tau\rangle}(x_0)}$$

Then

$$\limsup_{\varepsilon\to 0} c_\varepsilon \geqslant \frac{Card\, O_{\langle G,\tau\rangle}(x_0)}{nK(n,2)^n (f(x_0))^{\frac{n-2}{2}}} \tag{3.6.3}$$

We have according to lemma 3.4 (v_0 is given by the assumption ($\star$) of theorem 3.1)

$$\limsup_{\varepsilon\to 0} c_\varepsilon \leqslant c \leqslant \sup_{t\geqslant 0} J(tv_0) < \frac{Card\, O_{\langle G,\tau\rangle}(x_0)}{nK(n,2)^n (f(x_0))^{\frac{n-2}{2}}}$$

This is in contradiction with (3.6.3).
Thus, assumption (RA) is absurd, and theorem 3.1 is proved. ■

4. – Estimates and test functions.

We say here that a, f and h satisfy (C) at $x\in M$ if one of the following occurs

$$(C)\ \begin{cases} n\geqslant 4,\ h(x)>0 \\ n=4,\ h(x)=0 \text{ and } \dfrac{a(x)}{2}-\dfrac{Scal_g(x)}{12}<0 \\ n\geqslant 5,\ h(x)=0 \text{ and } \dfrac{2\,Scal_g(x)}{n-4}-\dfrac{8(n-1)\,a(x)}{(n-2)(n-4)}>\dfrac{\Delta_g f(x)}{f(x)} \\ n\geqslant 5,\ h(x)=0,\ \dfrac{2\,Scal_g(x)}{n-4}-\dfrac{8(n-1)\,a(x)}{(n-2)(n-4)}=\dfrac{\Delta_g f(x)}{f(x)} \text{ and } \Delta_g h(x)<0 \end{cases}$$

We prove the following

PROPOSITION 4.1. – *Let (M, g) be a compact Riemannian n-manifold (with or without boundary), $n \geqslant 4$, and denote by $Isom(M, g)$ the isometry group of (M, g). Let G be a subgroup of $Isom(M, g)$ and τ be an involutive isometry of (M, g) such that G and τ commute weakly. We consider a, f and h three smooth G-invariant and τ-invariant functions defined on M. We assume that f is positive. Then, given a point x in the interior of M such that $O_{\langle G, \tau\rangle}(x)$ is finite and $\tau(O_G(x)) \cap O_G(x) = \emptyset$, there exists a function $v \in \mathcal{H}$, $v \not\equiv 0$, such that*

$$(\star) \qquad \sup_{t \geqslant 0} \{J(tv)\} < \frac{Card\ O_{\langle G, \tau\rangle}(x)}{nK(n, 2)^n (f(x))^{\frac{n-2}{2}}}$$

if a, f and h satisfy (C) at x.

For $P \in M$ and $k \in \mathbb{N}^*$, let $\psi_{P, k}$ be defined as follows

$$\forall Q \in \overline{B}_P(\delta) \qquad \psi_{P, k}(Q) = \left(\frac{1}{k} + \frac{1 - \cos \alpha r}{\alpha^2}\right)^{1 - \frac{n}{2}} - \left(\frac{1}{k} + \frac{1 - \cos \alpha\delta}{\alpha^2}\right)^{1 - \frac{n}{2}}$$

$$\forall Q \in M \setminus \overline{B}_P(\delta) \qquad \psi_{P, k}(Q) = 0$$

where $r = d(P, Q)$, $Scal_g(P) = n(n-1)\,\alpha^2$, and δ is such that $|\alpha|\delta \leqslant \pi$ and less than the injectivity radius of M (see Aubin [3]). In such an expression we use the conventions that if $Scal_g(P) < 0$, $\cos \alpha r = \cosh i\alpha r$, and that if $Scal_g(P) = 0$, $\frac{1 - \cos \alpha r}{\alpha^2} = \frac{r^2}{2}$, where $Scal_g$ stands for the scalar curvature of g. We assume here that x is a point of the interior of M such that

$$\begin{cases} \tau(O_G(x)) \cap O_G(x) = \emptyset \\ Card\, O_G(x) < +\infty \end{cases}$$

We set $O_G(x) = \{x_1, \ldots, x_m\}$ and define the function $\Psi_{x, k}$ by

$$\Psi_{x, k} = \sum_{i=1}^{m} \left(\frac{\psi_{x_i, k}}{\|\psi_{x_i, k}\|_{p+1}} - \frac{\psi_{\tau(x_i), k}}{\|\psi_{\tau(x_i), k}\|_{p+1}} \right)$$

where δ is chosen small enough such that

$$\begin{cases} supp\, \psi_{x_i, k} \cap supp\, \psi_{x_j, k} = \emptyset \text{ if } i \neq j \\ supp\, \psi_{x_i, k} \cap supp\, \psi_{\tau(x_j), k} = \emptyset, \ \forall i, j \end{cases}$$

(where *supp* stands for the support of a function). By construction, $\Psi_{x, k}$ is G-invariant and τ-antisymmetrical. Moreover

$$J(t\Psi_{x,k}) = Card\, O_{\langle G,\tau\rangle}(x)\ J\left(t\frac{\psi_{x_1,k}}{\|\psi_{x_1,k}\|_{p+1}}\right)$$

so that we only have to compute $J\left(t\frac{\psi_{P,k}}{\|\psi_{P,k}\|_{p+1}}\right)$. As one easily check,

$$\begin{aligned}
J\left(t\frac{\psi_k}{\|\psi_k\|_{p+1}}\right) = {} & \frac{1}{K(n,2)^2}\frac{t^2}{2} - \frac{1}{p+1}f(P)\,t^{p+1} - C_1'\,h(P)\,k^l t^{q+1} \\
& - \frac{1}{k}\left[\frac{Scal_g(P)\,t^2}{n(n-4)K(n,2)^2} - \frac{4(n-1)a(P)t^2}{n(n-2)(n-4)K(n,2)^2} - \frac{\Delta_g f(P)\,t^{p+1}}{2n}\right] \\
& + C_1'\left[\frac{h(P)(n-2)(q+1)\,Scal_g(P)}{8n(n-1)} - \frac{h(P)(n-2)\,Scal_g(P)}{4(n-1)((n-2)\,q-4)}\right. \\
& \left. - \frac{\Delta_g h(P)\,n}{(n-2)\,q-4}\right]k^{l-1}t^{q+1} \\
& + o(k^{l-1})\,g_1(t)
\end{aligned}$$

for $n \geqslant 5$, and for $n = 4$

$$\begin{aligned}
J\left(t\frac{\psi_k}{\|\psi_k\|_{p+1}}\right) = {} & \frac{1}{K(4,2)^2}\frac{t^2}{2} - \frac{1}{4}f(P)\,t^4 + \frac{\log k}{k}\left(\frac{a(P)}{2} - \frac{Scal_g(P)}{12}\right)C_1 t^2 \\
& - C_1'\,h(P)\,k^l t^{q+1} + O\left(\frac{1}{k}\right)g_2(t)
\end{aligned}$$

where the g_i's, $i = 1, 2$, are smooth functions on $\mathbb{R}^+$ such that $g_i(0) = 0$, $C_1 > 0$ and $C_1' > 0$ are two constants independent of k, and $l = \frac{(n-2)(q-1)}{4} - 1$, (so that $l \in (-1; 0)$). First of all, before proving proposition 4.1, we state the following (elementary) lemma, useful in the proof of proposition 4.1 (for the proof we refer to Djadli [9])

LEMMA 4.2. – *Let $1 < q < p = \frac{n+2}{n-2}$ and $A > 0$, $B > 0$ be three given real numbers. For $k \in \mathbb{N}^*$ large, let also $A(k)$, $B(k)$ and $C(k)$ be real numbers such that*

$$\begin{cases} A(k) \to A \\ B(k) \to B \\ C(k) \to 0 \end{cases}$$

as $k \to +\infty$. *We define*

$$F(t, k) = A(k)\, t^2 - B(k)\, t^{p+1} - C(k)\, t^{q+1}$$

Then, for k large, one has that there exists t_k *such that*

$$F(t_k, k) = \max_{t \geqslant 0} F(t, k)$$

with the additional property that if $t_0 = \left(\frac{2A}{(p+1)B} \right)^{\frac{1}{p-1}}$, *then* $t_k \to t_0$ *as* $k \to +\infty$. *Furthermore, if*

$$A(k) = A + O(k^s), \;\; B(k) = B + O(k^s) \;\; and \;\; C = O(k^s)$$

for some $s < 0$, *then* $t_k = t_0 + O(k^s)$.

Proof of Proposition 4.1. – Suppose first that $n \geqslant 4$ and $h(x) > 0$. Taking $P = x$, we have (with the notations of lemma 4.2)

$$J\left(t \frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}} \right) = F(t, k) = A(k)\, t^2 - B(k)\, t^{p+1} - C(k)\, t^{q+1}$$

where,

$$A(k) = \frac{1}{2} \int_M \left\{ \left| \nabla \left(\frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}} \right) \right|^2 + a \left(\frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}} \right)^2 \right\} > 0$$

$$B(k) = \frac{1}{p+1} \int_M f \left(\frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}} \right)^{p+1} > 0$$

$$C(k) = \frac{1}{q+1} \int_M h \left(\frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}} \right)^{q+1}$$

and (see Djadli [9])

$$\lim_{k \to +\infty} A(k) = \frac{1}{2K(n, 2)^2} = A > 0$$

$$\lim_{k \to +\infty} B(k) = \frac{1}{p+1} f(x) = B > 0$$

$$\lim_{k \to +\infty} C(k) = 0$$

Let t_k and t_0 be as in lemma 4.2. Then

$$J\left(t_k \frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}}\right) = \frac{1}{K(n,2)^2}\frac{t_k^2}{2} - \frac{1}{p+1} f(x)\, t_k^{p+1} - C_1''\, h(x)\, k^l t_k^{q+1} + o(k^l)$$

where $C_1'' > 0$. Since $h(x) > 0$, one can write for k large

$$J\left(t_k \frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}}\right) < \frac{1}{K(n,2)^2}\frac{t_k^2}{2} - \frac{1}{p+1} f(x)\, t_k^{p+1} \leqslant \frac{1}{K(n,2)^2}\frac{t_0^2}{2} - \frac{1}{p+1} f(x)\, t_0^{p+1}$$

As a consequence

$$J\left(t_k \frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}}\right) < \frac{1}{nK(n,2)^n (f(x))^{\frac{n-2}{2}}}$$

and ($\star$) of theorem 3.1 is satisfied. Assume now that $n \geqslant 5$,

$$h(x) = 0 \text{ and } \frac{2\, Scal_g(x)}{n-4} - \frac{8(n-1)\, a(x)}{(n-2)(n-4)} > \frac{\Delta_g f(x)}{f(x)}$$

Then,

$$J\left(t \frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}}\right) = \frac{1}{K(n,2)^2}\frac{t^2}{2} - \frac{1}{p+1} f(x)\, t^{p+1} - \frac{1}{k}\left[\frac{Scal_g(x)\, t^2}{n(n-4)\, K(n,2)^2}\right.$$
$$\left. - \frac{4(n-1)\, a(x) t^2}{n(n-2)(n-4)\, K(n,2)^2} - \frac{\Delta_g f(x)\, t^{p+1}}{2n}\right] + o\left(\frac{1}{k}\right) g_1(t)$$

According to lemma 4.2, one can write that $t_k = t_0 + O\left(\frac{1}{k}\right)$. Hence,

$$J\left(t_k \frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}}\right) = \frac{1}{K(n,2)^2}\frac{t_0^2}{2} - \frac{1}{p+1} f(x)\, t_0^{p+1} - \frac{1}{k}\left[\frac{Scal_g(x)\, t_0^2}{n(n-4)\, K(n,2)^2}\right.$$
$$\left. - \frac{4(n-1)\, a(x)\, t_0^2}{n(n-2)(n-4)\, K(n,2)^2} - \frac{\Delta_g f(x)\, t_0^{p+1}}{2n}\right] + o\left(\frac{1}{k}\right)$$

Since

$$\frac{Scal_g(x)\, t_0^2}{n(n-4)\, K(n,2)^2} - \frac{4(n-1)\, a(x)\, t_0^2}{n(n-2)(n-4)\, K(n,2)^2} - \frac{\Delta_g f(x)\, t_0^{p+1}}{2n} =$$
$$\frac{2Scal_g(x)}{n-4} - \frac{8(n-1)\, a(x)}{(n-2)(n-4)} - \frac{\Delta_g f(x)}{f(x)} > 0$$

it follows that for k large

$$J\left(t_k \frac{\psi_{x,k}}{\|\psi_{x,k}\|_{p+1}}\right) < \frac{1}{K(n,2)^2}\frac{t_0^2}{2} - \frac{1}{p+1} f(x)\, t_0^{p+1} \leqslant \frac{1}{nK(n,2)^n (f(x))^{\frac{n-2}{2}}}$$

Here again $(\star)$ of theorem 3.1 is satisfied. As one can easily check, the proof that $(\star)$ holds in the two remaining cases of (C) is similar. This proves proposition 4.1. ∎

5. – Specific results.

Let (M, g) be a compact Riemannian n-manifold (with or without boundary), such that $n \geqslant 4$, and let $Isom(M, g)$ be its isometry group. We consider G a subgroup of $Isom(M, g)$ and τ an involutive isometry of (M, g) such that G and τ commute weakly. Let a, f and h be three smooth, G-invariant and τ-invariant functions on M such that $\Delta_g + a$ is coercive on $\mathcal{H}$ (see section 3) and f is positive. The following theorem holds

THEOREM 5.1. – *Assume that*

$(H1)$ $\inf\limits_{x \in M} \dfrac{Card\, O_{\langle G,\tau\rangle}(x)}{(f(x))^{\frac{n-2}{2}}}$ *is achieved at a point $x_{\min}$ of the interior of M*

$(H2)$ $\tau(O_G(x)) \cap O_G(x) = \emptyset$, $\forall x$ *in the interior of M of finite orbit under $\langle G, \tau\rangle$*

Then (E) possesses a nodal G-invariant and τ-antisymmetrical solution $u \in C^2(M)$, if either all the orbits under the action of $\langle G, \tau\rangle$ are infinite, or if for all x in the interior of M having a finite orbit under the action of $\langle G, \tau\rangle$, a, f and h satisfy (C) of paragraph 4 at x. Moreover, if we assume that $\mathcal{F}_\tau$ splits M into two subsets Ω_1 and Ω_2 stable under the action of G, then we can choose u such that the zero set of u is exactly the set $\mathcal{F}_\tau \cup \partial M$, where $\mathcal{F}_\tau$ is the set of the fixed points of τ.

REMARK 1. – The theorem still holds under weaker assumptions. We may only suppose $(H1)$ and that if $O_{\langle G,\tau\rangle}(x_{\min})$ has finite cardinality

$$\begin{cases} \tau(O_G(x_{\min})) \cap O_G(x_{\min}) = \emptyset \\ h(x_{\min}) \geqslant 0 \end{cases}$$

and a, f and h satisfy (C) at $x_{\min}$ (if $O_{\langle G,\tau\rangle}(x_{\min})$ is infinite, all the orbits under the action of $\langle G, \tau\rangle$ are infinite).

PROOF OF THEOREM 5.1. – According to proposition 4.1, there exists $v_{\min} \in \mathcal{H}$, $v_{\min} \not\equiv 0$, such that

$$\sup_{t \geq 0} \{J(tv_{\min})\} < \frac{Card\ O_{\langle G, \tau\rangle}(x_{\min})}{nK(n, 2)^n (f(x_{\min}))^{\frac{n-2}{2}}}$$

It follows that for all $x \in M$ of finite orbit under the action of $\langle G, \tau\rangle$,

$$\sup_{t \geq 0} \{J(tv_{\min})\} < \frac{Card\ O_{\langle G, \tau\rangle}(x)}{nK(n, 2)^n (f(x))^{\frac{n-2}{2}}}$$

The result follows from theorem 3.1. ■

5.2. *Specific results for bounded domains of euclidean spaces.*

We let here Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 4$, equipped with the Euclidean metric ξ. We begin with two remarks.

REMARK 2. – If Ω is starshaped, say at 0, obstructions to the existence of nodal solutions hold. Assume that $a \geq 0$, $\partial_r a \geq 0$, $\partial_r f \leq 0$, $h \leq 0$ and $\partial_r h \leq 0$, one of these inequalities being strict. We claim that (E) has no nodal solution. For all $u, v \in C^2(M)$, see Kazdan and Warner [15],

$$-2\Delta_\xi u \langle \nabla u, \nabla v\rangle = div\{2\langle \nabla u, \nabla v\rangle \nabla u - |\nabla u|^2 \nabla v\} - |\nabla u|^2 \Delta_\xi v$$

$$-2\langle \nabla u, Hess(v)\, \nabla u\rangle$$

where $Hess(v)$ denotes the hessian matrix of v. Taking u to be a nodal solution of (E) and $v(x) = \frac{1}{2} r^2$, we get by integrating by parts

$$\int_{\partial\Omega} (\partial_\nu u)^2 \langle x, \nu\rangle\, d\sigma = -2\int_\Omega \langle \nabla u, x\rangle (\Delta_\xi u)\, dv(\xi) - (n-2)\int_\Omega u(\Delta_\xi u)\, dv(\xi)$$

where ν is the unit outer normal $\partial\Omega$. Since u is a solution of (E)

$$\int_{\partial\Omega} (\partial_\nu u)^2 \langle x, \nu\rangle\, d\sigma = 2\int_\Omega \langle \nabla u, x\rangle (au - f|u|^{p-1}u - h|u|^{q-1}u)\, dv(\xi) +$$

$$(n-2)\int_\Omega u(au - f|u|^{p-1}u - h|u|^{q-1}u)\, dv(\xi)$$

Integrating by parts we get

$$\int_{\partial\Omega} (\partial_\nu u)^2 \langle x, \nu\rangle \, d\sigma = -2 \int_{\Omega} a u^2 \, dv(\xi) - \int_{\Omega} r(\partial_r a)\, u^2 \, dv(\xi) +$$

$$\frac{n-2}{n} \int_{\Omega} r(\partial_r f)\, |u|^{p+1} dv(\xi) - \frac{(n-2)(q+1)-2n}{q+1} \int_{\Omega} h |u|^{q+1} dv(\xi) +$$

$$\frac{2}{q+1} \int_{\Omega} r(\partial_r h)\, |u|^{q+1} dv(\xi)$$

and the claim easily follows.

REMARK 3. – As another remark, note that our results improve previous results of Demengel and Hebey [7] in the case of the 2-laplacian. On the one hand, $(\star)$ is localized. On the other hand, we do prescribe the zero set of the solutions.

The following proposition easily follows from theorem 3.1

PROPOSITION 5.3. – *Let Ω be a solid torus of $\mathbb{R}^3$, obtained by rotation around the z-axis of a ball centered on the y-axis. Let G be the group of the rotations around the z-axis and let τ be the orthogonal symmetry with respect to the (x, y)-plane (denoted by Π). We assume that a, f and h are three smooth G- and τ-invariant functions such that f is positive and $\Delta_\xi + a$ is coercive on $\mathcal{H}$. Then (E) admits a nodal solution $u \in C^2(\Omega)$, G-invariant and τ-antisymmetrical, whose zero set is exactly $\partial\Omega \cup (\Omega \cap \Pi)$.*

In somewhat more difficult situations, for example when all points have finite orbit, we can use theorem 5.1. As an example, the following holds

PROPOSITION 5.4. – *Let Ω be either a ball or an annulus of $\mathbb{R}^n$ centered at 0, $n \geqslant 4$, and τ be the antipodal map $\tau(x) = -x$. We assume that a, f and h are three smooth τ-invariant functions such that f is positive and $\Delta_\xi + a$ is coercive on $\mathcal{H}$. We also assume that there exists x_0 in the interior of Ω such that*

$$\begin{cases} f(x_0) \geqslant 2^{\frac{2}{n-2}} f(0) & (\textit{in the case where } \Omega \textit{ is a ball}) \\ f(x_0) \geqslant f(x) & \forall x \in \partial\Omega \end{cases}$$

Then (E) admits a nodal solution $u \in C^2(\Omega)$, τ-antisymmetrical if for all $x \neq 0$ (it is always the case when Ω is an annulus), a, f and h satisfy the condition (C) at x.

Once again, using theorem 5.1, we can easily deal with situations where almost all the points have an infinite orbit. As a corollary, we get multiplicity when a, f and h are assumed to be constants.

PROPOSITION 5.5. – *Let Ω be an annulus of $\mathbb{R}^n$, $n \geq 4$, centered at 0, and Π be some hyperplane of $\mathbb{R}^n$. We denote by $\Pi^\perp$ the orthogonal complement of Π such that $0 \in \Pi^\perp$, by G_Π the group of rotations around $\Pi^\perp$, by τ_Π the orthogonal symmetry with respect to Π and by $\langle G_\Pi, \tau_\Pi \rangle$ the group generated by G_Π and τ_Π. Let a, f and h be three smooth $\langle G_\Pi, \tau_\Pi \rangle$-invariant functions such that f is positive, $\Delta_\xi + a$ is coercive on $\mathcal{H}$, the set of the functions of $W^{1,2}(\Omega)$ which are G_Π-invariant and τ_Π-antisymmetrical. We also assume that there exists $x_0 \in (\Omega \cap \Pi^\perp) \backslash (\partial\Omega \cap \Pi^\perp)$ such that for all $x \in \partial\Omega \cap \Pi^\perp$, $f(x_0) \geq f(x)$. Then (E) admits a nodal solution $u \in C^2(\Omega)$, G_Π-invariant and τ_Π-antisymmetrical, whose zero set is exactly $(\Pi \cap \Omega) \cup \partial\Omega$, if for all $x \in (\Omega \cap \Pi^\perp) \backslash (\partial\Omega \cap \Pi^\perp)$, a, f and h satisfy (C) at x. In the particular case where a, f and h are three positive real numbers, (E) possesses an infinity of nodal solutions.*

REMARK 4. – Here again, in the spirit of remark 1, propositions 5.4 and 5.5 do hold under weaker assumptions.

5.6. *Specific results for the standard sphere.*

Let (S^n, st) be the standard unit sphere of $\mathbb{R}^{n+1}$. One knows exactly $Isom\,(S^n, st)$, and τ splits S^n into two domains, stable under the action of G, if and only if τ is an orthogonal symmetry with respect to an hyperplane of $\mathbb{R}^{n+1}$. As an easy consequence of theorem 5.1, the following proposition holds

PROPOSITION 5.7. – *Let Π be an hyperplane of $\mathbb{R}^{n+1}$, and τ be the symmetry with respect to Π. Let also G be the subgroup of $Isom\,(S^n, st)$ generated by the rotations around $\Pi^\perp$. We assume that a, f and h are three smooth $\langle G, \tau \rangle$-invariant functions, such that $\Delta_{st} + a$ is coercive and f is positive. We also assume that for all $P \in S^n \cap \Pi^\perp$, a, f and h satisfy (C) at P. Then (E) possesses a nodal solution $u \in C^2(S^n)$, G-invariant and τ-antisymmetrical, whose zero set is exactly $S^n \cap \Pi$.*

The following multiplicity result is a straightforward consequence of proposition 5.7. Its last part was proved in Djadli [9]

PROPOSITION 5.8. – *We assume that a, f and h are three positive numbers in the case $n \geq 5$ and that $a \in (0; 2)$ is a constant, f is a positive constant and h is a nonnegative constant if $n = 4$. Then to each hyperplane Π of $\mathbb{R}^{n+1}$, we can associate a nodal solution u_Π, invariant under the action of the subgroup of $Isom\,(S^n, st)$ which lets fixed $S^n \cap \Pi^\perp$. Moreover the zero set of u_Π is* *exactly $\Pi \cap S^n$. In particular, (E) admits an infinity of nodal solutions. On the contrary, (E) has an unique positive solution if $a < \dfrac{n(n-2)}{4}$.*

As an ending remark, we point out that more important zero sets can be prescribed. Let σ_1 and σ_2 be two orthogonal symmetries of (S, st) with respect to two hyperplanes Π_1 and Π_2 such that $\Pi_1^\perp \subset \Pi_2$. Let also a, f and h be three smooth σ_1- and σ_2-invariant functions such that f is positive, $\Delta_{st} + a$ is coercive on the subset of $W^{1,2}(M)$ whose functions are σ_1- and σ_2-antisymmetrical, and such that

$$\exists x_0 \in S^n \backslash (S^n \cap (\Pi_1 \cup \Pi_2)) \text{ such that } \forall x \in S^n \cap (\Pi_1 \cup \Pi_2) \; f(x_0) \geq 2^{\frac{2}{n-2}} f(x)$$

Mimicking what was done in the proof of theorem 3.1 and the proof of theorem 5.1, one can prove that (E) possesses a σ_1- and σ_2-antisymmetrical nodal solution whose zero set is exactly $S^n \cap (\Pi_1 \cup \Pi_2)$ if for all $x \in S^n \backslash (S^n \cap (\Pi_1 \cup \Pi_2))$, a, f and h satisfy (C) at x. No particular difficulties are involved here.

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Zindine Djadli: Université de Cergy-Pontoise, Département de Mathématiques
Site de Saint-Martin, 2 Avenue Adolphe Chauvin - BP 222 Pontoise
F 95302 Cergy-Pontoise Cedex, France. Zindine.Djadli@math.u-cergy.fr

Antoinette Jourdain: Université de Cergy-Pontoise, Département de Mathématiques
Site de Saint-Martin, 2 Avenue Adolphe Chauvin - BP 222 Pontoise
F 95302 Cergy-Pontoise Cedex, France. Antoinette.Jourdain@math.u-cergy.fr

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