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Groups in Which the Prime Graph is a Tree.

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Sunto. – *Il «prime graph» $\Gamma(G)$ di un gruppo finito G è definito nel modo seguente: l'insieme dei vertici è $\pi(G)$, cioè l'insieme dei primi che dividono l'ordine del gruppo e due vertici p, q costituiscono un lato (e si indica $p \sim q$) se esiste un elemento in G di ordine pq. Si studiano i gruppi G tali che il grafo $\Gamma(G)$ è un albero, dimostrando che, in questo caso, $|\pi(G)| \leq 8$.*

Summary. – *The prime graph $\Gamma(G)$ of a finite group G is defined as follows: the set of vertices is $\pi(G)$, the set of primes dividing the order of G, and two vertices p, q are joined by an edge (we write $p \sim q$) if and only if there exists an element in G of order pq. We study the groups G such that the prime graph $\Gamma(G)$ is a tree, proving that, in this case, $|\pi(G)| \leq 8$.*

1. – Introduction.

Given a finite group G, we construct its prime graph, $\Gamma(G)$, as follows: the vertices of $\Gamma(G)$ are the primes dividing the order of G and two vertices p, q are joined by an edge (we write $p \sim q$) if there is an element in G of order pq.

We denote the set of all the connected components of the graph $\Gamma(G)$ by $\{\pi_i(G), \text{ for } i = 1, 2, \ldots, t(G)\}$ and, if the order of G is even, we denote by $\pi_1(G)$ the component containing 2. We also denote by $\pi(n)$ the set of all primes dividing n, if n is a natural number.

The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. It turned out that $\Gamma(G)$ is not connected if and only if the augmentation ideal of G is decomposable as a module (see [7]). This is an example of an application of the properties of the prime graph of a finite group G. In general, we can study how some properties of the graph influence the structure of the group.

The structure of a finite group G such that $\Gamma(G)$ is not connected has been determined by Gruenberg and Kegel, in un unpublished article. Moreover all the simple groups G such that $\Gamma(G)$ is not connected have been described in [15], [8] and [9] and the almost simple ones in [10]. The diameter has also been studied: it has been proved in [11] that the diameter of $\Gamma(G)$ is less or equal 5 for any finite group G.

In this paper we study the finite groups G such that $\Gamma(G)$ is a *tree*, that is a connected graph without loops. We say that a graph is a *forest* if any connected component of the graph is a tree.

If G is a soluble group, we prove that $|\pi(G)| \leqslant 4$ (Proposition 2) and in the case $|\pi(G)| = 4$, the Fitting length of G is determined, in a more general situation.

PROPOSITION 3. – *Let G be a soluble group with* $\mathrm{diam}(\Gamma(G)) = 3$. *Then either $l_F(G) \leqslant 3$ or G has a normal section isomorphic to H, and $l_F(G) \leqslant 4$.*

We then consider the simple non abelian groups S such that $\Gamma(S)$ is a forest, describing them in List A. We show that no $\Gamma(S)$ is a tree. We classify all almost simple groups G with $\Gamma(G)$ a tree in Lemma 3. We show an example of a simple group S such that for any group H with $S \leqslant H \leqslant Aut(S)$, we have that $\Gamma(H)$ is not a tree, while there exists a group G with S as a composition factor and $\Gamma(G)$ a tree (see Example 3).

We study the general case, proving that if $\Gamma(G)$ is a tree, then G has the following structure.

THEOREM 5. – *Let G be a finite group such that $\Gamma(G)$ is a tree. If R is the soluble radical of G, then*

i) $\overline{G} = G/R$ is an almost simple group, that is $S \leqslant \overline{G} \leqslant \mathrm{Aut}(S)$, with S a finite simple non abelian group of List A;

ii) if $\overline{G} \neq 1$, then $|\pi(R)| \leqslant 3$.

We then describe the «non-modular» situation, that is groups G such that $\pi(R)$ is not contained in $\pi(\overline{G})$.

LEMMA 6. – *Let R be the soluble radical of G, with $\Gamma(G)$ a tree.*

i) If R is a t-group for some prime t not dividing $|\overline{G}|$, then $\overline{G}$ is one of the groups of List B;

ii) if $\pi(R) = \{p_1, p_2, p_3\}$, then $\pi(R) \nsubseteq \pi(\overline{G})$ and either

- *$p_2 = 2$, $p_3 = 3$ and $S = PSL(2, r)$, with $r = 7, 9, 17$ or $r = 2^a 3^b + 1$ and $r + 1 = 2t$ for some prime t, or*
- *$p_2 = 2$ and $S = A_5$;*

iii) if $\pi(R) = \{p_1, p_2\}$, and p_1 does not divide $|G/R|$, then $p_2 = 2$, and either

- *$S = A_5$ or $PSL(2, r)$, with $r = 7, 9, 17$ or $r = 2^a 3^b + 1$ and $r + 1 = 2t$ for some prime t, or*
- *$S = Sz(8)$, $Sz(32)$ or $PSL_2(8)$, or*

• *$S = A_7$, M_{11}, M_{22}, $B_2(3)$, $G_2(3)$, $U_4(3)$ or $PSL_2(q)$ and $|\pi_1(S)| \leq 2$ and $|\pi_i(S)| = 1$ for any $i > 2$.*

We show with an example how to construct most of them (see Example 3).

We prove that $|\pi(G)| \leq 8$ (Theorem 6) and show with an example that this bound is the best possible. In fact we prove that there is an extension G of a Suzuki group with a field automorphism, with $\Gamma(G)$ a tree and $|\pi(G)| = 8$ (see Example 2). If $\Gamma(G)$ a tree and $|\pi(G)| = 8$, we conjecture that the only non abelian composition factor of G is a Suzuki group. This conjecture is related to the unsolved problem in Number Theory, known as «twin prime problem».

2. – Soluble groups.

We begin with soluble groups. We say that a graph is a *forest* if any connected component of the graph is a tree. We observe that if $\pi(G)$ contains only two primes and $\Gamma(G)$ is connected, then $\Gamma(G)$ is a tree. We recall Proposition 1 of ([11]).

PROPOSITION 1 ([11]). – *Let G be a finite soluble group. If p, q, r are three different primes of $\pi(G)$, then G contains an element of order the product of two of these primes.*

We also need to point out some facts about soluble groups in which the prime graph is not connected. We recall that G is a *Frobenius group* if it has a subgroup H such that $H \cap H^x = 1$ for all $x \in G \setminus H$. Then $G = HF$, where F is the Fitting subgroup of G and H is called a *Frobenius complement*.

LEMMA 1. – *If G is soluble and $\Gamma(G)$ has more than two connected components, then G is either a Frobenius or a 2-Frobenius group and has exactly two components, one of which consists of the primes dividing the lower Frobenius complement; moreover G/F_2 is a cyclic group.*

PROOF. – The first statement is the corollary to Theorem A of [15]. A group is called 2-*Frobenius* if F_2 and G/F_1 are Frobenius groups, where $F_1 = \mathrm{Fit}(G)$ and $F_2/F_1 = \mathrm{Fit}(G/F_1)$. To prove the second statement we observe that if G is a soluble Frobenius group, then $G = F_2$. If G is a 2-Frobenius group, then F_2/F_1 is either cyclic or the product of an odd order cyclic group C by a generalized quaternion group Q_{2^n} (see for example 10.5.6 of ([14]). In the first case, G/F_2 is both a subgroup of $\mathrm{Aut}(F_2/F_1)$, which is abelian, and a Frobenius complement of G/F_1, which implies G/F_2 cyclic. In the second case $G/F_2 \leq \mathrm{Aut}(C) \times \mathrm{Out}(Q_{2^n})$. Now $\mathrm{Out}(Q_{2^n})$ is either an abelian 2-group, if $n > 3$, or S_3 if $n = 3$. Since $(|G/F_2|, |F_2/F_1|) = 1$, the prime 2 does not divide the order of G/F_2, which is therefore abelian and hence cyclic as before.

We can now prove

PROPOSITION 2. – *Let G be a finite soluble group. If $\Gamma(G)$ is a forest, then* $|\pi(G)| \leq 4$.

PROOF. – We first suppose that $\Gamma(G)$ is connected. We observe that if there are three distinct primes $p, q, r \in \pi(G)$ such that $p \sim q \sim r \sim p$, then $\Gamma(G)$ cannot be a tree.

Then for any prime $p \in \pi(G)$, there are at most two primes q_1, q_2 such that $q_1 \sim p \sim q_2$. In fact, if there exists q_3 such that $p \sim q_3$, then by applying Proposition 1 of [11] to q_1, q_2, q_3, we obtain, for example, $q_1 \sim q_2$ and this contradicts the fact that $\Gamma(G)$ is a tree.

Then $\Gamma(G)$ must be a chain. If $q_1, \ldots, q_n$ are the primes in $\pi(G)$, we can order them in a way such that $d(q_i, q_{i+j}) = j$, for any $i = 1, \ldots, n-1$ and $j = 1, \ldots, n-i$. If $n \geq 5$, we apply again Proposition 1 of [11] to q_1, q_3, q_5, and obtain, for example, $q_1 \sim q_3$ and this contradicts again the fact that $\Gamma(G)$ is a tree. This proves the Proposition in the case in which $\Gamma(G)$ is connected.

Let now $F = \mathrm{Fit}(G)$ be the Fitting subgroup of G and F_2 be the subgroup of G such that $F_2/F = \mathrm{Fit}(G/F)$. If $\Gamma(G)$ is not connected, then by Lemma 1 we know that the connected components of $\Gamma(G)$ are

$$\pi_1(G) = \pi(|F|) \cup \pi(|G/F_2|) \qquad \pi_2(G) = \pi(|F_2/F|).$$

Then $|\pi_2(G)| \leq 2 \geq |\pi(F)|$ since we suppose that $\Gamma(G)$ is a tree. If p, q are two distinct primes that divide $|F|$ and if there exists $r \in \pi_1(G) \backslash \{p, q\}$, then r divide $|G/F_2|$ and is coprime with $|F_2|$. Let $O_p(G)$ be the p-Sylow subgroup of F and $xO_p(G)$ an element of order r in $G/O_p(G)$ that acts on $F_2/O_p(G)$. Since $F_2/O_p(G)$ is not a nilpotent group, $xO_p(G)$ must fix some element. From the fact that r is in $\pi_1(G)$, the only possibility is that $xO_p(G)$ fixes an element of order q. Therefore we have $r \sim q$. In exactly the same way we can prove that $r \sim p$. But, since pq divides $|F|$, we also have $p \sim q$, against our hypothesis that $\Gamma(G)$ is a tree.

Then $|F| = p^n$. If there exist two other primes r, s in $\pi_1(G) \backslash \{p\}$ we can consider again an element x of order r of $G \backslash F_2$ acting on F_2. Since the action can not be fixed points free, the elements that can be fixed by x must have order p^i, that is $r \sim p$ in $\pi(G)$. Similarly $s \sim p$ and from the fact that G/F_2 is cyclic by lemma , we conclude that $r \sim s$, contradicting our hypothesis again.

We conclude that also $|\pi_1(G)|$ can not contain more than two primes and therefore $|\pi(G)| \leq 4$.

We would like to examine now the soluble groups in which the prime graph is a tree and which have exactly 4 prime divisors. It is easy to see that in this case the diameter of the prime graph is 3 and we want to prove that the Fitting length is bounded. Since it can be proved for the more general case of a soluble group in which the diameter of the prime graph is 3, we here give this proof. Before stating the Proposition we want to introduce the following group H and its subgroups T and V:

$$H = \langle a, b, x, y : a^4 = b^4 = x^4 = y^3 = 1,\ a^2 = b^2 = x^2, [b, a] = a^2,$$

$$[x, a] = [y, b] = ab, [y, a] = b^3, [x, y] = y^2, [x, b] = a^3 b \rangle;$$

$$T = \langle x, y \rangle; \qquad V = \langle a, b \rangle = \mathrm{Fit}(H).$$

Then H is the non-splitting covering of S_4, $H = VT$, where $V = \langle a, b \rangle = \mathrm{Fit}(H) \cong Q_8$, the quaternion group of order 8 and $H/V \cong S_3$.

If G is a soluble group, then the Fitting subgroup of G is not trivial. We can therefore define the Fitting series of a soluble group G, as follows:

$$F_0(G) = 1, \qquad \frac{F_n(G)}{F_{n-1}(G)} = \mathrm{Fit}\left(\frac{G}{F_{n-1}(G)}\right), \qquad \text{for } n \geqslant 1 .$$

Since G is a finite group, there exists a least $n \in \boldsymbol{N}$ such that $F_n(G) = G$. Then n is exactly the Fitting lenght of the group G, that we denote by $l_F(G)$.

PROPOSITION 3. – *Let G be a soluble group with* $\mathrm{diam}(\Gamma(G)) = 3$. *Then either $l_F(G) \leqslant 3$ or G has a normal section isomorphic to H, and $l_F(G) \leqslant 4$.*

PROOF If *diam* $(\Gamma(G)) = 3$, there exist two primes $p_1, p_4 \in \pi(G)$ such that $d(p_1, p_4) = 3$; therefore there exist two other primes p_2, p_3 such that

$$\overset{p_1}{\cdot}\!-\!\!-\!\!-\!\overset{p_2}{\cdot}\!-\!\!-\!\!-\!\overset{p_3}{\cdot}\!-\!\!-\!\!-\!\overset{p_4}{\cdot}\,.$$

As $p_1 \nsim p_4$, we can apply Proposition 1 to p, p_1, p_4 for any prime $p \in \pi(G)$. We therefore obtain $p \sim p_1$ or $p \sim p_4$. Moreover $p \sim p_1$ implies that $p \nsim p_4$ because, otherwise, $d(p_1, p_4) \leqslant 2$, against our hypothesis. If $\pi = \{p \in \pi(G) : p \sim p_1\} \cup \{p_1\}$, then $\pi' = \pi(G) \backslash \pi$ is exactly the set $\{p \in \pi(G) : p \sim p_4\} \cup \{p_4\}$.

We apply Proposition 1 to p, q, p_4 for $p, q \in \pi$ and obtain $p \sim q$; similarly for π'. Therefore both π and π' determine complete subgraphs of $\Gamma(G)$.

We consider now the following sets:

$$\pi_1 = \{p \in \pi : d(p, p_4) = 3\},$$
$$\pi_2 = \{p \in \pi : d(p, p_4) = 2\},$$
$$\pi_3 = \{p \in \pi' : d(p, p_1) = 2\},$$
$$\pi_4 = \{p \in \pi' : d(p, p_1) = 3\}.$$

We observe that $p_i \in \pi_i$, for $i = 1, 2, 3, 4$. If we set $\Psi = \pi_1 \cup \pi_4$, then $\Psi' = \pi_2 \cup \pi_3$.

If $O_\Psi(G) \neq 1$, then, for example, $O_{\pi_1}(G) \neq 1$. If x is an element of G of order p_4, x acts fixed-point-freely on $O_\pi(G) \geqslant O_{\pi_1}(G) \neq 1$. Therefore $O_\pi(G)$ is nilpotent and it is contained in the Fitting subgroup $\mathrm{Fit}(G) = F$ of G. Since p divides $|O_\pi(G)|$, for some $p \in \pi_1$, and $O_\pi(G) \leq F$, the Fitting subgroup F is a π-group and we can conclude that $O_\pi(G) = F$.

Let F_2 be the subgroup of G such that $F_2/F = \mathrm{Fit}(G/F)$. Then F_2/F is a nilpotent π'-group. Let Q be a q-Sylow subgroup of G, for $q \in \pi'$. Q acts fixed-point-freely on $O_{\pi_1}(G)$ because $q \nsim p$ for any $p \in \pi_1$. Therefore Q is a cyclic or a generalized quaternion group. Then F_2/F is either a cyclic group or the product of a cyclic group $\overline{C}$ of odd order by a generalized quaternion group $\overline{Q} \cong Q_{2^n}$. If $n \neq 3$, G/F_2 is a nilpotent group and therefore $l_F(G) \leq 3$. If $n = 3$, then G/F_2 is isomorphic to a subgroup of a direct product of an abelian group and a group isomorphic to S_3, the outer automorphism group of Q. If G/F_2 has not a normal subgroup isomorphic to S_3, then again $l_F(G) \leq 3$. Otherwise G/F has a normal section isomorphic to H, the Fitting length of G/F_2 is equal to 2 and therefore $l_F(G) = 4$.

We can now suppose $O_{\Psi}(G) = 1$. Then for any $p \in \Psi$ the p-Sylow subgroups of G are cyclic or generalized quaternion. In fact if $p \in \pi_1$, then for any $q \in \pi'$ we have that $p \nsim q$. Let P be a p-Sylow subgroup of G: if $O_{\pi'}(G) \neq 1$, then P acts fixed-points-freely on $O_{\pi'}(G)$. If $O_{\pi'}(G) = 1$, then $O_\pi(G) \neq 1$, as G is soluble. Moreover, since $O_p(G) = 1$ by our assumption, p does not divide $|O_\pi(G)|$. Therefore P acts fixed-points-freely on $O_{\pi,\pi'}(G)/O_\pi(G)$. In both cases P is a cyclic or a generalized quaternion group. The same argument holds if $p \in \pi_4$.

We consider now a $\{p_1, p_4\}$-Hall subgroup of G. We can suppose that its Fitting subgroup is a p_1-group. Since G is soluble, $N = O_{\Psi'}(G)$ is non-trivial. Let N_i be a π_i-Hall subgroup of N, $i = 2, 3$; then $N = N_2 N_3$ and, applying the Frattini argument to N and N_2, we obtain $G = NN_G(N_2)$. As p_1 and p_4 do not divide $|N|$, there exists a $\{p_1, p_4\}$-Hall subgroup U of G such that $U \leq N_G(N_2)$. If x is an element of order p_4 of U, x acts fixed-points-freely on N_2, which is therefore nilpotent. Then $N_2 = Dir Q_i$, with $i = 1, \ldots, r$, $Q_i \in \mathrm{Syl}_{q_i}(N_2)$ and $q_i \in \pi_2$ for any $i = 1, \ldots, r$. If $L = \Omega_1(\mathrm{Fit}(U))$, then, by the above remark, $L = < l >$ is a cyclic group. If Q_i is an abelian group, by Theorem 2.3 of [G], we have $Q_i = C_{Q_i}(L) \times [Q_i, L]$. If $[Q_i, L] \neq 1$ and the action of U on $[Q_i, L]$ is fixed-points-free, U should be a Frobenius complement and its centre should be non-trivial by Theorems 8.5, 8.8 of [H1]. But, as $\pi(U) = \{p_1, p_4\}$, $\Gamma(U)$ is not connected and the centre of U is trivial. Therefore there must exist a non-trivial p_1-element $x \in U$ such that $C_{[Q_i, L]}(x) \neq 1$. This means that $C_{[Q_i, L]}(L) \neq 1$, while $C_{Q_i}(L) \cap [Q_i, L] = 1$. Thus L centralizes Q_i.

If Q_i is not abelian, we consider the abelian group $Q_i/\mathrm{Frat}(Q_i)$. By the above argument $[Q_i/Frat(Q_i), L] = 1$. By Burnside's Theorem we also have that $[Q_i, L] = 1$. Then L centralizes N_2.

Let $n = n_2 n_3 \in N$, with $n_2 \in N_2$, $n_3 \in N_3$ and $l \in L$, then $[n_2 n_3, l] = [n_3, l]$ and therefore $[N, L] = [N_3, L]$.

We can also suppose that $[N_3, L] \leq N_3$. In fact as L permutes the π_3-Hall subgroups of N and $|L|$ is coprime with $|N: N_N(N_3)|$ which divides $|N_2|$, L must fix a π_3-Hall subgroup of N. Since L acts fixed-points-freely on N_3, N_3 is nilpotent and therefore $N_3 = \mathrm{Dir}\, R_i$, with $i = 1, \ldots, s$, $R_i \in \mathrm{Syl}_{r_i}(N_3)$ and $r_i \in \pi_3$ for any $i = 1, \ldots, s$. As $C_{R_i}(L) = 1$, by Theorem 3.5 of [G], $R_i = [R_i, L]$ for any $i = 1, \ldots, s$. Then $[N, L] = [N_3, L] = N_3$ is a normal subgroup of N, moreover it is also characteristic. Let $\widetilde{F}$ be the normal subgroup of G such that $\widetilde{F}/N = \mathrm{Fit}(G/N)$. Then $\widetilde{F}/N$ is a nilpotent Ψ'-group. By the above remark, all its Sylow subgroups are cyclic or generalized quaternion. Moreover if 2 divides $|\widetilde{F}/N|$, then the element of order 2 of $\widetilde{F}/N$ should be central in G/N. Therefore $\mathrm{diam}(\Gamma(G/N)) = 2$, while both p_1 and p_4 divide $|G/N|$ and their distance is 3. Therefore $\widetilde{F}/N$ is a cyclic group of odd order and $G/\widetilde{F}$ is an abelian group. Since $\widetilde{F}/N$ is nilpotent it must be either a π_1-group or a π_4-group. We prove that it is not a π_4-group. In fact let $\widetilde{U}$ be a Ψ'-Hall subgroup of G, containing U. The connected components of $\Gamma(\widetilde{U})$ are exactly π_1 and π_4. If we suppose that $\widetilde{F}/N$ is a π_4-group and set K a π_1-Hall subgroup of $\widetilde{U}$, by Corollary A of [W], for any $g \in \widetilde{U}$, we have $K \cap K^g = 1$. In particular, if $K \geq \mathrm{Fit}(U)$ and g is an element of order p_4 of U. But this is false because $\mathrm{Fit}(U) \lhd U$ and therefore $\widetilde{F}/N$ is a π_1-group. As $N/N_3 \cong N_2$ is a π_2-group, we have that $\widetilde{F}/N_3$ is a π-group and therefore an element of order p_4 acts fixed-points-freely on $\widetilde{F}/N_3$. By Thompson's Theorem, F/N_3 is nilpotent and the Fitting length of G is less or equal 3.

EXAMPLE 1. – *We give an example of a soluble group G, whose prime graph is a tree and has diameter equal to* 3. *Its Fitting length $l_F(G)$ is* 4 *and it has a normal section isomorphic to H.*

We consider the following example of a fixed-point-free action of H, the group defined before the preceding Proposition, on $P = \langle u, v : u^7 = v^7 = [u, v] = 1 \rangle$, an elementary abelian group of order 49. The action is the following

$$u^a = u^2 v^2, \quad u^b = u^{-1} v, \quad u^x = u^5 v, \quad u^y = u^4 v^2;$$

$$v^a = uv^5, \quad v^b = u^5 v, \quad v^x = u^2 v^2, \quad v^y = v^2.$$

Now we consider $F = P \times Q$, where $Q = \langle s, t : s^{13} = t^{13} = [s, t] = 1 \rangle$ is an elementary abelian group of order 169. We can define an action of H on Q in the following way:

$$s^a = s = s^b, \quad s^x = t, \quad s^y = s^3, \quad t^a = t = t^b, \quad t^x = s, \quad t^y = t^9.$$

Let $G = FH$, where the action of H on P and Q are the ones just described.

Then the Fitting length of G is 4, $\operatorname{diam}(\Gamma(G)) = 3$ and $\Gamma(G)$ is the following tree

$$\Gamma(G) = \overset{3}{\cdot}\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\cdot}\!\!-\!\!\!-\!\!\!-\!\!\overset{13}{\cdot}\!\!-\!\!\!-\!\!\!-\!\!\overset{7}{\cdot}.$$

3. – The general case.

For the study of the general case, we begin with the finite non abelian simple groups. We first prove a numerical Lemma.

LEMMA 2. – *Let $q = p^f$, where p is a prime and f a natural number; then*

i) *if f is even,* $3 \mid (2^f - 1)$, *if f is odd* $3 \mid (2^f + 1)$;

ii) $|\pi(q^2 - 1)| \leqslant 2 \Leftrightarrow q = 2, 3, 4, 5, 7, 8, 9, 17$;

iii) $|\pi((q^2 - 1)/(3, q - 1)| \leqslant 2 \Leftrightarrow q = 2, 3, 4, 5, 7, 8, 9, 16, 17, 25, 49, 97$ *or* $q = p$, $p - 1 = 3 \cdot 2^a$, $p + 1 = 2t$, $a \geqslant 2$ *and t an odd prime;*

iv) $|\pi((q^2 - 1)/(3, q + 1)| \leqslant 2 \Rightarrow q = 2^f$, *$f$ a prime or* $q = 3, 9$ *or* $q = p$ *and* $p + 1 = 3 \cdot 2^a$;

v) $|\pi((q-1)/(2, q-1)| \leqslant 2 \geqslant |\pi((q+1)/(2, q-1)| \Rightarrow q =$ $4, 9, 16, 81$ *or* $q = p^f$, $f = 1$ *or an odd prime.*

PROOF. – *i)* Let $f = 2c$, then $3 = (2^2 - 1)$ divides $2^{2c} - 1$. If f is odd, $3 = 2 + 1$ divides $(2^f + 1)$.

ii) We first suppose $p = 2$. If f is even we have $q + 1 = t^m$ for some prime t and $q - 1 = 3^n$. Let $f = 2c$ and suppose that c is even, then 3 does not divide $2^c + 1$ and therefore $q - 1$ cannot be a power of 3. Then c is odd but then $c = 1$ and $q = 4$.

If f is odd we have $q + 1 = 3^m$ and $q - 1 = t^n$ for some prime t. From lemma 4 of [10] we know that $((2^f + 1)/(2 + 1), 2 + 1) = (f, 2 + 1) = (f, 3)$. If r is an odd prime that divide f, and $r \neq 3$, then $(2^r + 1) \mid (2^f + 1)$ and $2^r + 1$ is not a power of 3.

Then $f = 3^s$. If we suppose $s \geqslant 2$, we can write $f = 9 \cdot 3^{s-2}$. But then $(2^9 + 1) = 19 \cdot 3^3$ divides $2^f + 1$, that therefore can not be a power of 3.

Let now p be an odd prime, then either $q - 1 = 2^f$ and $q + 1 = 2 \cdot t^m$, or $q - 1 = 2 \cdot t^m$ and $q + 1 = 2^f$, for some prime t.

In the first case $q = 2^f + 1$ and, if f is odd, we conclude that $q = 3$ or 9. If f is even, $f - 1$ is odd and again we can conclude that $t = 3$ and $q = 5$ or 17.

In the second case $q = 2^f - 1$ and, if f is even, $p = 3$ and therefore $q = 3$. If f is odd, we have $t = 3$ and $q = 7$.

iii) If $(3, q-1) = 1$ then we are in case *ii*). We can therefore suppose that 3 divides $(q-1)$. If $p = 2$, then $f = 2c$. If $c = c_1 c_2$ is not a prime, then we have three distinct primitive Zsigmondy's divisor respectively of $(4^{c_1} - 1)$, $(4^{c_1 c_2} - 1)$ and $(4^{c_1 c_2} + 1)$. If $c = 2$, then we add $q = 16$ to the list. If $c > 1$, then we apply the same argument to $(4^c - 1)$, $(4 + 1)$ and $(4^c + 1)$. We can now suppose p odd and $q \geqslant 11$. We first suppose $q - 1 = 3 \cdot 2 \cdot s^{\alpha}$ and $q + 1 = 2^{\beta}$, and therefore $2^{\beta - 1} - 1 = 3 s^{\alpha}$. But then we are in case ii), since $\beta - 1$ is even. The only case we have to add is $q = 31$. Then $q - 1 = 3 \cdot 2^{\alpha}$ and $q + 1 = 2 \cdot t^{\beta}$. Then either $f = 2$ or $f = 1$. If $f = 2$, then we are again in case ii), and we get $q = 25$ and 49. It can be easily seen that $t^{\beta} - 1 = 3 \cdot 2^{\alpha - 1}$ and again we conclude $\beta = 1$ (the case $\beta = 2$ does not give any new value for q).

iv) The proof is similar to the one of *iii*).

v) Suppose $\pi((q-1)/(2, q-1)) = \{r, t\}$. If $q = p^{f_1 f_2}$, and $q - 1 > 2$, then by Zsigmondy's argument, we can conclude that f is a prime or $f = 1$. Moreover if $f = 2$, we conclude by ii). We now consider $p = 2, 3$ and suppose $(f_1, f_2) = 1$, then

$$\frac{p^{f_1} - 1}{p - 1} = r^{\alpha}, \qquad \frac{p^{f_2} - 1}{p - 1} = t^{\beta}, \qquad \frac{p^{f_1 f_2} - 1}{p^{f_1} - 1} = t^{\beta} r^{\alpha},$$

which is clearly impossible. Then $q = p^{f^2}$, and f a prime. We then use the second inequality and Zsigmondy's argument to conclude that f cannot be odd. We therefore get $q = 16$ or 81.

We give now a list of simple non abelian groups S, such that the connected components of $\Gamma(S)$ are trees, as we prove in proposition 4. We recall that if G is not soluble, the connected components $\pi_i(G)$, $i > 2$ are complete, that is if $r_1, r_2 \in \pi_i(G)$ then $r_1 \sim r_2$ (see Lemma 5 and 6 of [15]).

LIST A:

A_5, A_6, A_7, A_8; M_{11}, M_{22};
$PSL_4(3)$; $B_2(3)$, $G_2(3)$, $U_4(3)$, $U_5(2)$, ${}^2F_4(2)'$;
$PSL_2(q)$ with $|\pi((q-1)/(2, q-1))| \leqslant 2 \geqslant |\pi((q+1)/(2, q-1))|$,
$PSL_3(q)$ with $|\pi((q^2+q+1)/(3, q-1))| \leqslant 2 \geqslant |\pi((q^2-1)/(3, q-1))|$,
$PSU_3(q)$ with $|\pi((q^2-q+1)/(3, q+1))| \leqslant 2 \geqslant |\pi((q^2-1)/(3, q+1))|$,
$Sz(q^2)$ with $|\pi(q^2 - \sqrt{2}q + 1)| \leqslant 2 \geqslant |\pi(q^2 + \sqrt{2}q + 1)|$, $|\pi(q^2 - 1)| \leqslant 2$, where $q^2 = 2^f$, or $q = 2^{f^2}$ with f an odd prime;
Ree(q^2) with $|\pi(q^2 - \sqrt{3}q + 1)| \leqslant 2 \geqslant |\pi(q^2 + \sqrt{3}q + 1)|$, $|\pi(q^2 - 1)| \leqslant 2 \geqslant |\pi(q^2 + 1)|$, where $q^2 = 3^f$, with f an odd prime.

PROPOSITION 4. – *Let G be a finite non abelian simple group. Then $\Gamma(G)$ is a forest if and only if G is in List A. Therefore, in any case, $\Gamma(G)$ is not a tree.*

PROOF. – The second statement follows immediately from the first. In fact if G is not in List A and the first statement is true, then $\Gamma(G)$ can not be a tree. If G is in List A, then $\Gamma(G)$ is not connected (see [15], [8] and [9]).

If G is not a group in List A, we prove there exists three different primes $p, q, r \in \pi(G)$ such that $p \sim q \sim r \sim p$. We use the classification of finite simple groups.

If G is a sporadic group, $G \neq M_{11}, M_{22}$, we can check in the Atlas [1] that $2 \sim 3 \sim 5 \sim 2$.

If $G = A_n$, the alternating group on n symbols, and $n \geq 9$, we can choose the following elements, with their respective orders:

$$|(12)(34)(567)| = 6, \quad |(12)(34)(56789)| = 10 \quad \text{and} \quad |(123)(45678)| = 15\,.$$

Then also in this case we have $2 \sim 3 \sim 5 \sim 2$.

Let now $G = {}^d L_n(q)$ be a finite simple group of Lie type of rank n, defined over the finite field K of order $q = p^f$. We observe that if $n \leq m$, then ${}^d L_m(q)$ contains an isomorphic copy of ${}^d L_n(q)$. It will therefore be sufficient to prove the existence of three primes with that property for the groups of minimal Lie rank, for any type.

We note that if the rank of G is greater or equal 3 or G is of type G_2, then $q^2 - 1$ divides the order of a maximal torus T. Similarly, if $G = B_2$, there exists a maximal torus of order $(q^2-1)/(2, q-1)$, but $\pi((q^2-1)/(2, q-1)) = \pi(q^2-1)$. Then if $q \geq 11$ and $q \neq 17$, by Lemma 1ii), we know that $|\pi(q^2-1)| > 2$. We recall that the maximal tori are, in particular, abelian groups and therefore if their order is divisible by more than three primes, the prime graph of G can not be a tree. Therefore, if rank of G is greater or equal 3, or G is of type B_2 or G_2, and $q \geq 11$, $q \neq 17$, $\Gamma(G)$ is not a tree.

For the remaining primes, that is $q = 2, 3, 4, 5, 7, 8, 9, 17$ it is enough to make some calculations on the order of the centralizers of involutions or on the order of other maximal tori.

For the groups of type $PSL_2(q)$, $PSL_3(q)$, $PSU_3(q)$, $Sz(q)$ and $\mathrm{Ree}(q)$ it is sufficient to recall which are the connected components.

We now examine the *almost simple* groups G, that is $S \leq G \leq \mathrm{Aut}(S)$, with S a finite simple non abelian group.

LEMMA 3. – *Let G be an almost simple group such that $\Gamma(G)$ is a tree. Then G is one of the following:*

$\mathrm{Aut}(A_6)$

$PSL_2(p^f) < \alpha >$, where p is a prime greater than 3, f is an odd prime and α is a field automorphism of order f.

$PGL(4, 3) \leq G \leq \mathrm{Aut}(PSL(4, 3))$

$\mathrm{Aut}(B_2(3))$,

$PSU(4,3)\langle\delta\rangle \leqslant G \leqslant \mathrm{Aut}(PSU(4,3))$, where δ is a diagonal automorphism of order 2.

$PGL(3,4) \leqslant G \leqslant PGL(3,4)\langle\alpha\rangle$ with α a graph-field automorphism of order 2.

$PSL(3,q)\langle\alpha\rangle$, with $q = 9, 25, 49$ and α a field automorphism of order 2.

$PGU(3,8) \leqslant G \leqslant \mathrm{Aut}(PSU(3,8))$.

$Sz(2^{f^2})\langle\alpha\rangle$ with α a field automorphism of order f.

PROOF. – It is enough to consider the groups in List A and check the connected components of the prime graph of the subgroups of their automorphism's group, using [10].

We recall the definition of *soluble radical* of a group G: it is the maximal soluble normal subgroup of G.

We can now describe the structure of the groups in which the prime graph is a tree.

THEOREM 5. – *Let G be a finite group such that $\Gamma(G)$ is a tree. If R is the soluble radical of G, then*

i) *$\overline{G} = G/R$ is an almost simple group, that is $S \leqslant \overline{G} \leqslant Aut(S)$, with S a finite simple non abelian group of List A;*

ii) *if $\overline{G} \neq 1$, then $|\pi(R)| \leqslant 3$.*

PROOF. – *i)* Let R be the soluble radical of G, $\overline{G} = G/R$ and $\overline{F} = F^*(\overline{G})$ the generalized Fitting subgroup of $\overline{G}$. We note that the Fitting subgroup of $\overline{G}$ is trivial and therefore $\overline{F} \cong \prod_i^n S_i$, with S_i simple non abelian groups. If $n > 1$, there exist two distinct odd primes $p, q \in \pi(S_1)$, such that $2 \sim q \sim p \sim 2$. But this contradicts our hypothesis that $\Gamma(G)$ is a tree. Then $\overline{F} \cong S$ is a simple non abelian group and, by Lemma 4, S must be one of List A.

ii) We can now suppose both R and $\overline{G}$ non trivial. Let S be the simple group of List A such that $S \leqslant \overline{G} \leqslant \mathrm{Aut}(S)$. We prove that

$$\text{if } P \in \mathrm{Syl}_p(R) \text{ is a cyclic or generalized quaternion group,}$$

$$(*)\qquad \text{then } p \sim s \ \forall\ s \in \pi(S).$$

If $N = N_G(P)$, then by the Frattini argument, $G = RN$. If $C = C_G(P)$ is contained in R, then N/C has a factor isomorphic to a non abelian simple group: this contradicts the fact that $N/C \leqslant Aut(P)$, which is abelian. Then $RC/R = \overline{C}$ is a normal non trivial subgroup of $\overline{G}$. Therefore $\overline{C}$ must contain an isomorphic copy of S and s divides the order of C, for any $s \in \pi(S)$. We also prove that

$$(**)\qquad \text{if } p \in \pi(R),\ 2 \nsim p, \qquad \text{then } 2 \sim s\,\forall\, s \in \pi(S).$$

Let p be a prime in $\pi(R)$ such that $2 \nsim p$ and P a p-Sylow subgroup of R. If $N = N_G(P)$, then by the Frattini argument, $G = RN$. If $C = C_G(P)$ is not contained in R, then $RC/R = \overline{C}$ is a normal non trivial subgroup of $\overline{G}$. Therefore $\overline{C}$ must contain an isomorphic copy of S and 2 divides the order of C, contradicting our hypothesis. Then $C \leqslant R$, that is $C \leqslant N \cap R$. Since $\overline{G} \cong N/N \cap R$, N/C has a quotient isomorphic to $\overline{G}$. Therefore 2 divides $|N/C|$ and there exists an element x in N, such that x acts fixed points free on P and $|xC| = 2$. This implies that xC is a central element in N/C (see [13]). Then for any $s \in \pi(N/C)$, and therefore for any $s \in \pi(S)$, we have $2 \sim s$ in N/C and therefore in G.

If $\Gamma(R)$ is not connected and $\pi_2(R)$ contains two distinct primes p, q, we know, by Lemma 1, that $p, q \in \pi(|F_2/F|)$, where F_2 is the second Fitting subgroup of R, and F_2 is a Frobenius group. Therefore both the p- and the q-Sylow subgroups are cyclic or generalized quaternion.

If $\Gamma(R)$ is connected, then by the proof of Lemma 2 ii) we get that also in this case we have two primes p, q in $\pi(R)$ such that both the p- and the q-Sylow subgroups are cyclic or generalized quaternion.

We can apply $(*)$ to p and q, but this gives a loop in $\Gamma(G)$. Therefore $\Gamma(R)$ contains at most 3 primes.

We now want to list all the almost simple groups G such that $|\pi_i(G)| = 1$ for any i.

LIST B:

A_5, S_5 $A_6 \cong PSL(2, 9)$, $PSL(2, 9)\langle\alpha\rangle$, with α a graph automorphism, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 17)$, $PSL(3, 4)$, $Sz(8)$, $Sz(32)$.

LEMMA 4. – *Let G be an almost simple group such that $|\pi_i(G)| = 1$ for any i, then G belongs to the List B.*

PROOF. – $2, 3 \in \pi_1(S)$ for any group of List A, except for A_5, A_6, $PSL(3, 4)$ and the two families $PSL(2, q)$ and $Sz(2^f)$.

For the groups $PSL(2, q)$, with q even, we must have $|\pi(q-1)| \leqslant 1 \geqslant |\pi(q+1)|$, that is $|\pi(q^2-1)| \leqslant 2$. If q is odd, then we must have $|\pi((q-1)/2| \leqslant 1 \geqslant |\pi(q+1)/2|$, that is $|\pi(q^2-1)| \leqslant 2$, since 8 divides q^2-1. Therefore we are in case ii) of Lemma 2.

If $G \cong Sz(q)$, then $q = 2^f$ with f an odd prime, and $\pi_3(G) \cup \pi_4(G) = \pi(q^2+1)$. Since 5 divides $2^{2f}+1$, we can suppose that $\pi_3(G) = \{5\}$. But $((4^f+1)/5, 4+1) = (f, 4+1)$ (see Lemma 4 ii) of [10]), and therefore the highest power of 5 dividing $|G|$ is 25. We conclude that $q = 8$ or $q = 32$.

For the almost simple groups which are not simple, it is enough to consider the automorphism group of these simple groups.

We now want to examine more closely the situation in which $\pi(R) \nsubseteq \pi(G/R)$. To do this we shall often need actions which are «nearly» fixed points free.

When «nearly» means «except some elements of order a prime p», we have the following definition. A group acts p'*-semiregularly* over a finite dimensional $\mathcal{F}$-vector space V, $\mathcal{F}$ a field of characteristic t, if every nontrivial p'-element acts without fixed points over $V\setminus\{0\}$. The action is said *separable* if t does not divide $|G|$, *inseparable* otherwise. If the action is inseparable, then of course $p = t$ (see [6]). The groups acting p'-semiregularly have been classified by Fleischmann, Lempken and Tiep in [6].

Another result that we shall need concerns the action of Frobenius groups.

LEMMA 5. – *Let G be a group acting faithfully on a vector space V defined over a field of characteristic t. If $O_p(G) \neq 1$, for some $p \neq t$, then for any $q \in \pi(G/O_p(G))$, we have either $p \sim q$ or $t \sim q$.*

PROOF. – If $Q \in \mathrm{Syl}_q(G)$, we define $N = O_p(G)\,Q$. Then if $p \nsim q$, N is a Frobenius group acting faithfully on V. Therefore by Lemma 1, iv) of [12] there exists $w \in V$ such that for any $x \in Q$, we have $w^x = w$, that is $t \sim q$.

We fix some notations. If $\Gamma(G)$ is a tree, then by Theorem 5 we know the structure of G. Then we define

$R =$ the soluble radical of G,
$S =$ the only non abelian simple factor of G,
$\overline{G} = G/R$,
$N =$ the normal subgroup of G such that $N/R \cong S$.

We finally want to remark that if we have a faithful action of a group G on a vector space V, an element $g \in G$ acts fixed points free over V if and only if $(\chi_{|H}, 1_H)_H = 0$, where $(.,.)$ is the inner product of characters, $H = \langle g \rangle$ and χ is the character defined by the action.

LEMMA 6. – *Let R be the soluble radical of G, with $\Gamma(G)$ a tree.*

i) *If R is a t-group for some prime t not dividing $|\overline{G}|$, then $\overline{G}$ is one of the groups of List B;*

ii) *if $\pi(R) = \{p_1, p_2, p_3\}$, then $\pi(R) \nsubseteq \pi(\overline{G})$ and either*

- *$p_2 = 2$, $p_3 = 3$ and $S = PSL(2, r)$, with $r = 7, 9, 17$ or $r = 2^a 3^b + 1$ and $r + 1 = 2t$ for some prime t, or*
- *$p_2 = 2$ and $S = A_5$;*

iii) *if $\pi(R) = \{p_1, p_2\}$, and p_1 does not divide $|G/R|$, then $p_2 = 2$, and either*

- *$S = A_5$ or $PSL(2, r)$, with $r = 7, 9, 17$ or $r = 2^a 3^b + 1$ and $r + 1 = 2t$ for some prime t, or*
- *$S = Sz(8)$, $Sz(32)$ or $PSL_2(8)$, or*

• *$S = A_7$, M_{11}, M_{22}, $B_2(3)$, $G_2(3)$, $U_4(3)$ or $PSL_2(q)$ and $|\pi_1(S)| \leq 2$ and $|\pi_i(S)| = 1$ for any $i > 2$.*

PROOF. – *i*) Let R be a t-group for some prime t not dividing $|\overline{G}|$. Let V be $\Omega_1(Z(R))$ and C be $C_G(V)$. Then $R \leq C \Downarrow G$, and therefore either $C \geq N$, and therefore any connected component of $\Gamma(N)$ can contain at most one prime, that is $S \in \mathrm{List}\,B$, or $C = R$ and then we have an action of $\overline{G}$ on V.

It can be checked in the Atlas, that if $S \cong A_7$, A_8, M_{11}, M_{22}, $A_3(3)$; $B_2(3)$, $G_2(3)$, $PSU_4(2)$, $PSU_4(3)$, $PSU_5(2)$, ${}^2F_4(2)'$, then $2 \sim 3 \sim t \sim 2$.

For the groups $PSL(2, p^f)$, the character table can be easily deduced from the one of $SL(2, p^f)$, which can be found in paragraph 38 of [5]. From these tables, one can see that if a connected component of $\Gamma(S)$ contains two primes s_1, s_2, then $s_1 \sim t \sim s_2$. Therefore we only need to consider the groups of List B.

The same is true for the Suzuki groups $Sz(q)$. In fact we want to prove that for any character χ we have that $(1_H, \chi_{|H}) \neq 1$, for $H = T_1 = \langle x \rangle$ or $H = T_2 = \langle y \rangle$ with $|x| = q + \sqrt{2q} + 1$ and $|y| = q - \sqrt{2q} + 1$. We first consider the case $H = T_1$. From the table of characters of the Suzuki groups [2], it is easy to check that this is true for any character $\chi \neq \Theta_l$. We only prove this case. We first observe that the inner product of Π and Θ_l is

$$0 = (\Pi, \Theta_l) = q^2(q-1)(q-\sqrt{2q}+1) - q^2(q-1)(q-\sqrt{2q}+1)(\Sigma_{1 \neq g \in T_1} \Theta_l(g))/4$$

$$\Rightarrow \Sigma_{1 \neq g \in T_1} \Theta_l(g) = 4 \Rightarrow (1_H, \Theta_l|_H) = \Theta_l(1) + \Sigma_{1 \neq g \in T_1} \Theta_l(g) > 0 .$$

The proof for T_2 is exactly the same. Then $Sz(q)$ must belong to List B.

We observe that a p-Sylow subgroup of S acts on R. Since the 3-Sylow subgroups of $PSL(3, q)$, $PSU(3, q)$ and $Ree(3^f)$ are not cyclic, they cannot act fixed points free. Therefore for these groups, we have $2 \sim 3 \sim t \sim 2$.

Therefore the only groups which admit a representation in coprime characteristic are the groups in List B. Moreover for any group in List B we have such a representation: if $H \in$ List B, and t is a prime not dividing $|H|$, there exists a group $G = VH$, where V is an elementary abelian t-group V on which H acts, such that $t \sim s$, for any $s \in \pi(H)$.

ii) We first suppose that

$$\Gamma(R) = \overset{p_1}{\cdot} \!\!-\!\!-\!\!-\!\! \overset{p_3}{\cdot} \qquad \overset{p_2}{\cdot}$$

This implies that the p_2-Sylow subgroups of R are cyclic or generalized quaternion groups. Then by $(*)$ of Theorem 5, we have $p_2 \sim s$ for any $s \in \pi(S)$. Therefore we can suppose that p_1 does not divide $|S|$. If R is a 2-Frobenius group and F is the Fitting subgroup of R, then R/F is a Frobenius group and we can suppose that the Frobenius complement is a p_3-group. Therefore we can apply $(*)$ of Theorem 5 both to the prime p_2 and the prime p_3. This gives a contradiction, since $|\pi(S)| \geqslant 3$ for any simple group S. The same is true if the Frobenius complement is a p_1-group. Then R is a Frobenius group, with Frobenius kernel F, a $\{p_1, p_2\}$-Hall subgroup of R. Let P_1 be a p_1-Sylow subgroup of R, $V_1 = \Omega_1(Z(P_1))$ and $C_1 = C_G(V_1)$. If C_1 is not contained in R, then C_1 must contain N, but this gives a contradiction. Then $C_1 = F$ and $\Gamma(N/F)$ is connected. If p_3 does not divide $|S|$, repeating the same argument, we get two actions of N/F over V_1 and $V_3 = \Omega_1(Z(P_3))$, with P_3 the p_3-Sylow subgroup of R. Since $\Gamma(N/F)$ is connected, one of these actions must be fixed-points free. Therefore $N/F \cong SL_2(5)$ and $p_2 = 2$. Otherwise p_3 divides $|S|$ and then the action of N/F must be p_3'-semiregular and separable. Therefore by proposition 5.2 of [6], $p_3 = 3$ and $N/F \cong SL(2, r)$ with $r = 7, 9, 17$ or $r = 2^a \cdot 3^b + 1$ and $r + 1 = 2 \cdot t$ for some prime t. Or $N/F \cong SL(2, 5)$ and p_3 can be any prime. We conclude that $SL(2, r) \leqslant G/F \leqslant GL(2, r)$ and any of these groups can be realised, by theorems 5.2 and 4.1 of [6].

We now suppose that $\Gamma(R)$ is connected and $p_1 \sim p_3 \sim p_2$. If $O_{p_1, p_2}(R) \neq 1$, then for example $O_{p_1}(R) \neq 1$, and a p_2-Sylow P_2 of R acts fixed points-free over $O_{p_1}(R)$. Otherwise we can consider the quotient modulo $O_{p_3}(R)$ and use a similar argument. We can then suppose that, for example, the p_2-Sylow subgroups of R are cyclic or generalized quaternion. Then by $(*)$ of Theorem 5, we have $p_2 \sim s$ for any $s \in \pi(S)$ and therefore p_1 does not divide $|S|$. If $O^{p_1}(R) \neq R$, then $G/O^{p_1}(R)$ is as in case 1. But this is not possible because in case 1 we should have $s \sim p_1$ for any $s \in \pi(S)$, giving a loop. Therefore $O^{\{p_3, p_2\}}(R) < R$ and we consider $\widetilde{G} = G/O^{\{p_3, p_2\}, p_1}(R)$. We recall that $p_1 \sim p_3 \sim p_2 \sim s$, for any $s \in \pi(S)$. If p_3 does not divide $|S|$, then we have a fixed points free action of G/C_1 over $V_1 = \Omega_1(Z(P_1))$, where P_1 is the p_1-Sylow subgroup of $\widetilde{G}$ and $C_1 = C_{\widetilde{G}}(V_1)$. If p_3 divides $|S|$, we have a p_3'-semiregular and separable action and we can conclude as before.

In both cases, we have seen that $\pi(R)$ is not contained in $\pi(\overline{G})$ and the diameter of $\Gamma(G)$ is less or equal 3.

iii) We suppose that $\Gamma(R)$ is not connected, $\pi(R) = \{p_1, p_2\}$, and that p_1 does not divide G/R. As in case ii), we can prove that R is a Frobenius group. We first suppose that $P_1 = \mathrm{Fit}(R) = \mathrm{Fit}(G)$, with $P_1 \in Syl_{p_1}(R)$. Then by $(*)$ of the proof of Theorem 5 we have $p_2 \sim s$ for any $s \in \pi(S)$. If $V_1 = \Omega_1(Z(P_1)$, then $C_1 = C_G(V_1))$ is a normal subgroup of G. If C_1 is not contained in R, then $C_1 \geqslant N$, and this gives a loop in $\Gamma(G)$, since $\pi(S)$ contains at least 3 primes. Therefore $C_1 = P_1$ and $\Gamma(N/C_1)$ is connected and the action of N/C_1 over V_1 must be s'-semiregular for some prime $s \in \pi(S)$. Then by Lemma 1.3(iv) of [6], we can apply Proposition 5.2 of [6], concluding as in the previous case.

If $P_2 = \mathrm{Fit}(R)$ is the p_2-Sylow subgroup of R, we can consider the group G/P_2 and we are in case i), that is $S \in \mathrm{List}\,B$. Moreover we have an action of G/C_2 over $V_2 = \Omega_1(Z(P_2))$, where $C_2 = C_G(V_2)$. Since $\Gamma(G/C_2)$ is connected, we have $C_2 = P_2$ and the action must be p_2'-semiregular and inseparable. By theorem 4.1 and following of [6], we have $p_2 = 2$ and $G/R \cong Sz(8)$, $Sz(32)$ or $PSL(2, 8)$. Any of these representations exist.

We now turn to the case $\Gamma(R)$ connected. Since R is soluble, we can suppose that $R = FH$, with F a normal p_1- (or p_2)-Sylow subgroup of R and H a p_2 (or p_1)-Sylow subgroup of R, and $\Gamma(R)$ connected. (It is enough to consider $O^{p_1, p_2}(R)$ or $O^{p_1, p_2, p_1}(R)$ and make the quotient). If F is a p_2-Sylow subgroup, then the group G/F is as in case i) and $\Gamma(G/F)$ is connected. Moreover we have a p_2'-semiregular action of G/F on F and therefore $p_2 = 2$ and $S = Sz(8)$, $Sz(32)$ or $SL(2, 8)$.

Then F is the p_1-Sylow subgroup of R and we define $V_1 = \Omega_1(Z(F))$ and $C_1 = C_G(V_1)$. If C_1 is not contained in R; then using the same argument as before we get again $p_2 = 2$ and $S = Sz(8)$, $Sz(32)$ or $SL(2, 8)$. Then $C_1 \leqslant R$ and we consider the action of G/C_1 over V_1. If $p_2 \neq 2$, then $2 \sim p_1$, since $O_2(G/C_1) = 1$. The same is true for p_2, and then we get a loop in $\Gamma(G)$. Then $p_2 = 2$. If $C_1 = R$, then $S \in \mathrm{List} B$ and again $S = Sz(8)$, $Sz(32)$ or $SL(2, 8)$. If $C_1 < R$, then $O_2(G/C_1) \neq 1$, and we can apply Lemma 3, obtaining either $s \sim 2$ or $s \sim p_1$. Therefore $|\pi_i(S)| = 1$ for $i > 1$ and $|\pi_1(S)| \leqslant 2$. Then S can only be one of these groups A_7, M_{11}, M_{22}, $PSL(2, q)$, $B_2(3)$, $G_2(3)$, $U_4(3)$. In any case $diam(\Gamma(G)) \leqslant 3$.

THEOREM 6. – *If $\Gamma(G)$ is a tree, then* $|\pi(G)| \leqslant 8$.

PROOF. – If G is a soluble group, then $|\pi(G)| \leqslant 4$ by Proposition 2.

If G is an almost simple group, then $|\pi(G)| \leqslant 6$, except when $G = Sz(2^{f^2}) < \alpha >$ with α a field automorphism of order f, in which case $|\pi(G)| \leqslant 8$.

If $\pi(R) \nsubseteq \pi(\overline{G})$ then by Lemma 6, $|\pi(G)| \leqslant 6$.

Therefore the only case we have to consider is when $\pi(R) \subseteq \pi(\overline{G})$. If $S \in$ List A, then $|\pi(S)| \leqslant 7$, except when $S = \mathrm{Ree}(q^2)$ and $|\pi(S)| \leqslant 8$. We prove that $|\pi(G) \setminus \pi(S)| \leqslant 1$. If this set is not empty, then S must be a finite simple group of Lie type and there exists an element α of $G \setminus S$ conjugated to a field automorphism of order a prime f that does not divide the order of S (see [10]). We also know that 2 divides $|C_S(\alpha)|$. Therefore if there are two distinct primes f_1 and f_2 in $\pi(G) \setminus \pi(S)$, we have $f_1 \sim 2 \sim f_2 \sim f_1$, against the hypothesis that $\Gamma(G)$ is a tree.

For the case $S = \mathrm{Ree}(q^2)$, we can also prove that $|\pi(G) \setminus \pi(S)|$ is empty. We recall that the characteristic of the field over which $\mathrm{Ree}(q)$ is defined is 3, that is $q = 3^m$. Therefore, if $f \in \pi(G) \setminus \pi(S)$ and f is the order of a field automorphism α, then 3 divides $|C_S(\alpha)|$. Then we have $f \sim 2 \sim 3 \sim f$, against the hypothesis that $\Gamma(G)$ is a tree.

We now show some examples. First we prove that the bound of the theorem 6 is the best possible.

EXAMPLE 2. – *This is an example of an almost simple group in which the prime graph is a tree and* $|\pi(G)| = 8$.

Let $S = {}^2B_2(q^2)$, with $q^2 = 2^9$, and $G = S\langle \alpha \rangle$ where α is a field automorphism of S of order 3. Then the connected components of $\Gamma(S)$ are described in [8] and are the following

$\pi_1(S) = \{2\}$,
$\pi_2(S) = \pi(q^2 - 1) = \{7, 73\}$,
$\pi_3(S) = \pi(q^2 - \sqrt{2}q + 1) = \{13, 37\}$,
$\pi_4(S) = \pi(q^2 + \sqrt{2}q + 1) = \{5, 109\}$.

Moreover, since $C_S(\alpha) \cong {}^2B_2(2^3)$, we have that $\pi(C_S(\alpha)) = \{2, 5, 7, 13\}$ and therefore $\Gamma(G)$ is a tree and $|\pi(G)| = 8$.

If $|\pi(G)| = 8$, then the only other possibility for S are the Ree groups $\mathrm{Ree}(q^2)$, when $q^2 = 3^f$, with f a prime. Let f be a prime, $f \leq 100$, $\mathrm{Ree}(q^2)$ belongs to the List A if and only if $f \leq 11$ and in these cases $|\mathrm{Ree}(q^2)| \leq 7$. Moreover the problem of finding a prime f such that $3^f - 1 = 2t$ and $3^f + 1 = 4r$, for some primes r, t, is equivalent to the «twin prime problem», which is still unsolved. We therefore don't know if there is any example of this type.

We now show an example with $R \neq 1$.

EXAMPLE 3. – *This is an example of a group G such that* $|\pi(R) \setminus \pi(\overline{G})| = 2$ *and* $\Gamma(G)$ *is a tree, showing that also the bound in Theorem* 6 ii) *is the best possible. Moreover this is an example of a group* $S \in \mathrm{List} A$ *such that for any group* H *with* $S \leq H \leq \mathrm{Aut}(S)$, *we have that* $\Gamma(H)$ *is not a tree, while there exists a group G with S as a composition factor and* $\Gamma(G)$ *a tree.*

We consider the group $M \cong SL(2, 5)$ and its complex character $\chi = \theta_1$ (see par. 38 of [5]). Let ϱ be the corresponding complex representation of M of degree 4 and V be the $M\mathbb{C}$-module. Easy calculations shows that $(\chi_{|H}, 1_H)_H = 0$ for any cyclic subgroup H of M except for the subgroups of order 3. By theorem 3.8 (see chapter 3) of [3], for any prime p coprime with the order of M, there exists a field $\boldsymbol{F}_q$, $q = p^n$ such that for any irreducible representation of M in $GL(n, \boldsymbol{C})$, there is an irreducible representation of M in $GL(n, \boldsymbol{F}_q)$.

Let then $Q = F_q^4$, q as just described and suppose also that $p \neq 29$. Then Q is an elementary abelian p-group and we can make M acts $3'$-semiregularly over Q, while an element of order 3 of M centralizes a non trivial element of Q.

Let now P be an elementary abelian group of order 29^2, then there is a fixed points free action of M over P (see exercise 3.4.11 of [4]). If $F = P \times Q$, then M acts over F in the way we have just described. Let $G = FM$, then the soluble radical R of G is $F\langle z \rangle$, where $\langle z \rangle = Z(M)$ is a cyclic group of order 2, $\pi(R) \setminus \pi(\overline{G}) = \{29, p\}$ and

$$\Gamma(G) = \overset{29}{\cdot}\!-\!\!-\!\!-\!\overset{p}{\cdot}\!-\!\!-\!\!-\!\overset{3}{\cdot}\!-\!\!-\!\!-\!\overset{2}{\cdot}\!-\!\!-\!\!-\!\overset{5}{\cdot}$$

is a tree. We have already observed that neither $\Gamma(A_5)$ nor $\Gamma(S_5)$ are trees.

In a similar way we can construct examples for the groups described in Lemma 6 i), ii) and the first two cases of iii).

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