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On the Lyapunov Exponent and Exponential Dichotomy for the Quasi-periodic Schrödinger Operator.

R. FABBRI (*)

Sunto. – *In questo lavoro viene studiato l'esponente di Lyapunov $\beta(E)$ per l'operatore di Schrödinger in una dimensione con potenziale quasi periodico. Indicato con $\Gamma \subset \mathbb{R}^k$ l'insieme delle frequenze le cui componenti sono razionalmente indipendenti e considerato $0 \leqslant r < 1$, si fa vedere come $\beta(E)$ risulti zero sul complementare in $\Gamma \times C^r(\mathbb{T}^k)$ dell'insieme $\mathcal{D}$ in cui si ha dicotomia esponenziale (D.E.). Le tecniche ed i metodi usati sono basati sulle proprieta' del numero di rotazione e della D.E. per l'operatore considerato.*

Summary. – *In this paper we study the Lyapunov exponent $\beta(E)$ for the one-dimensional Schrödinger operator with a quasi-periodic potential. Let $\Gamma \subset \mathbb{R}^k$ be the set of frequency vectors whose components are rationally independent. Let $0 \leqslant r < 1$, and consider the complement in $\Gamma \times C^r(\mathbb{T}^k)$ of the set $\mathcal{D}$ where exponential dichotomy holds. We show that $\beta = 0$ is generic in this complement. The methods and techniques used are based on the concepts of rotation number and exponential dichotomy.*

1. – Introduction.

The purpose of this paper is to study the positivity of the Lyapunov exponent for linear differential equations where we consider cocycles based on flows rather than cocycles based on a discrete dynamical system. In particular, we refer to the one-dimensional Schrödinger operator with a quasi-periodic potential. The quasi-periodic Schrödinger operator has been intensively studied in the last 25 years, in the discrete and in the continuous formulation. In the theory of discrete 1-dimensional Schrödinger operator, Lyapunov exponents can decide if the spectrum of such operator is or is not absolutely continuous. In this paper, we will study the self-adjoint differential operator

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(Schrödinger operator)

$$H = -\frac{d^2}{dt^2} + q(t)$$

on $L^2(\mathbb{R})$, where $q(t)$ is a quasi-periodic function with at least two basic frequencies. The real-valued function $q(t)$ is called quasi-periodic with k frequencies if there exists a continuous, real-valued function Q on the k-torus $\mathbb{T}^k$ and a vector of frequencies $\underline{\gamma} = (\gamma_1, \ldots, \gamma_k) \in \mathbb{R}^k$ whose components are rationally independent and a point $\psi = (\psi_1, \ldots, \psi_k) \in \mathbb{T}^k$ such that

$$q(t) = Q(\psi_1 + \gamma_1 t, \ldots, \psi_k + \gamma_k t) = Q(\tau_t(\psi))$$

where τ_t: $\mathbb{T}^k \to \mathbb{T}^k$ given by $\tau_t(\psi) = \psi + \gamma t$ is the Kronecker quasi periodic flow defined on the torus. So what we consider is the operator

$$H_\psi : -\frac{d^2}{dt^2} + Q(\psi + \gamma t)$$

on $L^2(\mathbb{R})$ with $\psi \in \mathbb{T}^k$, $Q : \mathbb{T}^k \to \mathbb{R}$ continuous and $\gamma \in \mathbb{R}^k$ with components rationally independent.

We want to study the Lyapunov exponent of the equation

$$H_\psi \phi = E\phi$$

where $E \in \mathbb{R}$. Let's give the exact definition of this object. If we fix $E \in \mathbb{R}$, we can write the equation $H_\psi \phi = E\phi$ in a system form given by:

$$\begin{bmatrix} \phi \\ \phi' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -E + Q(\psi + \gamma t) & 0 \end{bmatrix} \tag{1}$$

where we put $A(\psi + \gamma t) = \begin{pmatrix} 0 & 1 \\ -E + Q(\psi + \gamma t) & 0 \end{pmatrix}$ with $A(\psi + \gamma t) \in sl(2, \mathbb{R})$. So the system (1) becomes equivalent to the system;

$$\underline{x}' = A(\psi + \gamma t)\, \underline{x} \tag{2}$$

where $\underline{x} = \begin{pmatrix} \phi \\ \phi' \end{pmatrix}$. Let $\Phi(t) = \Phi_{\psi, E}(t)$ be the fundamental matrix of the system (1) such that $\Phi(0) = Id$. Then we can define the Lyapunov exponent β of the system (1)

$$\beta(E) = \lim_{t \to \infty} \frac{1}{t} \ln \|\Phi(t)\| \tag{3}$$

where $\|\cdot\|$ indicates any norm on the set of 2×2 matrices. For each $E \in \mathbb{R}$, the limit exists and takes on the same value $\beta(E)$ for Lebesgue-a.a. $\psi \in T^k$; see, e.g., [JPS] for basic facts about Lyapunov exponents. Sometimes it is useful to suppress the parameter E and study the system

$$\begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ Q(\psi + \gamma t) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix}. \tag{4}$$

In this case $\beta = \beta(Q)$ is defined by setting $E = 0$ in (3).

To formulate our result exactly, let $\Gamma \subset \mathbb{R}^k$ be the set of vector of frequencies $\gamma = (\gamma_1, \ldots, \gamma_k)$ whose components are independent over $\mathbb{Q}$. Then Γ, with the topology induced by $\mathbb{R}^k$, is a Baire space and hence it can be given a metric d with respect to which (Γ, d) is a complete metric space [Ch]. Let us define $C^r = C^r(\mathbb{T}^k)$ and consider the Banach space $(C^r, \|\cdot\|_r)$ where $Q \in C^r$ if and only if

$$\|Q\|_r = \|Q\|_0 + \sup_{\psi} \frac{|Q(\psi_1) - Q(\psi_2)|}{|\psi_1 - \psi_2|^r}$$

is finite. Let $W_r = \Gamma \times C^r$ be the product topological space. Our main result is the following

THEOREM 1.1. – *Suppose* $0 \leqslant r < 1$. *There is a residual subset* $W_* \subset W_r$ *with the following property: if* $w = (\gamma, Q) \in W_*$ *then either equations* (1) *admit an E. D. or the Lyapunov exponent* $\beta = \beta(w)$ *equals zero.*

We will give the exact definitions of the concept of exponential dichotomy (E.D) and some other ones in the next section. Let's observe that this Theorem can be viewed as a development of a result of Mañè [Ma].

We can refer to many papers concerning the study of the Lyapunov exponent for the random Schrödinger operator. In the papers of [S], [CS], [FSW] we have certain C^2 open sets of potentials for which the Lyapunov exponent for the corresponding discrete cocycle is positive. We have also papers of Eliasson [E1] and Moser-Pöschel [MP] where the study of the quasi periodic Schrödinger operator with smooth potential give us some information about the Lyapunov exponent. We want also to mention the paper of [FJP] where we have given a result of genericity of the positive Lyapunov exponent for a residual subset of frequencies contained in Γ in the C^r topology with $r \geqslant 0$ for the operator H_ψ on a set of locally positive measure other than results about the spectrum of such operator.

The question of the positivity of the Lyapunov exponent is a well studied subject: if we have exponential dichotomy for the system (1), this implies that the corresponding Lyapunov exponent is positive. It is not always true the viceversa ([M1], [M2]), so it is interesting to determine the class of system for which β is zero. In a recent paper ([FJ1]), we have proved an analogue of the Theorem 1.1 for a quasi-periodic $SL(2, \mathbb{R})$-valued cocycle defined on the torus $\mathbb{T}^k$.

We would like also to mention some other results concerning the positivity of the Lyapunov exponent; in a paper of Kotani ([K]), the positivity of β is studied for a general base flow $\{Y, \tau_t\}$ and in another one of Nerurkar ([N]), we have a result of density for $\beta > 0$ for continuous $SL(2, \mathbb{R})$-valued cocycles using some control-theoretic techniques. We have also the papers of Herman ([H]) and Wojthowski ([W1], [W2]) where we have some eximates for the Lyapunov exponents for some special integer $Sl(2, \mathbb{R})$-valued cocycles and those of Young ([Y1], [Y2]) for certain integer cocycles.

We would like to finish this Introduction by saying that the techniques used are based on the concepts of rotation number and exponential dichotomy associated to our system; in the proof of the Theorem 1.1 we will use some calculations due to Moser for the study of the spectrum of the almost periodic Schrödinger operator [Mo].

2. – Preliminaries.

In this chapter we want to introduce the definitions and the concepts we will use for the proof of our main Theorem. First of all let's recall what we mean by rotation number. It can be defined for any quasi-periodic Schrödinger operator or more generally for any «random» operator, see [JM]. Consider the quasi-periodic operator

$$H_\psi \colon -\frac{d^2}{dt^2} + Q(\psi + \gamma t)$$

with the corresponding system

$$\begin{pmatrix} \phi \\ \phi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -E + Q(\psi + \gamma t) & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \phi' \end{pmatrix}.$$

For fixed $E \in \mathbb{R}$ and $\psi \in \mathbb{T}^k$ if we consider the polar coordinates, we have that $\vartheta(t) = \arctan \dfrac{\phi'}{\phi}$, where $\begin{pmatrix} \phi(t) \\ \phi'(t) \end{pmatrix}$ is any non trivial solution of (1), satisfies the equation:

$$\vartheta' = [-E + Q(\psi + \gamma t)] \cos^2 \vartheta - \sin^2 \vartheta .$$

So we can define the rotation number $\alpha(E)$ by

$$\alpha(E) = -\lim_{t \to \infty} \frac{\vartheta(t)}{t} .$$

This limit exists for all $\psi \in \mathbb{T}^k$ and it does not depend on the choice of the initial

value $\vartheta(0)$ nor on ψ. [JM] We have also that the function $E \to \alpha(E)$ is continuous, non negative monotone non-decreasing, and it increases exactly on the spectrum of H_ψ for all $\psi \in \mathbb{T}^k$. We have also the «gap-labelling formula» ([GJ] [JM]):
if E belong to a resolvent interval of H_ψ, then $\alpha(E)$ is such that

$$2\alpha(E) = \sum_{j=1}^{k} n_j \gamma_j$$

where $n \in \mathbb{Z}^k$.

In the Theorem 1.1 we have referred to the exponential dichotomy (for some references [Co], [P1], [P2]). Let's give the definition of this concept.

DEFINITION 2.1. – *Let* $Q \in C^r(\mathbb{T}^k)$, $\gamma \in \mathbb{R}^k$ *any frequency vector. We say that the equations* (1) *admit an exponential dichotomy (E.D.) over* $\mathbb{T}^k$ *if there exists a continuous, projection-valued function* $P : \mathbb{T}^k \to \mathrm{Proj}(\mathbb{R}^2)$ *and some positive constants* K, δ *such that*

$$\|\Phi_\psi(t)\, P_\psi\, \Phi_\psi(s)^{-1}\| \leq K e^{-\delta(t-s)} \qquad t \geq s\,;$$

$$\|\Phi_\psi(t)(I - P_\psi)\, \Phi_\psi(s)^{-1}\| \leq K e^{\delta(t-s)} \qquad t \leq s\,.$$

If these conditions hold we have that $\dim P_\psi = 1$ $\forall \psi \in \mathbb{T}^k$.

We can study the spectrum of the operator H_ψ using the concept of E.D. We have indeed [J1] that, if $\gamma \in \Gamma$, E belongs to the resolvent $\mathbb{R} \setminus \sigma(H_\psi)$ of H_ψ if and only if equations (1) admit an exponential dichotomy over $\mathbb{T}^k$. If the components of the frequency vector γ are not rationally independent this is true if we consider the E.D. on the closure of the orbit $\{\psi + \gamma t \mid t \in \mathbb{R}\} \subset \mathbb{T}^k$.

3. – Proof of the Theorem 1.1.

Let $\mathcal{D} = \{(\gamma, Q) \in \Gamma \times C^r \mid (1)$ *has E.D.* $\forall \psi \in \mathbb{T}^k\}$.

What we want to show is that there exists a residual subset W_* contained in $\Gamma \times C^r$ such that, if $w = (\gamma, Q) \in W_* \setminus \mathcal{D}$ then $\beta(w) = 0$ with $0 \leq r < 1$. For the proof we will use some results we have obtained in the general case when the coefficient matrix $A \in sl(2; \mathbb{R})$ [F], [FJ1] inspired by the techniques and methods used by Johnson [J2] and Moser [Mo]. We will consider two cases: $r = 0$ and $0 < r < 1$.

First case: $r = 0$. We have the following

PROPOSITION 3.1. – *Let* F_p *be the set defined by:* $F_p = \left\{ w = (\gamma, Q) \in W_0 \mid (1) \right.$ *has Lyapunov exponent* $\left. \beta(w) \geq \frac{1}{p} \right\}$ *with* $p \geq 1$. *Then we have that* F_p *is closed in* W_0.

This proposition can be proved using the semicontinuity properties of β. To prove the Theorem 1.1 we will show that $F_p \setminus \mathcal{D}$ is a nowhere dense set in W_0, $\forall p \geq 1$.

PROOF OF THEOREM 1.1. – For contradiction consider an open set $V \subset F_p \setminus \mathcal{D}$ where V is given by $V = \widetilde{V}_1 \times V_2$ with $\widetilde{V}_1 = V_1 \cap \Gamma$ where V_1 is an open set in $\mathbb{R}^k$ and V_2 an open set in C^0. Let's take the pair $(\gamma, Q) \in V$; we have the following

PROPOSITION 3.2. – *There exists a vector $\gamma_* \in V_1$ and a function $Q_* \in V_2$ such that γ_* has rational components $\gamma_* = \left(\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}\right)$ with $T = l.c.m.(q_1, \ldots, q_k)$ such that, for each $\psi \in \mathbb{T}^k$, the logarithm $\ln \Phi_\psi(T)$ of the period matrix $\Phi_\psi(T)$ of the periodic system*

$$\begin{pmatrix} \phi \\ \phi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -E + Q_*(\psi + \gamma_* t) & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \phi' \end{pmatrix}$$

lies on the surface of a certain cone $\mathcal{C} \subset sl(2, \mathbb{R})$.

For the proof see [FJ1] where we have considered the case of the coefficient matrix $A \in sl(2, \mathbb{R})$. Let's pause briefly to explain the construction of the cone $\mathcal{C}$. Consider the periodic system

$$\underline{x} = A_*(\psi + \gamma_* t)\, \underline{x} \tag{5}$$

where $A_*(\psi + \gamma_* t) = \begin{pmatrix} 0 & 1 \\ -E + Q_*(\psi + \gamma_* t) & 0 \end{pmatrix}$ is periodic of period T. Suppose that, for some integer m, we have that $\alpha(\psi) = \dfrac{2\pi m}{T}$. This condition has a precise meaning that can be interpreted geometrically introducing the cone $\mathcal{C}$.

Let $\Phi_\psi(T)$ the period matrix of the periodic system (5): it satisfies one of the following conditions:

i) $\Phi_\psi(T) = Id$. This means that all the solutions of (5) are periodic of period T. In a spectral sense this implies that E defines a closed gap for the operator H_ψ (elliptic case).

ii) $\Phi_\psi(T)$ has eigenvalue 1 and a unique one-dimensional eigenspace. This means that E is an endpoint of a resolvent interval for H_ψ (parabolic case).

iii) $\Phi_\psi(T)$ has two eigenvalues, this implies that E is in the interior of a resolvent interval for H_ψ (hyperbolic case).

To introduce the cone $\mathcal{C}$, let's observe that $\Phi_\psi(T) \in SL(2, \mathbb{R})$, if we set $D_0 \subset SL(2, \mathbb{R})$ as the set of the matrices Φ elliptic, parabolic or hyperbolic, we have that there exists $D \subset sl(2, \mathbb{R})$ with D open with $D_0 \subset D$ and a single-valued, real-analytic branch $\ln : D \to sl(2, \mathbb{R})$ such that $\ln \Phi_\psi(T) \in \mathcal{C}$ where $\mathcal{C} \subset sl(2, \mathbb{R})$ is the cone given by $\mathcal{C} = \{(a, b, c) \mid c^2 = a^2 + b^2\}$ where (a, b, c) corresponds to the matrix $\begin{pmatrix} a & b-c \\ b+c & -c \end{pmatrix}$. This means that:

1) $\ln \Phi_\psi(T) = (0, 0, 0)$ closed gap case;

2) $\ln \Phi_\psi(T) \in \mathcal{C}$ but $\ln \Phi_\psi(T) \neq (0, 0, 0)$ parabolic case.

Let's observe that $\Phi_\psi(T)$ is hyperbolic if and only if $\ln \Phi_\phi(T)$ lies in the exterior of $\mathcal{C}$. For the introduction and use of the cone see [MP], [J2], [Mo].

To summarize what we did until now: we have taken $(\gamma, Q) \in V \subset F_p \setminus \mathcal{D}$, then we have determined $(\gamma_*, Q_*) \in V$ such that for the corresponding periodic system

$$\begin{pmatrix} \phi \\ \phi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -E + Q_*(\psi + \gamma_* t) & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \phi' \end{pmatrix}$$

holds the Proposition 3.2. Note that if we are in the cone the Lyapunov exponent $\beta(E)$ of the periodic system equals zero. Now, from the property of semicontinuity of $\beta(E)$ we have the following

PROPOSITION 3.3. – *Let's consider $\varepsilon > 0$, if $\gamma \in \Gamma$ is sufficiently close to γ_*, then the Lyapunov exponent of the system*

$$\begin{pmatrix} \phi \\ \phi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -E + Q_*(\psi + \gamma t) & 0 \end{pmatrix}$$

is less than $\varepsilon : 0 \leqslant \beta < \varepsilon$.

But this gives us a contradiction, because if we choose $\dfrac{1}{p} = \varepsilon$ and $\gamma \in V_1$ we have $\beta < \varepsilon$. So it is not true that $F_p \setminus \mathcal{D}$ contains an open set, that is to say $F_p \setminus \mathcal{D}$ is nowhere dense in W_0. What it remains to prove is how we can determine the periodic matrix

$$A_*(\psi + \gamma_* t) = \begin{pmatrix} 0 & 1 \\ -E + Q_*(\psi + \gamma_* t) & 0 \end{pmatrix}$$

that is to say how to choose the perturbation Q_*. Let $\alpha(E, \gamma, Q)$ be the rotation number of the system

$$\begin{pmatrix} \phi \\ \phi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -E + Q(\psi + \gamma t) & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \phi' \end{pmatrix}.$$

For the property of the rotation number [J2] we have that, considering a sequence $\gamma_n \in \mathbb{Q}^k$ with $\gamma_n = \left(\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}\right)$ with $\gamma_n \to \gamma$ we have that

$$\alpha(E, \psi, \gamma_n, Q) \to \alpha(E, \gamma, Q)$$

and this convergence is uniform in $(E, \psi) \in I \times \mathbb{T}^k$ for every compact $I = [-\eta, \eta] \subset \mathbb{R}$. Moreover $\alpha(E, \gamma, Q)$ is strictly increasing for $E \in [-\eta, \eta]$; so, considering the sequence $\gamma_n \in \mathbb{Q}^k$ with $\gamma_n \to \gamma$ and ϱ such that $0 < \varrho < \eta$, we have that there exists $N_0 \geq 1$ such that for every $n \geq N_0$ we can determine an integer $m = m(n)$ and a continuous real valued function $E_n \colon \mathbb{T}^k \to \mathbb{R}$ such that

$$\alpha(E_n(\psi), \psi, \gamma_n, Q) = \frac{2\pi m}{T} \tag{6}$$

and

$$\|E_n\|_0 \leq \varrho\,. \tag{7}$$

We determine the continuous function $E(\psi) = E_n(\psi)$ by taking the smallest real value for which (6) holds. Such function is constant along every orbit: $E_n(\psi + \gamma_n t) = E_n(\psi)$ $\forall t \in [0, T]$, $\psi \in \mathbb{T}^n$. So, choosing

$$A_* = A(\psi) + E_n(\psi)\, J \qquad \text{with} \qquad J = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

we have obtained $A_*(\psi) \in V_2$. So, writing $\gamma_* = \gamma_n$, we have that $\alpha_*(\psi) = \frac{2\pi m}{T}$ with $m \in \mathbb{Z}$ and then we can apply the proposition 3.2 and the conclusions of the Proposition 3.3. This proves Therem 1 if $r = 0$.

4. – Second case.

Consider $r < 1$; what we want to find is the perturbation function E_n such that $\|E_n\|_r < \varepsilon$ if $\|E_n\|_0 < \varrho$. As before let's take $(\gamma, Q) \in V$ where $Q \leftrightarrow A = \begin{pmatrix} 0 & 1 \\ -E + Q() & 0 \end{pmatrix}$. We can suppose WLG that $Q \in C^1(\mathbb{T}^k)$ where

$$|Q|_1 = \sum_{i=1}^{k} \left| \frac{\partial Q}{\partial \psi_i} \right|_0 .$$

Fix $n \geq N_1$ and a corresponding integer m such that we have (6) and (7). Following [J2] and [FJ1] let b a number such that

$$b \in \left(\frac{2\pi(m - 1/2)}{T}, \frac{2\pi m}{T} \right)$$

and let $E_b(\psi)$ be the unique value $E \in \mathbb{R}$ such that

$$\alpha(\psi, E_b(\psi)) = b \,. \tag{8}$$

If $b \nearrow \frac{2\pi m}{T}$ we have $E_b(\psi) \to E_n(\psi)$. Following the discussion of the previous chapter we have that, for every $b \in \left(\frac{2\pi(m-1/2)}{T}, \frac{2\pi m}{T}\right)$, $\|E_b\|_0 < \varrho$. If we have that E_b is Lipschitz continuous with Lipschitz constant verifying

$$\operatorname{Lip}(E_b) \leqslant \|Q\|_1 \tag{9}$$

we can conclude that $\|E_n\|_r < \varepsilon$. Indeed from (9) we have that

$$\operatorname{Lip} E_n \leqslant \|Q\|_1$$

and for every $0 \leqslant r < 1$ one can verify that

$$\|E_n\|_r \leqslant \|E_n\|_0 (1 + \operatorname{Lip}(E_n)^r \cdot 2^{1-r}) \,. \tag{10}$$

Then, choosing ϱ sufficiently small, we have $\|E_n\|_r < \varepsilon$.

To prove (9) introduce the classical discriminant $\Delta = \Delta(\psi, E_n)$ of the system

$$x' = [A(\psi + \gamma_* t) + E_n J]\, x$$

where $\Delta(\psi, E_n) = \operatorname{tr} \Phi_\psi(T)$. If $E_n = E \in \sum_j$ where $\sum_j$ is an interval in the spectrum of the operator H_ψ, we have that $\Phi_\psi(T)$ is conjugate to the matrix $\begin{pmatrix} \cos \alpha T & -\sin \alpha T \\ \sin \alpha T & \cos \alpha T \end{pmatrix}$ and therefore

$$\Delta(E) = 2 \cos \alpha(E)\, T \,.$$

Now, we can apply some calculations of Moser [Mo] and Johnson [J2]; let's take $\psi \in \mathbb{T}^k$ and $E \in \operatorname{int} \sum_j$, the function

$$q_\psi(t) = Q_*(\psi + \gamma_* t)$$

is an element of the Banach space $C^r[0, T]$, so the functional Δ acts on $C^r[0, T]$. We have then

$$\frac{d\Delta}{d\varepsilon}(q_\psi, q) = \lim_{\varepsilon \to 0} \frac{\Delta(q_\psi + \varepsilon q) - \Delta(q_\psi)}{\varepsilon} \,.$$

Let $\Phi_\psi(t)$ be the fundamental matrix of the periodic system

$$x' = q_\psi(t)\, x$$

and $\Phi_\psi(t) + \varepsilon \Phi_1(t)$ that of the perturbed system

$$x' = [q_\psi(t) + \varepsilon q(t)]\, x$$

we have [Mo],[J2] and [FJ1]

$$\Phi_1(t) = \Phi_\psi(t) \int_0^T \Phi_\psi^{-1}(s)\, q(s)\, \phi_\psi(s)\, ds$$

and then

$$\frac{d\Delta}{d\varepsilon} = \operatorname{tr} \Phi_1(T) = \operatorname{tr} \Phi_\psi(T) \int_0^T \Phi_\psi^{-1}(s)\, q(s)\, \Phi_\psi(s)\, ds$$

where $q(t) = q_1(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Since

$$\Phi_\psi(T) = A^{-1} \begin{pmatrix} \cos \alpha T & -\sin \alpha T \\ \sin \alpha T & \cos \alpha T \end{pmatrix} A$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and writing $\Phi_\psi(t) = \begin{pmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{pmatrix}$ we have

$$\frac{d\Delta}{d\varepsilon} = -\sin \alpha T \int_0^T K_1(s)\, q_1(s)\, ds$$

where we can calculate explicitly $K_1(t)$ following Moser [Mo]

$$K_1(t) = 2u_1(t)\, u_2(t)(a_{12}\, a_{11} + a_{21}\, a_{22}) + u_1^2(t)(a_{12}^2 + a_{22}^2) +$$
$$+ u_2^2(t)(a_{11}^2 + a_{21}^2)\,.$$

We can observe that $K_1(t) > 0$, $\forall t \in [0, T]$. If we consider indeed the expression

$$B\left[\frac{u_1(t)}{u_2(t)}\right]^2 + 2C\frac{u_1(t)}{u_2(t)} + D$$

where

$$B = a_{12}^2 + a_{22}^2 > 0$$

$$C = a_{12}\, a_{11} + a_{21}\, a_{22}$$

$$D = a_{11}^2 + a_{21}^2 > 0$$

we have that the discriminant of this second degree trinomial is given by

$$-(a_{11}\, a_{22} - a_{12}\, a_{21})^2 = -(\det A)^2$$

that is always negative. In the general $sl(2, \mathbb{R})$ case ([F], [FJ1]) we obtain $K_i(t)$ with $i = 1, 2, 3$ because we have

$$q(t) = q_1(t)\, J_1 + q_2(t)\, J_2 + q_3(t)\, J_3$$

where $q_i(t)$ are scalar functions and

$$J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let us return to the equation (8); if we differentiate with respect to ψ_j with $j = 1, \ldots k$, we have

$$\frac{\partial \alpha}{\partial \psi_j} + \frac{\partial \alpha}{\partial E} \cdot \frac{\partial E_b}{\partial \psi_j} = 0 \quad \forall j = 1, \ldots k\,.$$

and then

$$\frac{\partial \alpha}{\partial E} = -\frac{1}{2T \sin \alpha T} \cdot \frac{\partial \Delta}{\partial E} = \frac{1}{2T} \int_0^T K_1(s)\, ds$$

and also

$$\frac{\partial \alpha}{\partial \psi_j} = -\frac{1}{2T \sin \alpha T} \frac{\partial \Delta}{\partial \psi_j} = \frac{1}{2T} \int_0^T K_1(t) \frac{\partial Q_*}{\partial \psi_j}(t)\, dt\,.$$

We have hence

$$\frac{\partial E_b}{\partial \psi_j} = \frac{-\int_0^T K_1(t) \frac{\partial Q_*}{\partial \psi_j}(\psi + \gamma_* t)\, dt}{\int_0^T K_1(t)\, dt}\,.$$

From the positivity of $K_1(t)$ we have

$$\left| \frac{\partial E_b}{\partial \psi_j} \right| \leqslant \left\| \frac{\partial Q_*}{\partial \psi_j} \right\|_0$$

and then, considering $\sup_\psi$ we have that (9) holds. Now we can apply the conclusions of the previous chapter and then prove completely the Theorem 1.1.

REFERENCES

[Ch] G. Choquet, *Lectures on Analysis*, vol I, II, III. Benjamin N.Y., 1969.

[CS] V. Chulaevsky - Ya. Sinai, *Anderson localization for the I-D discrete Schrödinger operators with two-frequency potential*, Comm. Math. Phys, **125** (1989), 91-112.

[Co] A. COPPEL, *Dichotomies in Stability Theory*, Lectures Notes in Mathematics, **629**, Springer-Verlag, New York/Heidelberg/Berlin (1978).

[DC-J] C. DE CONCINI - R. JOHNSON, *The algebraic-geometric AKNS potentials*, Erg. Th. Dyn. Sys. **7** (1992), 1-24.

[E1] L.H. ELIASSON, *Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation*, Commun. Math. Phys., **146** (1992), 447-482.

[E2] L.H. ELIASSON, *Discrete one-dimensional quasi-periodic Schrödinger operator with pure point spectrum*, Acta Math., **179** (1997), 153-196.

[F] R. FABBRI, *Genericità dell'Iperbolicità nei Sistemi Differenziali Lineari di Dimensione Due*, Ph.D. Thesis, Università di Firenze, 1997.

[FJ1] R. FABBRI - R. JOHNSON, *On the Lyapunov exponent of certain $SL(2, R)$-valued cocycles*, Diff. Eqns. and Dynam. Sys., **7**, **3** (1999), 349-370.

[FJ2] R. FABBRI - R. JOHNSON, *Genericity of exponential dichotomy for two-dimensional quasi-periodic linear differential systems*, Ann. Mat. Pura ed Appl., **178** (2000), 175-193.

[Fe] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, New York / Heidelberg / Berlin, 1967.

[FJP] R. FABBRI - R. JOHNSON - R. PAVANI, *On the Nature of the Spectrum of the Quasi-Periodic Schrödinger Operator*, Nonlinear Analysis RWA, **3** (2001), 37-59.

[FSW] J. FRÖHLICH - T. SPENCER - P. WITTWER, *Localization for a class of one dimensional quasi-periodic Schrödinger operators*, Commun. Math. Phys., **132** (1990), 5-25.

[GJ] R. GIACHETTI - R.JOHNSON, *Spectral theory of two-dimensional almost periodic differential operators and its relation to classes of nonlinear evolution equations*, Il Nuovo Cimento, **82** (1984), 125-168.

[H] M. HERMAN, *Une méthode pour minòrer les exposants de Lyapunov et quelques exemples montrant le caractèr local d'un theorèm d'Arnold et de Moser sur le tore en dimension 2*, Commun. Math. Helv., **58** (1983), 453-502.

[J1] R. JOHNSON, *Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients*, Journal of Differential Equations, **61** (1986), 54-78.

[J2] R. JOHNSON, *Cantor spectrum for the quasi-periodic Schrödinger equation*, Journal of Differential Equations, **91** (1991), 88-110.

[JM] R. JOHNSON - J. MOSER, *The rotation number for almost periodic potentials*, Comm. Math. Phys., **84** (1982), 403-438.

[JPS] R. JOHNSON - K. J. PALMER - G. R. SELL, *Ergodic properties of linear dynamical systems*, Siam J. Math. Anal., **18** (1987), 1-33.

[K] S. KOTANI, *Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Scrödinger operators*, Proc. Taniguchi Symp. S.S., Katata (1982), 225-247.

[MW] W. MAGNUS - S. WINKLER, *Hill's Equation*, New York-London-Sydney, Interscience Publishers, 1966.

[Ma] R. MAÑÉ, OSELEDEC'S THEOREM FROM THE GENERIC VIEWPOINT, PROC. INT. CONGR. MATH. 1983, WARSAW, 1296-1276.

[M1] V. MILLIONŠČIKOV, *Proof of the existence...almost periodic coefficients*, Differential Equations, 4 (1968), 203-205.

[M2] V. MILLIOŇČIKOV, *Typicality of almost reducible systems with almost periodic coefficients*, Differential Equations, **14** (1978), 448-450.

[Mo] J. MOSER, *An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum*, Comment. Math. Helvetici, **56** (1981), 198-224.

[MP] J. MOSER - J. PÖSCHEL, *An extension of a result by Dinaburg and Sinai on quasi-periodic potentials*, Comment. Math. Helvetici, **59** (1984), 39-85.

[N] M. NERURKAR, *Positive exponents for a dense class of continuous $SL(2, R)$-valued cocycles which arise as solutions to strongly accessible linear differential systems*, Contemp. Math., **vol. 215** (1998), AMS, 265-278.

[P1] K. PALMER, *Exponential dichotomies and transversal homoclinic points*, Jour. of Diff. Eqns., **55** (1984), 225-256.

[P2] K. PALMER, *Exponential dichotomies, the shadowing lemma and transversal homoclinic points*, Dynamics Reported, **Vol. 1** (1988), 265-306.

[SS] R. J. SACKER - G. R. SELL, *A spectral theory for linear differential systems*, Journal of Differential Equations, **27** (1978), 320-358.

[S] YA. SINAI, *Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential*, Journal of Statistical Physics, **46** (1987), 861-909.

[W1] M. WOJTKOWSKI, *Invariant families of cones and Lyapunov exponents*, Ergod. Th. & Dynam. Sys., **5** (1985), 145-161.

[W2] M. WOJTKOWKI, *Principles for the Design of Billiards with Nonvanishing Lyapunov Exponents*, Comm. Math. Phys., **105** (1986), 391-414.

[Y1] L. S. YOUNG, *Some open sets of nonuniformly hyperbolic cocycles*, Ergod. Th. & Dynam. Sys., **13** (1993), 409-415.

[Y2] L. S. YOUNG, *Lyapounov exponents for some quasi-periodic cocycles*, Ergod. Th. & Dynam. Sys., **17** (1997), 483-504.

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