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The Euler Characteristic of Standard CR Manifolds.

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Sunto. – *In questo lavoro si calcola la caratteristica di Eulero delle varietà CR standard.*

Standard CR manifolds were constructed in [MN97], starting from Levi-Tanaka algebras. Here we simply recall a few basic facts. Given a CR manifold M, the distribution $\mathcal{D}_{-1} = \mathcal{C}^\infty(M, HM)$ of its analytic tangent spaces defines at each point $x \in M$ a nilpotent graded Lie algebra $\mathfrak{m}(x) = \bigoplus_{-\mu(x) \leq p \leq -1} \mathfrak{g}_p$. The nilmanifold $N(\mathfrak{m}(x))$ associated to $\mathfrak{m}(x)$ has a natural CR structure. We say that M is *locally standard* if each point $x \in M$ has a neighborhood which is CR diffeomorphic to a neighborhood of the identity in $N(\mathfrak{m}(x))$. In [Tan67], [Tan70], N. Tanaka considers the maximal transitive pseudocomplex prolongation $\mathfrak{g}(x) = \bigoplus_{-\mu(x) \leq p} \mathfrak{g}_p$ of $\mathfrak{m}(x)$. The algebras $\mathfrak{g}(x)$ are finite dimensional if and only if some nondegeneracy condition is satisfied by the Levi form of M at x. To this construction there correspond homogeneous standard nondegenerate CR manifolds having a maximal group of CR automorphisms ([MN97], [MN]). The standard CR manifold associated to the Levi-Tanaka algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the Kleinian space $S_\mathfrak{g} = G/G_+$, where G is the connected and simply connected Lie group having tangent Lie algebra $\mathfrak{g}$ and G_+ is the (closed) analytic Lie subgroup of G whose tangent Lie algebra is $\mathfrak{g}_+ = \bigoplus_{p \geq 0} \mathfrak{g}_p$, see [MN97]. In particular $S_\mathfrak{g}$ is simply connected.

It was proved in [MN] that:

1. the manifold $S_\mathfrak{g}$ admits a (Mostow) fibration with a Euclidean fiber F, and a compact base B which is the Cartesian product of a Hermitean symmetric space and a standard CR manifold;

2. $S_\mathfrak{g}$ is compact if and only if $\mathfrak{g}$ is semisimple.

Indeed, $\mathfrak{g}$ admits a Levi-Malcev decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{L}$, where both the radical $\mathfrak{r}$ and the Levi subalgebra $\mathfrak{L}$ are graded and pseudocomplex. If R and L are the analytic subgroups of G corresponding to $\mathfrak{r}$ and $\mathfrak{L}$, respectively, then the fiber is isomorphic to R/R_+ and the compact base B is L/L_+. Here R_+ and L_+ are the analytic subgroups corresponding to $\mathfrak{r} \cap \mathfrak{g}_+$ and $\mathfrak{L} \cap \mathfrak{g}_+$, respectively.

When $S_{\mathfrak{g}}$ is compact, by Montgomery's theorem (cf. [Mon50]), any maximal compact (connected) subgroup K of G acts transitively on $S_{\mathfrak{g}}$ with isotropy $H = K \cap G_+$, and the Euler characteristic $\chi = \chi(S_{\mathfrak{g}})$ of $S_{\mathfrak{g}}$ is nonnegative (see [HS41]). Moreover, $\chi > 0$ if and only if $rkK = rkH$, and in this case

$$\chi = \frac{|W(K)|}{|W(H)|}, \tag{1}$$

where $W(\cdot)$ denotes the Weyl group associated to a compact connected group (see [Sam58]).

The following theorem, which is an easy consequence of [MN], reduces the computation of the Euler characteristic of $S_{\mathfrak{g}}$ to the case of a compact standard CR manifold.

THEOREM 1. – *Let $S_{\mathfrak{g}}$ be the standard CR manifold associated to a given Levi-Tanaka algebra $\mathfrak{g}$. Then $\chi(S_{\mathfrak{g}}) = \chi(B)$, where B is the base of a Mostow fibration $S_{\mathfrak{g}} \to B$ and is the Cartesian product of a Hermitean symmetric space and a compact standard CR manifold.*

We recall (see [MN]) that a compact standard CR manifold is the Cartesian product of *irreducible* CR manifolds, i.e. of standard CR manifolds associated to simple Levi-Tanaka algebras. The simple ideals of the Levi subalgebra in the pseudocomplex graded Levi-Malcev decomposition above give the Hermitean symmetric spaces (case of kind $\leqslant 1$) or the irreducible standard CR manifolds (case of kind $\geqslant 2$) to factorize B.

A real simple Lie algebra is said to be *of the complex type* if it admits a complex structure for which it is a complex Lie algebra (or, equivalently, if its complexification is not simple); otherwise we call it *of the real type*.

THEOREM 2. – *Let $S_{\mathfrak{g}}$ be a compact standard CR manifold. Then $\chi(S_{\mathfrak{g}}) \geqslant 0$ and $\chi(S_{\mathfrak{g}}) > 0$ if and only if $\mathfrak{g}$ is the direct sum of simple ideals which are Levi-Tanaka algebras of the complex type.*

When $\mathfrak{g} = \bigoplus_{-\mu \leqslant p \leqslant \mu} \mathfrak{g}$ is a simple Levi-Tanaka algebra, we have:

$$\chi(S_{\mathfrak{g}}) = \begin{cases} 0 & \text{if } \mathfrak{g} \text{ is of the real type}, \\ \dfrac{|W(\mathfrak{g})|}{|W(\mathfrak{g}_0)|} & \text{if } \mathfrak{g} \text{ is of the complex type}, \end{cases} \tag{2}$$

where $W(\cdot)$ denotes the Weyl group of a root system associated to the Lie algebra.

PROOF. – We can assume that $\mathfrak{g}$ is simple.

First we prove that if $\mathfrak{g}$ is of the real type, then $\chi(S_{\mathfrak{g}}) = 0$.

We consider the homogeneous CR manifold $M = G^{\mathbb{R}}/G_+^{\mathbb{R}}$ defined in Theorem 4.9 of [MN97]; here $G^{\mathbb{R}}$ is a linear Lie group having Lie algebra $\mathfrak{g}$ and $G_+^{\mathbb{R}}$ its (closed) analytic subgroup with Lie algebra $\mathfrak{g}_+$. Since $S_{\mathfrak{g}}$ is a finite covering of M, it suffices to show that $\chi(M) = 0$.

A maximal compact Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ generates a compact analytic Lie subgroup K of $G_+^{\mathbb{R}}$ which acts transitively on M (by Montgomery's theorem); the isotropy subgroup K_0 has Lie algebra $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_+ = \mathfrak{k} \cap \mathfrak{g}_0$.

If we denote by E the *characteristic element* of $\mathfrak{g}$ (i.e. the element of $\mathfrak{g}$ such that $\mathfrak{g}_p = \{X \in \mathfrak{g} \,|\, [E, X] = pX\}$), then $E \notin \mathfrak{k}$ (as $\kappa_{\mathfrak{g}}(E, E) > 0$, where $\kappa_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$).

Assume that $\mathrm{rk}(\mathfrak{k}) = \mathrm{rk}(\mathfrak{g})$. By direct inspection (see [MN97]), we observe that this is the case for all simple Levi-Tanaka algebras of the real type, unless $\mathfrak{g}$ is $\mathfrak{so}(l-1, l+1)$ with $l \geqslant 4$ and even.

Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. For each $A \in \mathfrak{k}_0$, $\mathrm{ad}_{\mathfrak{g}}(A)$ is semisimple; moreover $[\mathfrak{h}_0, E] = 0$. Hence $\mathfrak{h}_0$ is properly contained in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing E. It follows that $rk(\mathfrak{k}_0) < rk(\mathfrak{g}) = rk(\mathfrak{k})$, and then $\chi(M) = 0$.

Assume now that $\mathfrak{g} = \mathfrak{so}(l-1, l+1)$, with $l \geqslant 4$ and even. In this case $rk(\mathfrak{k}) = l - 1$. On the other hand, $\mathfrak{k}_0 \cong \mathfrak{so}(l-1) \oplus \mathfrak{so}(2)$ and thus $rk(\mathfrak{k}_0) = \frac{l}{2} < l - 1 = rk(\mathfrak{k})$.

Finally we turn to the case where $\mathfrak{g}$ is of the complex type. Denote by $\sqrt{-1}$ the complex structure on $\mathfrak{g}$ for which it becomes a complex Lie algebra. We note that $\kappa_{\mathfrak{g}}(\sqrt{-1}E, \sqrt{-1}E) < 0$, where E is the characteristic element and $\kappa_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$; hence $\sqrt{-1}E$ is contained in a maximal compact Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$.

We have:

$$\mathfrak{g}_0 = Z_{\mathfrak{g}}(\sqrt{-1}E) := \{X \in \mathfrak{g} \,|\, [X, \sqrt{-1}E] = 0\},$$

and therefore

$$\mathfrak{k}_0 := \mathfrak{k} \cap \mathfrak{g}_0 = Z_{\mathfrak{k}}(\sqrt{-1}E) := \{X \in \mathfrak{k} \,|\, [X, \sqrt{-1}E] = 0\}.$$

It follows that $rk(\mathfrak{k}_0) = rk(\mathfrak{k})$ and then $\chi(S_{\mathfrak{g}}) > 0$; moreover, $\mathfrak{k}_0$ is a real form of $\mathfrak{g}_0$ and we have:

$$W(K) = W(\mathfrak{k}) = W(\mathfrak{g}),$$

$$W(K_0) = W(\mathfrak{k}_0) = W(\mathfrak{g}_0),$$

where K_0 is the the connected component of the identity of $K \cap G_+$. ■

We recall (see [MN98]) that the gradation of a semisimple Levi-Tanaka algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is given by a partition of a system of simple roots $\mathcal{B}$ into two subset $\mathcal{B}_{-1}$ and $\mathcal{B}_0$:

$$\mathcal{B}_{-1} = \{\alpha \in \mathcal{B} \,|\, \alpha(E) = -1\},$$

$$\mathcal{B}_0 = \{\alpha \in \mathcal{B} \,|\, \alpha(E) = 0\},$$

where as usual we denote by E the characteristic element of $\mathfrak{g}$.

We set

$$\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}\},$$

where the indices $\alpha_{i_1}, \ldots, \alpha_{i_\nu}$ satisfy $1 \leqslant i_1 < \ldots < i_\nu \leqslant l$, $\nu \geqslant 2$, and the conditions described in [MN98].

For the nonexceptional simple Lie algebras of type A_l, B_l, C_l, D_l, respectively, we obtain:

$$A_l\colon\ \chi = \frac{(l+1)!}{i_1!(i_2-i_1)!\ldots(i_\nu - i_{\nu-1})!(l+1-i_\nu)!},$$

$$B_l\colon\ \chi = \frac{2^{i_\nu} l!}{i_1!(i_2-i_1)!\ldots(i_\nu - i_{\nu-1})!(l-i_\nu)!},$$

$$C_l\colon\ \chi = \frac{2^{i_\nu} l!}{i_1!(i_2-i_1)!\ldots(i_\nu - i_{\nu-1})!},$$

$$D_l\colon \chi = \frac{2^{\min\{i_\nu,\, l-1\}}\, l!}{i_1!(i_2-i_1)!\ldots(i_\nu - i_{\nu-1})!(l-i_\nu)!}.$$

For the case of exceptional simple Lie algebras of the complex type, we compute the Euler characteristic by using the explicit form of $\mathfrak{g}_0$ that is given in the tables at the end of [MN98].

To a $\mathfrak{g}$ of the complex type G_2 there corresponds a unique standard CR manifold $S_\mathfrak{g}$ for which

$$\chi = 12,$$

while for the complex type F_4 there are six nonisomorphic standard CR manifolds, classified in the table of page 348 of [MN98], and we have respectively:

1. 1 1 0 0 $\quad \chi = 2^6 \cdot 3$,

2. 1 1 0 1 $\quad \chi = 2^6 \cdot 3^2$,

3. 0 1 1 0 $\chi = 2^5 \cdot 3^2$,

4. 1 1 1 0 $\chi = 2^6 \cdot 3^2$,

5. 0 1 1 1 $\chi = 2^6 \cdot 3^2$,

6. 1 1 1 1 $\chi = 2^7 \cdot 3^2$.

Here and in the following the string of 1's and 0's refers to the standard labelling of the fundamental roots: a 1 means that the corresponding root belongs to $\mathcal{B}_{-1}$, a 0 that it belongs to $\mathcal{B}_0$.

Finally we give the Euler characteristic of the standard *CR* manifolds associated to simple Levi-Tanaka algebras of the complex types E6, E7 and E8 in the following tables.

E6

01.	100001	$\chi = 2 \cdot 3^3 \cdot 5$
02.	110000	$\chi = 2^4 \cdot 3^3$
03.	101000	$\chi = 2^4 \cdot 3^3$
04.	011000	$\chi = 2^3 \cdot 3^3 \cdot 5$
05.	001010	$\chi = 2^4 \cdot 3^3 \cdot 5$
06.	101001	$\chi = 2^4 \cdot 3^3 \cdot 5$
07.	001100	$\chi = 2^4 \cdot 3^3 \cdot 5$
08.	010100	$\chi = 2^5 \cdot 3^2 \cdot 5$
09.	111000	$\chi = 2^4 \cdot 3^3 \cdot 5$
10.	101010	$\chi = 2^5 \cdot 3^3 \cdot 5$
11.	110100	$\chi = 2^5 \cdot 3^3 \cdot 5$
12.	101100	$\chi = 2^5 \cdot 3^3 \cdot 5$
13.	100110	$\chi = 2^4 \cdot 3^4 \cdot 5$
14.	101011	$\chi = 2^6 \cdot 3^3 \cdot 5$
15.	011100	$\chi = 2^5 \cdot 3^3 \cdot 5$
16.	001110	$\chi = 2^4 \cdot 3^4 \cdot 5$
17.	110101	$\chi = 2^5 \cdot 3^4 \cdot 5$
18.	101101	$\chi = 2^5 \cdot 3^4 \cdot 5$
19.	111100	$\chi = 2^6 \cdot 3^3 \cdot 5$
20.	110110	$\chi = 2^5 \cdot 3^4 \cdot 5$
21.	101110	$\chi = 2^5 \cdot 3^4 \cdot 5$
22.	011110	$\chi = 2^5 \cdot 3^4 \cdot 5$
23.	111101	$\chi = 2^6 \cdot 3^4 \cdot 5$
24.	101111	$\chi = 2^6 \cdot 3^4 \cdot 5$
25.	111110	$\chi = 2^6 \cdot 3^4 \cdot 5$
26.	111111	$\chi = 2^7 \cdot 3^4 \cdot 5$

E7

01.	1000001	$\chi = 2^3 \cdot 3^3 \cdot 7$
02.	0000011	$\chi = 2^3 \cdot 3^3 \cdot 7$
03.	1100000	$\chi = 2^6 \cdot 3^2 \cdot 7$
04.	0100010	$\chi = 2^6 \cdot 3^3 \cdot 7$
05.	1000011	$\chi = 2^4 \cdot 3^3 \cdot 5 \cdot 7$
06.	0100011	$\chi = 2^7 \cdot 3^3 \cdot 7$
07.	1010000	$\chi = 2^6 \cdot 3^2 \cdot 7$
08.	0110000	$\chi = 2^6 \cdot 3^3 \cdot 7$
09.	0100100	$\chi = 2^6 \cdot 3^2 \cdot 5 \cdot 7$
10.	0000110	$\chi = 2^6 \cdot 3^3 \cdot 7$
11.	1010001	$\chi = 2^7 \cdot 3^3 \cdot 7$
12.	0100101	$\chi = 2^6 \cdot 3^3 \cdot 5 \cdot 7$
13.	0000111	$\chi = 2^7 \cdot 3^3 \cdot 7$
14.	0010100	$\chi = 2^7 \cdot 3^2 \cdot 5 \cdot 7$
15.	0101000	$\chi = 2^6 \cdot 3^2 \cdot 5 \cdot 7$
16.	1110000	$\chi = 2^7 \cdot 3^3 \cdot 7$
17.	1010010	$\chi = 2^6 \cdot 3^3 \cdot 5 \cdot 7$
18.	1000110	$\chi = 2^6 \cdot 3^3 \cdot 5 \cdot 7$
19.	0100110	$\chi = 2^6 \cdot 3^3 \cdot 5 \cdot 7$
20.	0010101	$\chi = 2^7 \cdot 3^3 \cdot 5 \cdot 7$
21.	0101001	$\chi = 2^8 \cdot 3^2 \cdot 5 \cdot 7$
22.	0011000	$\chi = 2^5 \cdot 3^3 \cdot 5 \cdot 7$
23.	0001100	$\chi = 2^7 \cdot 3^2 \cdot 5 \cdot 7$
24.	1010011	$\chi = 2^7 \cdot 3^3 \cdot 5 \cdot 7$
25.	1000111	$\chi = 2^7 \cdot 3^3 \cdot 5 \cdot 7$
26.	0100111	$\chi = 2^7 \cdot 3^3 \cdot 5 \cdot 7$

27. 1010100 $\chi = 2^8\cdot 3^2\cdot 5\cdot 7$
28. 0010110 $\chi = 2^7\cdot 3^3\cdot 5\cdot 7$
29. 1101000 $\chi = 2^6\cdot 3^3\cdot 5\cdot 7$
30. 0101010 $\chi = 2^7\cdot 3^3\cdot 5\cdot 7$
31. 0011001 $\chi = 2^7\cdot 3^3\cdot 5\cdot 7$
32. 0001101 $\chi = 2^7\cdot 3^3\cdot 5\cdot 7$
33. 1010101 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
34. 0010111 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
35. 1101001 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
36. 0101011 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
37. 1011000 $\chi = 2^6\cdot 3^3\cdot 5\cdot 7$
38. 0111000 $\chi = 2^6\cdot 3^3\cdot 5\cdot 7$
39. 0011010 $\chi = 2^6\cdot 3^4\cdot 5\cdot 7$
40. 1001100 $\chi = 2^7\cdot 3^3\cdot 5\cdot 7$
41. 0101100 $\chi = 2^8\cdot 3^2\cdot 5\cdot 7$
42. 0001110 $\chi = 2^7\cdot 3^3\cdot 5\cdot 7$
43. 1010110 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
44. 1101010 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
45. 1011001 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
46. 0111001 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
47. 0011011 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
48. 1001101 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
49. 0101101 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
50. 0001111 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
51. 0011100 $\chi = 2^7\cdot 3^3\cdot 5\cdot 7$
52. 1010111 $\chi = 2^9\cdot 3^3\cdot 5\cdot 7$
53. 1101011 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
54. 1111000 $\chi = 2^7\cdot 3^3\cdot 5\cdot 7$
55. 1011010 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
56. 0111010 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
57. 1101100 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
58. 1001110 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
59. 0101110 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
60. 0011101 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
61. 1111001 $\chi = 2^9\cdot 3^3\cdot 5\cdot 7$
62. 1011011 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
63. 0111011 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
64. 1101101 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
65. 1001111 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
66. 0101111 $\chi = 2^9\cdot 3^3\cdot 5\cdot 7$
67. 1011100 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
68. 0111100 $\chi = 2^8\cdot 3^3\cdot 5\cdot 7$
69. 0011110 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
70. 1111010 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
71. 1101110 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
72. 1011101 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
73. 0111101 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
74. 0011111 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
75. 1111011 $\chi = 2^9\cdot 3^4\cdot 5\cdot 7$
76. 1101111 $\chi = 2^9\cdot 3^4\cdot 5\cdot 7$
77. 1111100 $\chi = 2^9\cdot 3^3\cdot 5\cdot 7$
78. 1011110 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
79. 0111110 $\chi = 2^8\cdot 3^4\cdot 5\cdot 7$
80. 1111101 $\chi = 2^9\cdot 3^4\cdot 5\cdot 7$
81. 1011111 $\chi = 2^9\cdot 3^4\cdot 5\cdot 7$
82. 0111111 $\chi = 2^9\cdot 3^4\cdot 5\cdot 7$
83. 1111110 $\chi = 2^9\cdot 3^4\cdot 5\cdot 7$
84. 1111111 $\chi = 2^{10}\cdot 3^4\cdot 5\cdot 7$

E8

001. 11000000 $\chi = 2^{10}\cdot 3^3\cdot 5$
002. 00000011 $\chi = 2^7\cdot 3\cdot 5\cdot 7$
003. 10100000 $\chi = 2^{10}\cdot 3^3\cdot 5$
004. 10000011 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
005. 01100000 $\chi = 2^9\cdot 3^3\cdot 5\cdot 7$
006. 01000100 $\chi = 2^{10}\cdot 3^3\cdot 5\cdot 7$
007. 00000110 $\chi = 2^6\cdot 3^4\cdot 5\cdot 7$
008. 10100001 $\chi = 2^{10}\cdot 3^3\cdot 5\cdot 7$
009. 01001000 $\chi = 2^8\cdot 3^3\cdot 5^2\cdot 7$
010. 11100000 $\chi = 2^{10}\cdot 3^3\cdot 5\cdot 7$
011. 10100010 $\chi = 2^{10}\cdot 3^4\cdot 5\cdot 7$
012. 01000101 $\chi = 2^{10}\cdot 3^4\cdot 5\cdot 7$
013. 10000110 $\chi = 2^7\cdot 3^4\cdot 5^2\cdot 7$
014. 00000111 $\chi = 2^7\cdot 3^4\cdot 5\cdot 7$
015. 00101000 $\chi = 2^9\cdot 3^3\cdot 5^2\cdot 7$
016. 00001100 $\chi = 2^{10}\cdot 3^3\cdot 5\cdot 7$
017. 01010000 $\chi = 2^{10}\cdot 3^3\cdot 5\cdot 7$
018. 01000110 $\chi = 2^{10}\cdot 3^4\cdot 5\cdot 7$
019. 10100100 $\chi = 2^{10}\cdot 3^3\cdot 5^2\cdot 7$
020. 01001001 $\chi = 2^{10}\cdot 3^3\cdot 5^2\cdot 7$
021. 00110000 $\chi = 2^9\cdot 3^4\cdot 5\cdot 7$
022. 10100011 $\chi = 2^{11}\cdot 3^4\cdot 5\cdot 7$
023. 10000111 $\chi = 2^8\cdot 3^4\cdot 5^2\cdot 7$
024. 01001010 $\chi = 2^9\cdot 3^4\cdot 5^2\cdot 7$

025. 10101000 $\chi = 2^{10}\cdot 3^{3}\cdot 5^{2}\cdot 7$
026. 00101001 $\chi = 2^{11}\cdot 3^{3}\cdot 5^{2}\cdot 7$
027. 10001100 $\chi = 2^{10}\cdot 3^{3}\cdot 5^{2}\cdot 7$
028. 00001101 $\chi = 2^{10}\cdot 3^{4}\cdot 5\cdot 7$
029. 11010000 $\chi = 2^{10}\cdot 3^{4}\cdot 5\cdot 7$
030. 01010001 $\chi = 2^{10}\cdot 3^{3}\cdot 5^{2}\cdot 7$
031. 00011000 $\chi = 2^{9}\cdot 3^{3}\cdot 5^{2}\cdot 7$
032. 01000111 $\chi = 2^{11}\cdot 3^{4}\cdot 5\cdot 7$
033. 10100101 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
034. 00101010 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
035. 01001100 $\chi = 2^{10}\cdot 3^{3}\cdot 5^{2}\cdot 7$
036. 00001110 $\chi = 2^{10}\cdot 3^{4}\cdot 5\cdot 7$
037. 01010010 $\chi = 2^{11}\cdot 3^{3}\cdot 5^{2}\cdot 7$
038. 10110000 $\chi = 2^{10}\cdot 3^{4}\cdot 5\cdot 7$
039. 00110001 $\chi = 2^{9}\cdot 3^{4}\cdot 5^{2}\cdot 7$
040. 10100110 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
041. 01001011 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
042. 10101001 $\chi = 2^{12}\cdot 3^{3}\cdot 5^{2}\cdot 7$
043. 10001101 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
044. 00101100 $\chi = 2^{11}\cdot 3^{3}\cdot 5^{2}\cdot 7$
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060. 01011000 $\chi = 2^{10}\cdot 3^{3}\cdot 5^{2}\cdot 7$
061. 00011010 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
062. 10100111 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
063. 01001110 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
064. 10101100 $\chi = 2^{12}\cdot 3^{3}\cdot 5^{2}\cdot 7$
065. 00101101 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
066. 11110000 $\chi = 2^{11}\cdot 3^{4}\cdot 5\cdot 7$
067. 01110001 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
068. 10110010 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
069. 00110011 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
070. 11010100 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
071. 01010101 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
072. 10011001 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
073. 00111000 $\chi = 2^{9}\cdot 3^{4}\cdot 5^{2}\cdot 7$
074. 00011100 $\chi = 2^{11}\cdot 3^{3}\cdot 5^{2}\cdot 7$
075. 10101011 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
076. 10001111 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
077. 00101110 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
078. 11010011 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
079. 01110010 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
080. 01010110 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
081. 10110100 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
082. 00110101 $\chi = 2^{10}\cdot 3^{5}\cdot 5^{2}\cdot 7$
083. 11011000 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
084. 01011001 $\chi = 2^{12}\cdot 3^{3}\cdot 5^{2}\cdot 7$
085. 10011010 $\chi = 2^{10}\cdot 3^{5}\cdot 5^{2}\cdot 7$
086. 00011011 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
087. 01001111 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
088. 10101101 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
089. 11110001 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
090. 10110011 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
091. 11010101 $\chi = 2^{11}\cdot 3^{5}\cdot 5^{2}\cdot 7$
092. 01110100 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
093. 00110110 $\chi = 2^{10}\cdot 3^{5}\cdot 5^{2}\cdot 7$
094. 01011010 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
095. 10111000 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$
096. 00111001 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
097. 10011100 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
098. 00011101 $\chi = 2^{11}\cdot 3^{4}\cdot 5^{2}\cdot 7$
099. 10101110 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
100. 00101111 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
101. 11110010 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
102. 01110011 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
103. 11010110 $\chi = 2^{11}\cdot 3^{5}\cdot 5^{2}\cdot 7$
104. 01010111 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
105. 10110101 $\chi = 2^{11}\cdot 3^{5}\cdot 5^{2}\cdot 7$
106. 11011001 $\chi = 2^{12}\cdot 3^{4}\cdot 5^{2}\cdot 7$
107. 10011011 $\chi = 2^{11}\cdot 3^{5}\cdot 5^{2}\cdot 7$
108. 01111000 $\chi = 2^{10}\cdot 3^{4}\cdot 5^{2}\cdot 7$

109.	00111010	$\chi = 2^{10}\cdot 3^5\cdot 5^2\cdot 7$
110.	01011100	$\chi = 2^{12}\cdot 3^3\cdot 5^2\cdot 7$
111.	00011110	$\chi = 2^{11}\cdot 3^4\cdot 5^2\cdot 7$
112.	11110100	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
113.	01110101	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
114.	10110110	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
115.	00110111	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
116.	11011010	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
117.	01011011	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
118.	10111001	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
119.	10011101	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
120.	00111100	$\chi = 2^{11}\cdot 3^4\cdot 5^2\cdot 7$
121.	10101111	$\chi = 2^{13}\cdot 3^4\cdot 5^2\cdot 7$
122.	11110011	$\chi = 2^{13}\cdot 3^4\cdot 5^2\cdot 7$
123.	11010111	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
124.	01110110	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
125.	11111000	$\chi = 2^{11}\cdot 3^4\cdot 5^2\cdot 7$
126.	01111001	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
127.	10111010	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
128.	00111011	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
129.	11011100	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
130.	01011101	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
131.	10011110	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
132.	00011111	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
133.	11110101	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
134.	10110111	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
135.	11011011	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
136.	01111010	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
137.	01011110	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
138.	10111100	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
139.	00111101	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
140.	11110110	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
141.	01110111	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
142.	11111001	$\chi = 2^{13}\cdot 3^4\cdot 5^2\cdot 7$
143.	10111011	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
144.	11011101	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
145.	10011111	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
146.	01111100	$\chi = 2^{12}\cdot 3^4\cdot 5^2\cdot 7$
147.	00111110	$\chi = 2^{11}\cdot 3^5\cdot 5^2\cdot 7$
148.	11111010	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
149.	01111011	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
150.	11011110	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
151.	01011111	$\chi = 2^{13}\cdot 3^4\cdot 5^2\cdot 7$
152.	10111101	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
153.	11110111	$\chi = 2^{13}\cdot 3^5\cdot 5^2\cdot 7$
154.	11111100	$\chi = 2^{13}\cdot 3^4\cdot 5^2\cdot 7$
155.	01111101	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
156.	10111110	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
157.	00111111	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
158.	11111011	$\chi = 2^{13}\cdot 3^5\cdot 5^2\cdot 7$
159.	11011111	$\chi = 2^{13}\cdot 3^5\cdot 5^2\cdot 7$
160.	01111110	$\chi = 2^{12}\cdot 3^5\cdot 5^2\cdot 7$
161.	11111101	$\chi = 2^{13}\cdot 3^5\cdot 5^2\cdot 7$
162.	10111111	$\chi = 2^{13}\cdot 3^5\cdot 5^2\cdot 7$
163.	11111110	$\chi = 2^{13}\cdot 3^5\cdot 5^2\cdot 7$
164.	01111111	$\chi = 2^{13}\cdot 3^5\cdot 5^2\cdot 7$
165.	11111111	$\chi = 2^{14}\cdot 3^5\cdot 5^2\cdot 7$

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