Cristina Giannotti

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On the Range of Elliptic Operators Discontinuous at One Point.

CRISTINA GIANNOTTI

Sunto. – *Si considerano operatori uniformemente ellittici del secondo ordine in forma non variazionale, L, a coefficienti misurabili e limitati in $\mathbb{R}^d$ $(d \geq 3)$ e continui in $\mathbb{R}^d \setminus \{0\}$ e si prova il seguente risultato: se $\Omega \subset \mathbb{R}^d$ è un dominio limitato, allora $L(W^{2,p}(\Omega))$ è denso in $L^p(\Omega)$ per ogni $p \in (1, d/2]$.*

Summary. – *Let L be a second order, uniformly elliptic, non variational operator with coefficients which are bounded and measurable in $\mathbb{R}^d$ $(d \geq 3)$ and continuous in $\mathbb{R}^d \setminus \{0\}$. Then, if $\Omega \subset \mathbb{R}^d$ is a bounded domain, we prove that $L(W^{2,p}(\Omega))$ is dense in $L^p(\Omega)$ for any $p \in (1, d/2]$.*

1. – Introduction.

Let $\mathcal{L}$ be the class of second order, uniformly elliptic, non variational operators L with bounded measurable coefficients in $\mathbb{R}^d$ $(d \geq 3)$. Also, for any $p \in (1, +\infty)$ and any bounded domain $\Omega \subset \mathbb{R}^d$, let $\mathcal{R}(p, \Omega)$ be the subfamily of the operators satisfying $\overline{L(W^{2,p}(\Omega))} = L^p(\Omega)$. We recall that if L has second order coefficients continuous on $\overline{\Omega}$ then $L \in \mathcal{R}(p, \Omega)$ for every $p > 1$; actually if Ω is also smooth then it is known that $L(W^{2,p}_{\gamma_0}(\Omega)) = L^p(\Omega)$.

On the other hand, there exist operators with discontinuous coefficients which belong to $\mathcal{R}(p, \Omega)$ for every $p > 1$. Examples of this kind are the operator

$$S_\alpha := \alpha\Delta + (1 - d\alpha) \sum_{i,j=1}^{d} \frac{x_i x_j}{|x|^2} \frac{\partial^2}{\partial x_i \partial x_j}$$

which is discontinuous at the origin of $\mathbb{R}^d$ (see [2], [4], [7]) and the operator

$$\mathcal{U}_\alpha := \alpha\Delta + (1 - 3\alpha) \sum_{i,j=1}^{3} \frac{x_i x_j}{(x_1)^2 + (x_2)^2} \frac{\partial^2}{\partial x_i \partial x_j}$$

which is discontinuous on the x_3 axis of $\mathbb{R}^3$ (see [8], [1]).

At the same time, in [5] it is proved that there exists an elliptic operator L_0 in $\mathbb{R}^3$ with coefficients which are discontinuous on a circumference C and Hölder continuous on $\mathbb{R}^3 \setminus C$ and which is not in $\mathcal{R}(p, \Omega)$ for any $p > 2$ and $\Omega \supset C$. It is therefore natural to ask which conditions on the coefficients imply that an elliptic operator belongs to $\mathcal{R}(p, \Omega)$.

In [5] the following characterization of $\mathcal{R}(p, \Omega)$ has been given: $L \in \mathcal{R}(p, \Omega)$ if and only if there is no nontrivial solution to $L^\star u = 0$ with support in $\overline{\Omega}$. Moreover, it is proved that if L is Lipschitz continuous outside a closed set N of measure 0 and such that $\mathbb{R}^d \setminus N$ is connected, then L is in $\mathcal{R}(p, \Omega)$ for all $p > 1$ and any bounded domain Ω. This result is obtained as consequence of a unique continuation theorem for $L^\star$ in $\mathbb{R}^d \setminus N$.

The previously quoted operator L_0 shows that, in general, the condition on the Lipschitz continuity on $\mathbb{R}^d \setminus N$ of the coefficients cannot be removed. However we expect that this can be done if the set N is of some special kind. Note also that, if L has merely continuous coefficients, then the unique continuation for L and $L^\star$ fails and the technique of [5] cannot be used.

In this note, we consider the case of operators L which are continuous on $\mathbb{R}^d \setminus \{0\}$, with no other condition on the regularity of the coefficients. We show that these operators are in $\mathcal{R}(p, \Omega)$ for any $1 < p \leqslant d/2$ and any bounded domain $\Omega \subset \mathbb{R}^d$.

2. – Range of Elliptic Operators discontinuous at one point.

Let $\mathcal{L}$ be the family of the uniformly elliptic, second order operators of the form

$$L := \sum_{i,j=1}^{d} a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b^j \frac{\partial}{\partial x_j} + c ,$$

with bounded, measurable coefficients, defined in $\mathbb{R}^d$, $(d \geqslant 3)$, satisfying

$$\sum_{i,j=1}^{d} (a^{ij})^2 \leqslant M^2, \qquad \sum_{j=1}^{d} (b^j)^2 \leqslant M^2, \qquad -M \leqslant c \leqslant 0 ,$$

$$\sum_{i,j=1}^{d} a^{ij} \lambda_i \lambda_j \geqslant \alpha |\lambda|^2, \qquad \forall \lambda \in \mathbb{R}^d$$

for some positive constants M and α. We will use the summation convention and L will be written as $L = a^{i,j} \partial_{i,j} + b^j \partial_j + c$.

We begin with an approximation lemma.

LEMMA 1. – *Let D be a smooth bounded domain in $\mathbb{R}^d$ and let $L \in \mathcal{L}$ with continuous second order coefficients in $\overline{D}$. Let $1 < p < \infty$ and $s \in L^{p'}(D)$*

$(p' = p/(p-1))$, *satisfying*

$$\int_D sL\varphi\, dx = 0 \qquad \forall \varphi \in C_0^\infty(D).$$

Let $L_n \in \mathcal{L}$, $L_n = a_n^{i,j}\partial_{i,j} + b_n^j \partial_j + c_n$, *with coefficients in* $C^\infty(\overline{D})$ *and the same* α *and* M *as* L. *Let us also assume that*

$$(1) \qquad \|a_n^{i,j} - a^{i,j}\|_{C^0(\overline{D})} \to 0, \qquad b_n^j \to b^j, \qquad c_n \to c \quad \text{a.e. in } D.$$

Then there exists a sequence $\{s_n\}$, *with each* s_n *smooth on* $\overline{D}$ *and such that*

$$(2) \qquad L_n^\star s_n = 0 \ \text{ in } D, \qquad \lim_{n\to\infty} \|s_n - s\|_{L^{p'}(D)} = 0.$$

PROOF. – Let $W^{1-\frac{1}{p},p}(\partial D)$ be the Banach space of the traces on ∂D of the functions in $W^{1,p}(D)$ and let X^p its dual. By standard facts on elliptic equations (see e.g. [3] and [6]), it can be checked that there exists a unique $T_s^L \in X^p$ which satisfy

$$\int_D sLv\, dx = T_s^L\left(\frac{\partial v}{\partial N}\bigg|_{\partial D}\right) \qquad \forall v \in W_{\gamma_0}^{2,p}(D)$$

(N outward normal to ∂D) and

$$(3) \qquad k_1 \|s\|_{L^{p'}(D)} \leqslant \|T_s^L\|_{X^p} \leqslant k_2 \|s\|_{L^{p'}(D)}$$

where k_1 and k_2 are two constants depending on d, D α, M and the modulus of continuity of $a^{i,j}$. Note also that if L is smooth on $\overline{D}$ then s is smooth on $\overline{D}$, it satisfies the equation $L^\star s = \partial_{i,j}(a^{i,j}s) - \partial_j(b^j s) + cs = 0$ and we can write

$$T_s^L(\varphi) = \int_{\partial D} s a^{i,j} N_i N_j \varphi\, d\sigma \qquad \forall \varphi \in W^{1-\frac{1}{p},p}(\partial D).$$

Now, it is possible to find a sequence of smooth functions $\{t_n\}$ such that, if we set $T_n(\varphi) = \int_{\partial D} t_n a_n^{i,j} N_i N_j \varphi d\sigma$, then

$$(4) \qquad \lim_{n\to\infty} \|T_n - T_s^L\|_{X^p} = 0.$$

Moreover, if s_n is the (unique) solution to $L_n^\star s_n = 0$ in D with $s_{n|\partial D} = t_n$ then

$$(5) \qquad k_1 \|s_n\|_{L^{p'}(D)} \leqslant \|T_n\|_{X^p} \leqslant k_2 \|s_n\|_{L^{p'}(D)}.$$

By (1) and the fact that each $L_n \in \mathcal{L}$ with the same constants α and M as L, we may use the same k_1, k_2 in (5) and (3).

Let us now define

$$\|L_n - L\| := \sum_{i,j} \|a_n^{i,j} - a^{i,j}\|_{C^0(\overline{D})} + \sum_j \|b_n^j - b^j\|_{L^{p'}(D)} + \|c_n - c\|_{L^{p'}(D)}.$$

The limit (2) is proved if we can show that

$$\|s_n - s\|_{L^{p'}(D)} \leqslant k(\|L_n - L\| + \|T_n - T_s^L\|_{X^p}). \tag{6}$$

To prove this, let $f \in L^p(D)$ and $u \in W^{2,p}_{\gamma_0}(D)$ the solution to $Lu = f$ in D. Then

$$\left| \int_D (s_n - s)\, f dx \right| \leqslant \left| \int_D (sLu - s_n L_n u)\, dx \right| + \left| \int_D s_n (L_n u - Lu)\, dx \right|. \tag{7}$$

By (4) and (5), there exists a $k_3 > 0$ so that $\|s_n\|_{L^{p'}(D)} \leqslant k_3$. Therefore

$$\left| \int_D s_n (L_n u - Lu)\, dx \right| \leqslant k_3 \|(L_n - L)\, u\|_{L^p(D)} \leqslant k_4 \|L_n - L\| \, \|f\|_{L^p(D)}. \tag{8}$$

Moreover

$$\left| \int_D (sLu - s_n L_n u)\, dx \right| = \left| T_n \left(\frac{\partial u}{\partial N} \Big|_{\partial D} \right) - T_s^L \left(\frac{\partial u}{\partial N} \Big|_{\partial D} \right) \right| \leqslant \tag{9}$$

$$\|T_n - T_s^L\|_{X^p} \left\| \frac{\partial u}{\partial N} \right\|_{W^{1-(1/p),p}(\partial D)} \leqslant k_5 \|T_n - T_s^L\|_{X^p} \|u\|_{W^{2,p}} \leqslant k_6 \|T_n - T_s^L\|_{X^p} \|f\|_{L^p}.$$

From (7)-(9), (6) follows immediately. ■

LEMMA 2. – *Let $L \in \mathcal{L}$ with continuous second order coefficients and let D be a smooth bounded domain in $\mathbb{R}^d$. Let also $s \in L^{p'}(D)$ be a function satisfying $\int_D sL\varphi\, dx = 0$ for any $\varphi \in W_0^{2,p}(D)$. Then*

$$\int_D |s| L\varphi\, dx \geqslant 0 \qquad \forall \varphi \in W_0^{2,p}(D),\ \varphi \geqslant 0\,. \tag{10}$$

Moreover, if $\overline{supp(s)} \subset D$, the claim is true for every $\varphi \in W^{2,p}(D)$, $\varphi \geqslant 0$.

PROOF. – Let $\phi_\varepsilon(t)$ be the $C^{1,1}$ function

$$\phi_\varepsilon(t) = \begin{cases} |t| & \text{if } |t| \geqslant \varepsilon \\ \dfrac{\varepsilon}{2} + \dfrac{t^2}{2\varepsilon} & \text{if } |t| < \varepsilon\,. \end{cases}$$

Assume first that L has C^∞ coefficients and hence that $s \in C^\infty(\overline{D})$ and it satisfies $L^\star s = 0$. We have that $\phi_\varepsilon(s) \in C^{1,1}(D)$ and that

$$L^\star(\phi_\varepsilon(s)) = \begin{cases} \dfrac{1}{\varepsilon} a^{i,j} \partial_i s \partial_j s + \left(\dfrac{\varepsilon}{2} - \dfrac{s^2}{2\varepsilon}\right)(\partial_{i,j} a^{i,j} - \partial_j b^j + c) & \text{if } |s| < \varepsilon \\ 0 & \text{if } |s| \geqslant \varepsilon\,. \end{cases}$$

Now for any $\varphi \in C_0^\infty(D)$, $\varphi \geqslant 0$

$$\int_D \phi_\varepsilon(s)\, L\varphi\, dx = \int_D L^\star(\phi_\varepsilon(s))\, \varphi\, dx =$$

$$\int_{|s|<\varepsilon} \left\{ \frac{1}{\varepsilon} a^{i,j} \partial_i s \partial_j s + \left(\frac{\varepsilon}{2} - \frac{s^2}{2\varepsilon}\right)(\partial_{i,j} a^{i,j} - \partial_j b^j + c)\right\} \varphi\, dx \geqslant$$

$$\frac{\varepsilon}{2} \int_{|s|<\varepsilon} (\partial_{i,j} a^{i,j} - \partial_j b^j + c)\, \varphi\, dx - \frac{\varepsilon}{2} \int_{|s|<\varepsilon} |\partial_{i,j} a^{i,j} - \partial_j b^j + c| \varphi\, dx\,.$$

Since the last two terms tend to 0 as $\varepsilon \to 0$, it follows that

$$\int_D |s| L\varphi\, dx \geqslant 0 \qquad \forall \varphi \in W_0^{2,p}(D),\ \varphi \geqslant 0\,.$$

Now, assume that L has continuous second order coefficients and let $L_n \to L$, where each L_n is with smooth coefficients as in Lemma 1. So there exists $\{s_n\}$, $s_n \in C^\infty(\overline{D})$, such that $L_n^\star s_n = 0$ and $s_n \to s$ in $L^{p'}(D)$. Then

$$\left| \int_D |s| L\varphi\, dx - \int_D |s_n| L_n \varphi\, dx \right| \leqslant$$

$$\left| \int_D (|s| - |s_n|)\, L_n \varphi\, dx \right| + \left| \int_D |s| (L\varphi - L_n \varphi) \right| \leqslant$$

$$\leqslant \|s_n - s\|_{L^{p'}(D)} \|L_n \varphi\|_{L^p(D)} + \|s\|_{L^{p'}(D)} \|L_n - L\| \, \|\varphi\|_{W^{2,p}(D)} \xrightarrow{n\to\infty} 0\,.$$

From the previous discussion on the operators with smooth coefficients, (10) follows. The last claim is straightforward. ■

THEOREM 1. – *Let $L \in \mathcal{L}$ with continuous second order coefficients in $\mathbb{R}^d \setminus \{0\}$, $d \geqslant 3$. Then $L \in \mathcal{R}(p, \Omega)$ for any $1 < p \leqslant \frac{d}{2}$ and any bounded domain $\Omega \subset \mathbb{R}^d$.*

PROOF. – Clearly, we just need to consider the case $0 \in \overline{\Omega}$. In what follows, B_r denotes the open ball of radius r and center 0.

Let R be so large that $B_R \supset \overline{\Omega}$ and let $\varphi = 1 - (|\cdot|/R)^N$, which is smooth and positive on B_R and vanishing on ∂B_R. Moreover, from the expression

$$L\varphi = -\left(\frac{|x|}{R}\right)^{N-2} \cdot \left\{ \frac{N(N-2)}{R^2} \sum_{i,j=1}^{d} a^{i,j} \frac{x_i x_j}{|x|^2} - \frac{N}{R^2} \sum_{j=1}^{d} a^{j,j} - \frac{N}{R^2} \sum_{j=1}^{d} b^j x_j \right\} + c\varphi$$

it follows that $N > 2$ can be chosen large enough so that $L\varphi \leqslant -\left(\frac{|x|}{R}\right)^{N-2}$. Consider also some nonnegative $\psi \in C^{\infty}[0, +\infty)$ which is identically 0 on a neighbourhood of $[0, 1]$ and identically 1 on a neighbourhood of $[2, +\infty)$ and, for any $0 < \varepsilon < R/2$, let us define $\psi_{\varepsilon}(x) = \psi\left(\frac{|x|}{\varepsilon}\right)$. Finally, let $\varphi_{\varepsilon}(x) = \varphi(x)\, \psi_{\varepsilon}(x)$, which is smooth on B_R, equal to φ on $\mathbb{R}^d \setminus B_{2\varepsilon}$ and identically 0 on a neighbourhood of B_{ε}.

Let $s \in L^{p'}(\mathbb{R}^d)$ be a solution to $L^{\star} s = 0$, with support in $\overline{\Omega}$. By Lemma 2, we have that $0 \leqslant \int_{B_R} |s| L\varphi_{\varepsilon}\, dx = \int_{\Omega \setminus B_{\varepsilon}} |s| L\varphi_{\varepsilon}\, dx$ and hence that

$$0 \leqslant \int_{B_{2\varepsilon} \setminus B_{\varepsilon}} |s| L(\varphi\psi_{\varepsilon})\, dx + \int_{\Omega \setminus B_{2\varepsilon}} |s| L\varphi\, dx \leqslant$$

$$\int_{B_{2\varepsilon} \setminus B_{\varepsilon}} |s| L(\varphi\psi_{\varepsilon})\, dx - \int_{\Omega \setminus B_{2\varepsilon}} |s| \left(\frac{|x|}{R}\right)^{N-2} dx\,.$$

Then

$$0 \leqslant \int_{\Omega \setminus B_{2\varepsilon}} |s| \left(\frac{|x|}{R}\right)^{N-2} dx \leqslant$$

$$\int_{\varepsilon < |x| < 2\varepsilon} |s| \{\psi_{\varepsilon} L\varphi + \varphi(L - c)\, \psi_{\varepsilon} + 2a^{i,j} \partial_i \psi_{\varepsilon} \partial_j \varphi\}\, dx\,.$$

Since $\psi_{\varepsilon} \geqslant 0$ and $L\varphi \leqslant 0$ in $\varepsilon < |x| < 2\varepsilon$, we have

$$\int_{\Omega \setminus B_{2\varepsilon}} |s| \left(\frac{|x|}{R}\right)^{N-2} dx \leqslant \int_{\varepsilon < |x| < 2\varepsilon} |s| \{\varphi(L - c)\, \psi_{\varepsilon} + 2a^{i,j} \partial_i \psi_{\varepsilon} \partial_j \varphi\}\, dx \leqslant$$

$$\int_{\varepsilon < |x| < 2\varepsilon} |s| \left\{ \frac{2a^{i,j}}{\varepsilon} \partial_j \varphi \psi'_{\varepsilon} \frac{x_i}{|x|} + \frac{\varphi}{\varepsilon^2} \cdot \left(a^{i,j} \psi''_{\varepsilon} \frac{x_i x_j}{|x|^2} + \psi'_{\varepsilon} \left(a^{j,j} - \frac{a^{i,j} x_i x_j}{|x|^2} \right) \frac{\varepsilon}{|x|} + \varepsilon \psi'_{\varepsilon} b^j \frac{x_j}{|x|} \right) \right\} dx\,.$$

Therefore, there exists $C>0$, which depends only on d, M, α, r and N such that

$$\frac{2a^{i,j}}{\varepsilon}\partial_j\varphi\psi'_\varepsilon\frac{x_i}{|x|}+$$
$$\frac{\varphi}{\varepsilon^2}\left(a^{i,j}\psi''_\varepsilon\frac{x_ix_j}{|x|^2}+\psi'_\varepsilon\left(a^{j,j}-\frac{a^{i,j}x_ix_j}{|x|^2}\right)\frac{\varepsilon}{|x|}+\varepsilon\psi'_\varepsilon b^j\frac{x_j}{|x|}\right)\leqslant\frac{C}{\varepsilon^2}$$

in $\varepsilon<|x|<2\varepsilon$. Then

$$\int_{\Omega\setminus B_{2\varepsilon}}|s|\left(\frac{|x|}{R}\right)^{N-2}dx\leqslant\frac{C}{\varepsilon^2}\int_{\varepsilon<|x|<2\varepsilon}|s|\,dx\leqslant C\varepsilon^{(d/p)-2}\|s\|_{L^{p'}}|B_1|^{1/p}(2^d-1)^{1/p}.$$

Since $(d/p)-2\geqslant0$ as $\varepsilon\to0$, we get that $\int_\Omega|s|\left(\frac{|x|}{R}\right)^{N-2}dx=0$ and hence that $s\equiv0$ a.e. in $\mathbb{R}^d$. This implies the claim by the characterization of $\mathcal{R}(p,\Omega)$ in [5]. ■

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Cristina Giannotti, Dipartimento di Matematica e Fisica, Università di Camerino,
Via Madonna delle Carceri, 62032 Camerino (Macerata) Italy
e-mail: giannotti@campus.unicam.it

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