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On Trivially Semi-Metrizable and D-Completely Regular Mappings.

F. Cammaroto - G. Nordo - B. A. Pasynkov

Sunto. – *In questo lavoro si definiscono le nozioni di funzione* T*-simmetrizzabile,* T*-semi-metrizzabile e* T-D*-completamente regolare. Esse vengono caratterizzate in termini di parallelismo rispettivamente a spazi simmetrizzabili, semi-metrizzabili e* D*-completamente regolari. Viene inoltre provato che le sottofunzioni di funzioni* D*-completamente regolari (ossia le sottofunzioni di un prodotto per fibre di funzioni sviluppabili) coincidono (a meno di omemorfismi) con le sottofunzioni di prodotti per fibre di funzioni semi-metrizzabili.*

Summary. – *Trivially symmetrizable, trivially semi-metrizable and trivially* D*-completely regular mappings are defined. They are characterized as mappings parallel to symmetrizable, semi-metrizable and* D*-completely regular spaces correspondently. One shows that trivially* D*-completely regular mappings, i.e. submappings of fibrewise products of developable mappings coincide (up to homeomorphisms) with submappings of fibrewise products of semi-metrizable mappings.*

Introduction.

Generalizing the notions of symmetrizable and semi-metrizable spaces, the notions of trivially symmetrizable (T-symmetrizable, for short) and trivially semi-metrizable (T-semi-metrizable, for short) mappings are introduced in the first section of the paper. It is proved that a continuous mapping is T-symmetrizable (T-semi-metrizable) if and only if it is parallel to some symmetrizable (semi-metrizable) space.

In the second section, generalizing the notion of D-completely regular space [B_1, B_2] the trivially D-completely regular (T-D-CR, for short) mappings (as submappings of fibrewise products of developable mappings) are defined. T-D-CR mappings are characterized as mappings parallel to D-CR spaces and as submappings of fibrewise products of semi-metrizable mappings.

Preliminaries.

Throughout the paper «space» means «topological space» and a continuous mapping is a continuous mapping between spaces.

Let we have a continuous mapping $f : X \to Y$. Then (see [P]):

- f is called T_0- (respectively, T_1-) *mapping* if for any two distinct points $x_1, x_2 \in X$ with $fx_1 = fx_2$ there exist an open set O of X such that $|\{x_1, x_2\} \cap O| = 1$ (respectively, there exist neighborhoods O_i of x_i, $i = 1, 2$, such that $x_1 \notin O_2$, $x_2 \notin O_1$);
- a family $\mathcal{B}$ of open in X sets is called a *base for f at a point $x \in X$ if, for any neighborhood O of x, there exist $U \in \mathcal{B}$ and a neighborhood V of fx such that $x \in U \cap f^{-1}V \subset O$ and a family $\mathcal{B}$ of open sets in X is called base for f* if it is a base for f at any point $x \in X$ (hence, $\mathcal{B} \cup f^{-1}\tau$, where τ is the topology of Y, is a prebase of the topology of X);
- f is *parallel to a space F if there exists and embedding e of X in $Y \times F$ such that $f = p \circ e$, where p is the projection of $Y \times F$ onto Y.*

Let us note that, in the last situation, if q is the projection of $Y \times F$ onto F and $\mathcal{B}$ is a base of F, then $(q \circ e)^{-1}\mathcal{B}$ is, evidently, a base for f.

Let we have a family $\{f_\alpha\}_{\alpha \in A}$ of mappings $f_\alpha : X_\alpha \to Y$. The fibrewise product $\bigotimes_{\alpha \in A} f_\alpha$ of $\{f_\alpha\}_{\alpha \in A}$ is the mapping $f : X \to Y$ defined on the subspace

$$X = \left\{ x = (x_\alpha)_{\alpha \in A} \in \Pi = \prod_{\alpha \in A} X_\alpha :\ f_\beta(pr_\beta(x)) = f_\gamma(pr_\gamma(x)) \ \text{ for any } \beta, \gamma \in A \right\}$$

of Π by the relations $f = f_\alpha \circ pr_\alpha$ for every $\alpha \in A$ (where $pr_\alpha : \Pi \to X_\alpha$ is the α-th projection of the product Π).

The space X is called the *fan product* of the spaces X_α relatively to mappings f_α (with $\alpha \in A$). The febrewise product f is also called the *long projection* of the Fan product X, the restrictions $\pi_\alpha = pr_\alpha|_X$ (with $\alpha \in A$) are called the *short projections* of the fan product X and we have

$$f_\beta \circ \pi_\beta = f_\gamma \circ \pi_\gamma \qquad \text{for every } \beta, \gamma \in A .$$

1. – T-symmetrizable and T-semi-metrizable mappings.

Let us recall, for a set X, that:

- a function $d : X \times X \to \mathbb{R}_+$ (the set of non negative real numbers) is a *symmetric* on X if, for each $x, y \in X$, the following holds:

  (1) $d(x, y) = 0$ iff $x = y$;

  (2) $d(x, y) = d(y, x)$.

• a function $d: X\times X\to\mathbb{R}_+$ will be called a *pseudo-symmetric* on X if, for each $x, y\in X$, the following holds:

(1) $d(x, y) = 0$ if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) = 0$ implies $d(x, z) = d(y, z)$ for any $z\in X$.

Evidently, each symmetric is a pseudo-symmetric.

For a pseudo-symmetric d on X, $x\in X$ and $\varepsilon > 0$, the sets $B(x, \varepsilon) = \{y\in X: d(x, y) < \varepsilon\}$ are called ε-balls.

The *topology τ_d generated by a pseudo-symmetric d on X* is defined in the following way:

$$U\in\tau_d \quad \text{iff} \quad \forall x\in U\ \exists\varepsilon > 0 : B(x, \varepsilon)\subset U\,.$$

Hence, a subset F of the space (X, τ_d) is closed if and only if $d(x, F) = \inf\{d(x, y): y\in F\} > 0$ for any $x\notin F$.

A space (X, τ) is called *(pseudo-) symmetrizable* if there exists a (pseudo-) symmetric d on X such that $\tau = \tau_d$.

Evidently, every symmetrizable space is a T_1-space and every pseudo-symmetrizable space is an R_0-space (i.e. $x\in cl\{x'\}$ if and only if $x'\in cl\{x\}$ for any $x, x'\in X$).

Let us note that if d is a pseudo-symmetric for a space X then $cl\{x\} = \{y\in X: d(x, y) = 0\}$ for any $x\in X$.

Let d be a pseudo-symmetric on a set X. Then

$$xdx' \quad \text{iff} \quad d(x, x') \qquad \text{for any } x, x'\in X$$

is an equivalence relation on X.

Let X/d be the set of all equivalence classes on X with respect to the equivalence d and $\pi_d: X\to X/d$ be the correspondent canonical function.

For any $A, B\in X/d$, put $\overline{d}(A, B) = d(x, y)$ for some $x\in A$ and $y\in B$. Evidently, the function $\overline{d}$ is well-defined (i.e., it is independent of the choice of $x\in A$ and $y\in B$). It is clear that $d(x, y) = \overline{d}(\pi_d x, \pi_d y)$, $\overline{d}$ is a symmetric on X/d, $\pi_d\tau_d = \tau_{\overline{d}}$ and $\pi_d^{-1}\tau_{\overline{d}} = \tau_d$. Hence $\pi_d: (X, \tau_d)\to(X/d, \tau_{\overline{d}})$ is continuous and open.

A (pseudo-) symmetric d on a set X is called a *(pseudo-) semimetric* on X if:

(4) for any $x\in X$ and any $\varepsilon > 0$, there exists $O\subset B(x, \varepsilon)$ such that $x\in O$ and for any $y\in O$ may be find $\varepsilon(y) > 0$ such that $B(y, \varepsilon(y))\subset O$.

A topological space (X, τ) is called *(pseudo-) semi-metrizable* if there exists a (pseudo-) semi-metric d for the space (X, τ), i.e., d is a (pseudo-) semi-metric on X such that $\tau_d = \tau$.

Clearly, a (pseudo-) symmetric d for a space X is a (pseudo-) semi-metric for this space if and only if $x\in int\,B(x, \varepsilon)$ for any $x\in X$ and any $\varepsilon > 0$.

It is not difficult to check the following: let d be a pseudo-semi-metric for a space X and $H \subset X$, then $x \in \mathrm{cl}\, H$ if and only if $d(x, H) = 0$.

Obviously, if d is a pseudo-symmetric on a set X, then it is a pseudo-semi-metric on X if and only if $\overline{d}$ is a semi-metric on X/d. Hence, d is a pseudo-semi-metric for (X, τ_d) if and only if $\overline{d}$ is a semi-metric for $(X/d, \tau_{\overline{d}})$.

DEFINITION 1. – Let we have a mapping f of a set X to a space (Y, τ). Any pseudo-symmetric (pseudo-semi-metric) on X will be called a *trivial pseudo-symmetric* (*trivial pseudo-semi-metric*) *on* f. A trivial pseudo-symmetric (trivial pseudo-semi-metric) d on f will be called a *trivial symmetric* (*trivial semi-metric*) on f if $d|_{f^{-1}y}$ is a symmetric (semi-metric) for any $y \in Y$. For a trivial pseudo-symmetric d on f, the topology on X with the prebase $f^{-1}\tau \cup \tau_d$ will be called the topology *generated* (or *induced*) *by* d and will be denoted by $\tau(d, f)$.

Evidently, the mapping $f : (X, \tau(d, f)) \to (Y, \tau)$ is continuous and τ_d is a base for it.

If d is a trivial symmetric on f, then all fibres $f^{-1}y$ (with $y \in Y$) are T_1-spaces and so $f : (X, \tau(d, f)) \to (Y; \tau)$ is a T_1-mapping.

DEFINITION 2. – Let $f : (X, \theta) \to (Y, \tau)$ be a continuous mapping. A trivial (pseudo-) symmetric (respectively, a trivial (pseudo)- semi-metric) d on $f : X \to (Y, \tau)$ is called a *trivial* (*pseudo-*) *symmetric* (respectively, (*pseudo-*) *semi-metric*) for the continuous mapping $f : (X, \theta) \to (Y, \tau)$ if $\theta = \tau(d, f)$ (i.e., if τ_d is a base for f). A continuous mapping is called *trivially* (*pseudo-*) *symmetrizable* (respectively, *trivially* (*pseudo-*) *semi-metrizable*) if there exists a trivial (pseudo-) symmetric (respectively, trivial (pseudo-) semi-metric) for it.

REMARK 1. – Evidently, every trivial pseudo-symmetric (respectively, every trivial pseudo-semi-metric) for a T_0-mapping f is a trivial symmetric (respectively, trivial semi-metric) and so f is a T_1-mapping.

THEOREM 1. – *A continuous mapping $f : X \to Y$ is trivially symmetrizable (respectively, trivially semi-metrizable) if and only if f is parallel to a symmetrizable (respectively, semi-metrizable) space.*

PROOF. – Let f be trivially symmetrizable (trivially semi-metrizable). Take a trivial symmetric (trivial semi-metric) d for f. Then $F = (X/d, \tau_{\overline{d}})$ is a symmetrizable (semi-metrizable) space and $\pi_d \colon X \to F$ is such that $\pi_d^{-1}\tau_{\overline{d}} = \tau_d$. The diagonal product $e = f \triangle \pi_d \colon X \to Y \times F$ is continuous and $f = p \circ e$, $\pi_d = q \circ e$, where p and q are the projections of $Y \times F$ onto Y and F respectively. Since d is a symmetric on f, for any $y \in Y$, any $x \in f^{-1}y$ and for the closure $cl_d\{x\}$ of the set $\{x\}$ in the space (X, τ_d), we have that $f^{-1}y \cap cl_d\{x\} = \{x\}$. Hence, π_d and e are injective on each fibre $f^{-1}y$ (with $y \in Y$). Hence, also e is injective. Let $U \in \tau_d$, $O \in \tau$ and $V = f^{-1}O \cap U$. Since $U = \pi_d^{-1}\pi_d U$ and $\pi_d U$ is open in F, we see that the set $eV = e(e^{-1}p^{-1}O \cap \pi_d^{-1}\pi_d U) = e(e^{-1}p^{-1}O \cap e^{-1}q^{-1}\pi_d U) = ee^{-1}(p^{-1}O \cap q^{-1}\pi_d U \cap eX) = (p^{-1}O \cap q^{-1}\pi_d U) \cap eX$ is open in eX. Hence e is an embedding.

Now, let f be parallel to a symmetric (semi-metric) space F and let $\overline{d}$ a symmetric (semi-metric) for the space F. There exists an embedding $e: X \to Y \times F$ such that $f = p \circ e$, where p is the projection of $Y \times F$ onto Y. Let q be the projection of $Y \times F$ onto F. Then $e^{-1}q^{-1}\tau_{\overline{d}}$ is a base for f and the function $d(x, x') = \overline{d}(q \circ e(x), q \circ e(x'))$, for any $x, x' \in X$, is a trivial symmetric (trivial semi-metric) for f. ∎

2. – D-**complete regular mappings.**

The class of developable mappings was introduced in [CP] and it was proved there that a continuous mapping is developable if and only if it is parallel to a developable (and, by the definition of developable spaces in [CP], T_1-) space.

The following definition copies Brandenburg's definition of D-completely regular (D-CR, for short) spaces (see [B_1, B_2]).

DEFINITION 3. – A continuous mapping will be called *trivially D-completely regular* (T-D-CR, for short) if it may be embedded in the fibrewise product of a family of developable mappings $f_\alpha: X_\alpha \to Y$ (with $\alpha \in A$).

Let us note that, by the definition of developable mappings in [CP], all the mappings f_α them are T_1-mappings. Since the fibrewise product of T_1-mappings is a T_1-mapping and any submapping of a T_1-mapping is also a T_1-mapping, we have that all the T-D-CR mappings are T_1-.

LEMMA 1. – *Let a continuous mapping $f_\alpha: X_\alpha \to Y$ be parallel to a space F_α (for $\alpha \in A$). Then the fibrewise product $f = \bigotimes_{\alpha \in A} f_\alpha$ is parallel to the product $F = \prod_{\alpha \in A} F_\alpha$, i.e., up to homeomorphisms, f is a submapping of the projection p of $Y \times F$ onto Y.*

PROOF. – Let $e_\alpha: X_\alpha \to Y \times F_\alpha$ is an embedding such that $f_\alpha = p_\alpha \circ e_\alpha$, where p_α is the projection of $Y \times F_\alpha$ onto Y. Let us note that the fibrewise product $p = \bigotimes_{\alpha \in A} p_\alpha$ is the projection of the product $Y \times F$ to Y.

Identify X_α with $e_\alpha X_\alpha$ by means of e_α. Then f_α will be identified with the restriction $p_\alpha |_{X_\alpha \equiv e_\alpha X_\alpha}$ and the fibrewise product $f = \bigotimes_{\alpha \in A} f_\alpha |_{X_\alpha}$ will be identified with some submapping of p. ∎

It is known (see [B_1, H]) that, for a T_1-space, the following properties are equivalent:

(i) X is D-CR;

(ii) X can be embedded into the product of T_1- σ-spaces;

(iii) X can be embedded into the product of semi-stratifiable spaces;

(iv) X can be embedded into the product of perfect T_1-spaces;

(v) X can be embedded into some degree of the space D_1;

(vi) X can be embedded into the product of semi-metrizable spaces.

If a mapping f is parallel to a space F and F can be embedded into a space G, then, evidently, f is also parallel to G. The equivalences cited above imply the equivalence of conditions (2)-(7) in the following theorem:

THEOREM 2. – *For a continuous mapping $f: X \to Y$ the following conditions are equivalent:*

(1) *f is T-D-CR*;

(2) *f is parallel to a D-CR space*;

(3) *f is parallel to the product of T_1- σ-spaces*;

(4) *f is parallel to the product of semi-stratifiable spaces*;

(5) *f is parallel to the product of perfect T_1-spaces*;

(6) *f is parallel to some degree of the space D_1*;

(7) *f is parallel to the product of semi-metrizable spaces*;

(8) *f can be embedded in the fibrewise product of trivially semi-metrizable mappings.*

PROOF. – (1) $\Rightarrow$ (2) Let $f: X \to Y$ be embedded in the fibrewise product $g = \bigotimes_{\alpha \in A} g_\alpha$ of developable mappings $g_\alpha: Z_\alpha \to Y$ (with $\alpha \in A$). Then g_α is parallel to a developable space F_α and, by Lemma 1, g is parallel to $F = \prod_{\alpha \in A} F_\alpha$. Then f is also parallel to F and the space F is D-CR.

(7) $\Rightarrow$ (8) Let $f: X \to Y$ be parallel to the product $F = \prod_{\alpha \in A} F_\alpha$ of semimetrizable spaces F_α, i.e., there exists an embedding $e: X \to F$ such that $f = p \circ e$, where p is the projection of $Y \times F$ onto Y. Since p is the fibrewise product of the projections $p_\alpha: Y \times F_\alpha \to Y$ and every p_α is semi-metrizable (because it is parallel to the semi-metrizable space F_α), we have (8).

(8) $\Rightarrow$ (1) Let $f: X \to Y$ can be embedded in the fibrewise product $\bigotimes_{\alpha \in A} f_\alpha$ of semi-metrizable mappings f_α. By Theorem 1, we can suppose that f_α is a submapping of the projection $p_\alpha: Y \times F_\alpha \to Y$ for some semi-metrizable space F_α. Then f can be embedded in the fibrewise product of the projections

p_α, $\alpha \in A$, and so in the projection $p : Y \times F \to Y$ for $F = \prod_{\alpha \in A} F_\alpha$. By equivalences cited before, F is D-CR and so F can be embedded into the product G of developable spaces G_β (with $\beta \in B$). Hence, f is parallel to G and so, as above, it can be embedded in the fibrewise product of the projections q_β: $Y \times G_\beta \to Y$. These projections are developable because they are parallel to the developable spaces G_β. ■

COROLLARY 1. – *Let a continuous mapping $f : X \to Y$ be such that Y is a D-CR space. Then X is a D-CR space if and only if f is a T-D-CR mapping.*

REFERENCES

[B$_1$] H. BRANDENBURG, *Some characterizations of developable spaces*, Proc. A.M.S. **80** (1980), 157-161.

[B$_2$] H. BRANDENBURG, *On a class of nearness spaces and the epireflective hull of developable topological spaces*, in: Proceedings of the International Symposium on Topology and its Applications, Beograd (1977).

[CP] F. CAMMAROTO - B. A. PASYNKOV, *Some metrization theorems for continuous mappings*, to appear (1999).

[E] R. ENGELKING, *General Topology*, Heldermann, Berlin, 1989.

[H] N. C. HELDERMANN, *The category of D-completely regular spaces is simple*, Trans. A.M.S., **262** (1980), 437-446.

[P] B. A. PASYNKOV, *On extension to mappings of certain notions and assertions concerning spaces*, in: Mapping and Functors, Izdat. MGU, Moscow (1984), 72-102 (in Russian).

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