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Antoine Henrot, Hervé Maillot

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Optimization of the Shape and the Location of the Actuators in an Internal Control Problem.

ANTOINE HENROT (*) - HERVÉ MAILLOT

Sunto. – *Consideriamo un corpo Ω sottomesso ad una forza esterna data e del quale vogliamo controllare lo spostamento. Cerchiamo un rinforzo per minimizzare un funzionale che dipende dallo spostamento del corpo. L'insieme delle configurazioni ammissibili è un insieme di funzioni caratteristiche di sottodomini (un rinforzo ammissibile è un sottodominio con una rigidezza uguale ad uno) di volume prescritto. In tal caso, si ha bisogno di una versione rilassata del problema di ottimizzazione e si cerca una densità ottimale della rigidezza che non è, in generale, una funzione caratteristica. Diamo una caratterizzazione completa di questo elemento ottimale e dimostriamo alcuni risultati di regolarità. Quindi esibiamo condizioni sufficienti sui campi di forze per cui la distribuzione ottimale della rigidezza è una funzione caratteristica di un sottodominio. Studiamo il caso particolare di un corpo ed una forza radialmente simmetrici. Infine alcuni risultati numerici illustrano nel caso bidimensionale le proprietà enunciate.*

1. – Introduction.

Let an open bounded and connected domain Ω in $\mathbb{R}^N$ $(N = 2, 3)$ be given. In this paper, we study the following optimal design problem

$$\text{(1)} \qquad \operatorname{Opt}(\omega): \inf_{\omega \in O_C} J(\omega) = \int_\Omega |\nabla u|^2 + \chi_\omega u^2\, dx = \int_\Omega fu\, dx\,,$$

with $O_C = \{\omega \subset \Omega,\ |\omega| \leq C < |\Omega|\}$ and where u satisfies the following state equation:

$$\text{(2)} \qquad \begin{cases} -\Delta u(x) + \chi_\omega u(x) = f(x) & \text{in } \Omega\,, \\ u(x) = 0 & \text{on } \partial\Omega\,. \end{cases}$$

A physical interpretation could be the following: Ω is a two-dimensional membrane and f the third component of a vertical force acting on Ω. We want to reinforce a part of the membrane, that is a subdomain ω of given measure, with a stiffness equal to one, in such a way that the displacement u given by (2) mini-

(*) This work has been realized when the two authors were in the Mathematics Department of the Université de Franche-Comté (UMR CNRS 6623), Besançon France.

mizes the cost functional J given in (1). This problem could also be considered as a static model problem for classical control problems when we look for the optimal location of the actuators in some stabilization equations.

Concerning our optimization criterion, the term $\chi_\omega u^2$ can be seen as a cost term since $\chi_\omega u$ solves the following optimal control problem (where ω is a fixed datum):

$$\inf_{v \in L^2(\omega)} J(v) = \int_\Omega |\nabla u(v)|^2 + \chi_\omega v^2 \, dx ,$$

where $u(v)$ is the solution of the Dirichlet problem

$$\begin{cases} -\Delta u + \chi_\omega v = f & \text{in } \Omega , \\ \quad u \qquad\ \ = 0 & \text{on } \partial\Omega . \end{cases}$$

(See [Li] for details).

Notice that without constraint on the design set O_C, $\mathrm{Opt}(\omega)$ is trivial. Indeed, since $f \geqslant 0$, if $\omega_2 \subset \omega_1$ the associated solutions of the state equation satisfy $u_1 \leqslant u_2$ and we obtain $J(\omega_1) \leqslant J(\omega_2)$. Thus J is non-increasing with respect to the set inclusion and $\inf_{\omega \subset \Omega} J(\omega) = J(\Omega)$. It is a classical problem to look for an optimal domain, with a volume constraint, in the case of a non-increasing functional. G. Buttazzo and G. Dal Maso in a famous paper [Bu-Dm 2], proved that this problem has always a solution if we assume moreover that the functional *is lower semi-continuous for the γ-convergence.* Unfortunately, as we will see in the Remark 2.1, the functional defined in (1) turns out to be not l.s.c. for this convergence. So, we cannot apply their general result and we need to use a relaxed formulation. Since the shape variable appears in the lower order part of the elliptic state equation, the relaxation involves a simple convexification of the design set. For more general relaxation in optimal design, and in particular, when the relaxation involves the differential operator itself, we refer for example to [Bu-Dm 1], [Ch-Dm] or [Ko-St]. When one introduce a relaxation, a very interesting challenge is to decide whether one are able to find conditions on the data which insure that the relaxed solution is indeed a classical one, i.e. a domain. It is one interesting topic of this paper to present such results.

Let us now precise the notations used throughout this paper. Identifying every (class of) domain ω with its characteristic function χ_ω, the design set O_C is naturally identified with the closed subset of $L^\infty(\Omega)$

$$L_C = \left\{ l \in L^\infty(\Omega),\ 0 \leqslant l \leqslant 1,\ l(l-1) = 0 \text{ a.e.}, \int_\Omega l(x)\, dx = C \right\}.$$

In this context, $L^\infty(\Omega)$ will be endowed with the *weak-** topology. Since this topology does not provide compactness of the design set L_C (this set is not closed for the *weak*-* topology), we need to introduce some relaxation. This relaxation involves a simple convexification of the design set (see [He-Pi] or [Ma] for more details). Therefore our shape optimization relaxed problem reads:

$$\mathrm{Opt}\,(l): \inf_{l \in \tilde{L}_C} J(l) = \int_\Omega |\nabla u|^2 + lu^2\, dx = \int_\Omega fu\, dx\,, \tag{3}$$

with

$$\tilde{L}_C = \left\{ l \in L^\infty(\Omega),\ 0 \leqslant l \leqslant 1, \int_\Omega l(x)\, dx = C \right\},$$

and where u solves $P(l)$:

$$P(l) \qquad \begin{cases} -\Delta u + lu(x) = f(x) & \text{in } \Omega\,, \\ \quad u(x) \quad = \quad 0 & \text{on } \partial\Omega\,. \end{cases}$$

The sequel of the paper is organized as follows. The second section is devoted to the resolution of $\mathrm{Opt}\,(l)$ and the characterization of any minimizer. In section 3, we provide sufficient conditions on the leading term f for which the relaxed optimal design will in fact be a classical design. The special $2-D$ case where Ω and f are radially symmetric is considered in section 4. We prove existence of a radially symmetric minimizer which will be unique and classical if the forcing function f is non increasing. At last, in the last section a numerical method along with some examples are presented.

2. – The relaxed problem: existence and characterization of the solution.

The relaxed formulation previously stated leads to the following existence result.

THEOREM 2.1. – *J is convex and continuous on $L^\infty(\Omega)$ for the weak-* topology. In particular, there exists l^* in $\tilde{L}_C$ which realizes the minimum of J on $\tilde{L}_C$. Moreover*:

$$\inf_{\omega \in O_C} J(\omega) = \min_{l \in \tilde{L}_C} J(l) = J(l^*)\,. \qquad \blacksquare \tag{4}$$

PROOF. – The convexity of J comes from the variational formulation of $P(l)$: since u_l realizes the minimum in $H_0^1(\Omega)$ of $\dfrac{1}{2}\displaystyle\int_\Omega |\nabla v|^2 + lv^2\, dx - \int_\Omega fv\, dx$ which is an affine function with respect to l we have

$$-\frac{1}{2}J(l) = -\frac{1}{2}\int_\Omega |\nabla u_l|^2 + l u_l^2\, dx = \min_{v \in H_0^1(\Omega)} \left(\frac{1}{2}\int_\Omega |\nabla v|^2 + l v^2\, dx - \int_\Omega f v\, dx \right)$$

which is concave as an infimum of affine functions. Hence $J(l)$ is convex.

Now, let l_n be a sequence in $L^\infty(\Omega)$ converging *weak*-* to a function l. Let us denote by u_n the solution of $P(l_n)$ and by u the solution of $P(l)$. The function $v_n = u_n - u$ satisfies

$$\begin{cases} -\Delta v_n + l_n v_v = (l - l_n)\, u & \text{in } \Omega\,, \\ \qquad v_n = 0 & \text{in } \partial\Omega\,, \end{cases} \tag{5}$$

and the variational formulation of (5) gives:

$$\int_\Omega |\nabla v_n(l)|^2\, dx + \int_\Omega l_n v_n^2\, dx = \int_\Omega (l - l_n)\, u v_n\, dx\,. \tag{6}$$

Thanks to (6) and since l_n is bounded in $L^\infty(\Omega)$, v_n is bounded in $H_0^1(\Omega)$. So, by Rellich's theorem, there exists a subsequence v_{n_k} and a function $v \in H_0^1(\Omega)$ such that v_{n_k} converges strongly to v in $L^2(\Omega)$. Therefore uv_{n_k} converges strongly to uv in $L^1(\Omega)$ and since $l - l_n \rightharpoonup 0$ *weak** in $L^\infty(\Omega)$, the right hand side of (6) converges to 0. Moreover $\int_\Omega l_n v_n^2\, dx \geq 0$, then v_{n_k} converges (strongly) to 0 in $H_0^1(\Omega)$. Now 0 being the single accumulation point of (v_n), the whole sequence converges to 0 in $H_0^1(\Omega)$. The continuity of J follows immediately. The existence of a minimizer in $\widetilde{L}_C$ follows from the *weak*-* compactness of $\widetilde{L}_C$. (4) is an easy consequence of the fact that $\widetilde{L}_C$ is the closed convex hull of L_C and the fact that $\inf_{L_C} J = \inf_{O_C} J$. ■

REMARK 2.1. – We are now in position to prove that the functional J is not lower-semi continuous for the γ-convergence. Let $\Omega =]0, \pi[\times]0, 4[$ and ω_n be the sub-domain defined as

$$\omega_n = \{(x, y) \in \mathbb{R}^2;\ 0 < x < \pi,\ 0 < y < 2 + \sin(nx)\}\,.$$

The sequence ω_n converges in the Hausdorff sense to $\omega =]0, \pi[\times]0, 1[$. Since we are in two dimensions and ω_n is simply connected, the classical result of Šverak (see [Sv]) applies and ω_n γ-converges to ω. Now, it is classical to verify that the sequence of characteristic functions χ_{ω_n} converges in L^∞ *weak*-* to the function l defined by

$$l(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \omega =]0,\pi[\times]0,1[\\ \dfrac{1}{2} & \text{if } (x,y) \in]0,\pi[\times]1,3[\\ 0 & \text{if } (x,y) \notin]0,\pi[\times]0,3[\,. \end{cases}$$

According to Theorem 2.1, it follows that u_{ω_n} solution of (2), converge strongly in $H_0^1(\Omega)$ to u_l solution of $P(l)$ with l defined above. So,

$$J(\omega_n) = \int_\Omega f u_{\omega_n}\, dx \to \int_\Omega f u_l\, dx\,.$$

Now, since $l > \chi_\omega$, the maximum principle shows that $u_\omega > u_l$ in Ω. Therefore

$$J(\omega) = \int_\Omega f u_\omega\, dx > J(l) = \int_\Omega f u_l\, dx = \lim J(\omega_n)$$

what proves that J is not l.s.c. for the γ-convergence. ■

We are now interested in giving some properties or characterization of the optimal function l^*. Of course these properties will be obtained through the optimality conditions. First of all, let us express the first derivative of J.

LEMMA 2.1. – *The functional J is Frechet-differentiable at any point $l \in L^\infty(\Omega)$ and*

$$\langle J'(l), h\rangle = -\int_\Omega h u_l^2\, dx \tag{7}$$

where u_l is the solution of $P(l)$. ■

PROOF. – Let us fix h in $L^\infty(\Omega)$. Subtracting $P(l+h)$ and $P(l)$, we obtain:

$$-\Delta(u_{l+h} - u_l) + l(u_{l+h} - u_l) = -h u_{l+h}. \tag{8}$$

Multiplying (8) by $(u_{l+h} + u_l)$ and integrating upon Ω yield

$$\int_\Omega |\nabla u_{l+h}|^2 - |\nabla u_l|^2 + l u_{l+h}^2 - l u_l^2\, dx = -\int_\Omega h u_{l+h}(u_{l+h} + u_l)\, dx\,,$$

or

$$J(l+h) - J(l) + \int_\Omega h u_l^2\, dx = -\int_\Omega h u_l (u_{l+h} - u_l)\, dx\,,$$

and therefore

$$|J(l+h) - J(l) + \int_\Omega h u_l^2\, dx| \leq \|h\|_\infty \|u_{l+h} - u_l\|_2 \|u_l\|_2 . \tag{9}$$

Now, according to (8) and the Poincaré inequality

$$\|u_{l+h} - u_l\|_2 \leq C\|hu_{l+h}\|_2 \leq 2C\|h\|_\infty \|u_l\|_2 ,$$

and the result follows from (9). ■

In order to characterize the minimum, let us introduce the following definition.

DEFINITION 2.1. – *For any function $l \in \tilde{L}_C$, we denote by*

$$\begin{cases} \Omega_0 = \{x \in \Omega,\ l(x) = 0\} \\ \Omega^* = \{x \in \Omega,\ 0 < l(x) < 1\} \\ \Omega_1 = \{x \in \Omega,\ l(x) = 1\} . \end{cases} \quad ■$$

Of course, these sets are defined up to a set of zero measure and the equalities or inequalities make sense almost everywhere. Since we want to write the optimality condition, we have to characterize the tangent cone of $\tilde{L}_C$ in $L^\infty(\Omega)$.

LEMMA 2.2. – *The tangent cone $T'(l)$ to the set $\tilde{L}_C$ at the point l is the set of every function h in $L^\infty(\Omega)$ such that*

(i) $\int_\Omega h(x)\, dx = 0$,

(ii) $\|\chi_{Q_n^0} h_-\|_\infty \to 0$ *when* $n \to \infty$,

(iii) $\|\chi_{Q_n^1} h_+\|_\infty \to 0$ *when* $n \to \infty$, *where h_- (resp. h_+) is the negative (resp. positive) part of h and where $Q_n^0 = \{x \in \Omega,\ l(x) \leq 1/n\}$ and $Q_n^1 = \{x \in \Omega,\ l(x) \geq 1 - 1/n\}$.* ■

PROOF. – See [B.P.R.S] or [C.P.]. ■

REMARK 2.2. – The condition $h(x) \geq 0$ in Ω_0 and $h(x) \leq 1$ in Ω_1 is clearly necessary but not sufficient for h to be in $T'(l^*)$. This can be shown by some elementary examples. ■

REMARK 2.3. – The first order optimality condition is

$$\forall h \in T'(l),\ \langle J'(l), h\rangle \geq 0 . \tag{10}$$

Notice that, since J is convex and $\tilde{L}_C$ is convex, this necessary condition is also a sufficient one. Hence (10) turns out to give a characterization of the minimizers of J. ■

More precisely, we can prove the following result.

THEOREM 2.2. – *Let l^* be in $\tilde{L}_C$ and u^* solve $P(l^*)$. Let Ω_0, Ω_1, Ω^* be defined as above. Then l^* minimizes J if and only if:*

(i) *u^* is constant in Ω^* (as soon as $|\Omega^*| > 0$)*

(ii) *$\forall (x_0, x^*, x_1) \in \Omega_0 \times \Omega^* \times \Omega_1$, we have $u^*(x_0) \leqslant u^*(x^*) \leqslant u^*(x_1)$.* ■

PROOF. – Let l^* be a minimizer of J and let us denote by $\Omega_n^* = \{x \in \Omega,\ 1/n \leqslant l^* \leqslant 1 - 1/n\}$. We are going to prove that u^* is constant on Ω_n^*. Since $\Omega^* = \bigcup_{n>0} \Omega_n^*$ (increasing union), this will prove the first point. Let us assume, for a contradiction, that u^* is not constant on Ω_n^*. Then it is possible to find two measurable sets ω_1 and ω_2 in Ω_n^* such that

$$|\omega_1| = |\omega_2| \quad \text{and} \quad \int_{\omega_1} u^{*2}\, dx < \int_{\omega_2} u^{*2}\, dx\,. \tag{11}$$

Now taking

$$h(x) = \begin{cases} -1 & \text{in } \omega_1 \\ +1 & \text{in } \omega_2 \\ 0 & \text{elsewhere} \end{cases}$$

which belongs to $T'(l^*)$ (see Lemma 2.2), yields

$$\langle J'(l^*), h \rangle = -\int_\Omega h u^{*2}\, dx = -\int_{\omega_2} u^{*2}\, dx + \int_{\omega_1} u^{*2}\, dx < 0$$

by (11), what contradicts the optimality condition (10).

The second point is proved in a similar way by assuming that there exists a set of positive measure ω_0 in Ω_0 such that

$$u^*_{/\omega_0} > u^*_{/\Omega^*} = cst\,.$$

Then we select ω^* in Ω_n^*, with $|\omega_0| = |\omega^*|$ and we conclude by choosing the function h defined by $h = 1$ in ω_0, $h = -1$ in ω^*. We prove in the same way that $u^*_{/\omega_1} \geqslant u^*_{/\Omega^*}$.

Conversely, let us assume that (l^*, u^*) satisfy (i), (ii) and let us denote by c^* the (constant) value of u^* on Ω^*. Let h be in the tangent cone $T'(l^*)$. According to remark 2.2, h is nonnegative on Ω_0 and non positive on Ω_1, and

$$-\int_{\Omega} hu^{*2}\,dx = -\int_{\Omega_0} hu^{*2}\,dx - \int_{\Omega^*} hu^{*2}\,dx - \int_{\Omega_1} hu^{*2}\,dx \geqslant$$

$$-\int_{\Omega_0} hc^{*2}\,dx - \int_{\Omega^*} hc^{*2}\,dx - \int_{\Omega_1} hc^{*2}\,dx = -c^{*2}\int_{\Omega} h\,dx = 0\,.$$

Therefore, the first order optimality condition is satisfied and according to remark 2.3, l^* minimizes J. ■

REMARK 2.4. – The function u^* solution of $P(l^*)$ is in the Sobolev space $H^2(\Omega)$, so u^* is continuous (thanks to the Sobolev embedding $H^2 \hookrightarrow C^0$ in dimension $N = 1$, 2 or 3). In particular, because of the boundary condition, the set Ω_0 is non empty and contains a neighbourhood of $\partial\Omega$. Now let c^* be defined by

$$c^* = \sup_{x \in \Omega_0} u^*(x)\,. \tag{12}$$

If Ω^* is non empty, according to the theorem 2.2, $u^*_{/\Omega} = c^*$. Similarly, if Ω_1 is non empty, we have $c^* = \inf_{x \in \Omega_1} u^*(x)$. Moreover, the open set $\{x \in \Omega,\ u^*(x) < c^*\}$, included in Ω_0, (and which contains a neighbourhood of $\partial\Omega$) is connected when $\partial\Omega$ is connected. Indeed, if it was not, let ω be a connected component of this open set which does not meet the neighbourhood of $\partial\Omega$. According to (12), we have $u^* = c^*$ on $\partial\omega$, but $-\Delta u^* = f \geqslant 0$ in ω (since $l^* = 0$ in ω) and then by maximum principle, we should have $u^* > c^*$ in ω, which is impossible. ■

3. – Existence of an optimal domain.

In the context of shape optimization problems which involve relaxed formulation, a natural question arises: can we give some conditions on the data $(f, C, \ldots)$ in order to obtain a classical minimizer, *i.e.* a solution of the original problem. With the notations introduced in definition 2.1, it is equivalent to ask whether there exists a minimizer l^* with Ω^* empty. Let us mention that in many optimal design problems, it is difficult to exhibit such sufficient conditions (see for instance [Ch-Dm] or [Co-Uh]). So, the following theorem seems to be rather original in this context.

THEOREM 3.1. – *Let* u_0 *denotes the solution of the problem*

$$\begin{cases} -\Delta u_0 = f & in\ \Omega\,, \\ u_0 \quad = 0 & in\ \partial\Omega\,, \end{cases}$$

There exists a characteristic function χ_{Ω_1} *which is a minimum of the functional* J *if one of the following conditions holds.*

(i) $u_0 \leqslant f$ *in* Ω,

(ii) $f \leqslant -\Delta f$ *in* Ω.

(iii) $C > |\{x \in \Omega,\ u_0(x) > \alpha\}|$, *where* $\alpha = \inf\{f(x)$, *such that* $u_0(x) > f(x)\}$. *Moreover the minimum is unique.* ■

PROOF OF THE THEOREM 3.1. – We are going to prove that under the above conditions, the set Ω^* corresponding to a minimizer l^* is necessarily empty: it means that all the minimizers will correspond to characteristic functions. Now, since J is convex, if l_1^* and l_2^* minimize J, $tl_1^* + (1-t)\, l_2^*$ will be minimizers of J for all t in $]0,1[$. Since characteristic functions are extremal points of the convex $\widetilde{L}_C$, uniqueness of the minimum will follow.

Let l^* be a minimizer and assume that Ω^* is not empty. Then $u^* = c^* =$ *constant* on Ω^*. Since u^* is in H^2, it follows by a classical result (see *e.g.* [A-L]) that $u^*_{x_i}$ vanishes almost everywhere in the set $\{u^* = c^*\}$. Using the same argument yields $u^*_{x_i x_i} = 0$ almost everywhere in the set $\{u^*_{x_i} = 0\}$ and then in $\{u^* = c^*\}$. It follows that $-\Delta u^* = 0$ almost everywhere in Ω^*. But from the equation satisfied by u^*, it follows that

$$l^* u^* = l^* c^* = f \quad \text{in } \Omega^*. \tag{13}$$

Now, by maximum principle, we know that $u^* \leqslant u_0$ in Ω, so (13) yields

$$l^* = \frac{f}{c^*} \geqslant \frac{f}{u_0} \quad \text{in } \Omega^*.$$

Then if (i) holds, we have $l \geqslant 1$ in Ω^* which contradicts the definition of Ω^*. If (ii) holds, let us denote by v the function $f - u_0$. We have

$$-\Delta v = -\Delta f + \Delta u_0 = -\Delta f - f \geqslant 0 \quad \text{in } \Omega\,,$$

$$v = f \quad \text{on } \partial\Omega\,.$$

So $f - u_0 \geqslant 0$ in Ω and we can apply (i).

Finally, (13) together with the definition of Ω^* implies $f < c^*$. Then, Ω^* is necessarily included in the set $\{x \in \Omega, f(x) < u_0\}$ (since $u^* \leqslant u_0$ by maximum principle). Therefore, $\Omega^* \cup \Omega_1$ which is included in the set $\{x \in \Omega,\ u^* \geqslant c^*\}$ satisfies

$$\Omega^* \cup \Omega_1 \subset \{x \in \Omega,\ u^* > \alpha\} \subset \{x \in \Omega,\ u_0 > \alpha\} \tag{14}$$

Now, since the constraint $C = \int_\Omega l(x)\,dx = |\Omega_1| + \int_{\Omega^*} l(x)\,dx$ implies $C \leqslant |\Omega^* \cup \Omega_1|$, the assumption (iii) is incompatible with the inclusion (14). Hence Ω^* is empty as soon as (iii) holds. ■

In the case where there exists a classical solution χ_{Ω_1}, it would be interesting to discuss the regularity of the optimal domain Ω_1. It is a general challenge in shape optimization to get such regularity results, since we often only know that the optimum is a measurable set or a quasi-open set (see e.g. [Bu-Dm 2]). Even proving that the optimal domain is an open set would be interesting. For such a result in the case of the first eigenvalue of an elliptic operator, we refer to [Ha]. In our case, we are able to prove, thanks to a supplementary simple assumption on the data f, that the optimal domain is an open set which has a boundary with zero Lebesgue measure.

PROPOSITION 3.1. – *Let f satisfy one of the points* (i), (ii), (iii) *of the theorem* 3.1 *and assume moreover* (iv) *f is not constant on a set of positive measure.*

Then, the class of optimal domains Ω_1 such that $l^* = \chi_{\Omega_1}$ contains an open set $\widetilde{\Omega}_1$ with $|\partial \widetilde{\Omega}_1| = 0$. ■

PROOF. – Let us recall that the associated optimal state solves:

$$\begin{cases} -\Delta u^* + \chi_{\Omega_1} u^* = f & \text{in } \Omega\,, \\ u^* = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Moreover the level line $u^* = c^*$ is such that $c^* = \sup_{x \in \Omega_0} u^*(x) = \inf_{x \in \Omega_1} u^*(x)$. Now notice that $\Omega_1 = \{x \in \Omega,\ u^*(x) > c\} \cup (\{x \in \Omega,\ u^*(x) = c^*\} \cap \Omega_1)$. But $\{x \in \Omega,\ u^*(x) = c^*\} \cap \Omega_1 \subset \{x \in \Omega,\ f = c^*\}$. Indeed, if $|\{x \in \Omega,\ u^*(x) = c^*\} \cap \Omega_1| > 0$ we have $-\Delta u^* = 0$ and $f = l^* c^* = c^*$ in this set. Then $|\{x \in \Omega,\ u^*(x) = c^*\} \cap \Omega_1| = 0$ if (iv) holds. Moreover, u^* being continuous, $\{x \in \Omega,\ u^*(x) > c^*\}$ is an open set and $l^* = \chi_{\{x \in \Omega,\, u^*(x) > c^*\}}$. At last, since $\partial\Omega_1 \subset \{x \in \Omega,\ u^*(x) = c^*\}$ which has zero measure if (iv) holds, the boundary of Ω_1 has zero measure. ■

REMARK 3.1. – Notice that the maximization (under volume constraint) of J always leads to a classical solution. Indeed, the continuous functional

$$E(l) = \min_{v \in H_0^1(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla v|^2 + lv^2\, dx - \int_\Omega fv\, dx \right) = -\frac{1}{2} J(l),$$

as a concave function, reaches its minimum at the extremal points of the convex set $\widetilde{L}_C$ (*i.e.* the elements of L_C). ■

4. – The radial case.

In this section Ω is the unit ball $B(0, 1)$ of $\mathbb{R}^2$ and f is radially symmetric *i.e.* $f = f(r)$ where $r = |x|$.

The natural question is: «Are the minima necessarily radially symmetric?»

We are able to give a complete answer only in the case where f is non increasing (see below). But without hypothesis on f we can give the

PROPOSITION 4.1. – *The functional J admits at least one radially symmetric minimizer.*

PROOF. – Let l^* be a minimizer of J. For θ in $[0, 2\Pi]$, let us denote by l_θ^* the function obtained from l^* by the rotation of angle θ. Obviously l_θ^* minimizes J for every θ. Let us now consider the function $\bar{l}(x) = \dfrac{1}{2\Pi} \displaystyle\int_0^{2\Pi} l_\theta^*(x)\, d\theta$. It is a radially symmetric admissible function and we have (Jensen inequality)

$$J(\bar{l}) \leqslant \frac{1}{2\Pi} \int_0^{2\Pi} J(l_\theta^*)\, d\theta = \min_{\widetilde{L}_C} J(l),$$

since J is convex. Then $\bar{l}$ is a minimizer of J. ■

REMARK 4.1. – To prove existence of a radial minimizer, we can also use the classical tool of spherical rearrangement. The main point would be to use the optimality conditions, and more precisely the fact that the level set $\{u > c^*\}$ is transformed into the corresponding level set of the rearranged function which allow us to compare the integrals.

Let us come back to the above question. Clearly, looking for the radially symmetric minimizers is a problem much easier. Indeed, it consists in looking for the solution of a linear O.D.E. which satisfies the optimality condition given in the theorem 2.2.

More precisely, let us consider first the case where $f > 0$ in Ω. Since $-\Delta u = f$ in Ω_0, u cannot be constant in a set of positive measure (see the proof of the theorem 3.1). Then, according to the remark 2.4, the set Ω_0 is a connected ring of exterior boundary $\partial\Omega$:

$$\Omega_0 = \{x \in \Omega,\ R_0 < |x| < 1\}.$$

It remains to find the location of Ω_1 and Ω^* (each of them could be empty) which would also be rings or union of rings centered at 0. Working in polar coordinates, the problem becomes:

Find a subdivision of $]0, 1[$: $]0, 1[= \Omega_1 \cup \Omega^* \cup \Omega_0$ with $\Omega_0 =]R_0, 1[$ such that the (unique) solution of

$$(15)\qquad \begin{cases} -u'' - \dfrac{1}{r}u' + u = f(r) & \text{in } \Omega_1, \\ u = constant = c^* & \text{in } \Omega^*, \\ -u'' - \dfrac{1}{r}u' = f(r) & \text{in } \Omega_0 =]R_0, 1[, \\ u'(0) = 0,\ u(1) = 0. \end{cases}$$

satisfies the optimality condition

$$\sup_{\Omega_0} u = c^* = \inf_{\Omega_1} u\,.$$

When f is given by a simple expression, (15) can be solved almost explicitely and then the above program can be performed, using for instance a sotfware like Maple.

When f vanishes, the situation is slightly more complicated since Ω_0 can have connected components inside the set $\{f = 0\}$. Nevertheless, the situation is similar: we have to check with a larger number of cases. As an illustration of this point, let us give the following example.

We take $f = a\chi_{[R,1]}$, $0 < R < 1$, $a > 0$. Using the maximum principle together with the optimality conditions (see the proof of the theorem 4.1 below) it is easy to verify that for such a f, the minimizer l^* is a characteristic function. It remains to find its location. We have to check with four cases that are

- $l^* = \chi_{[0, R_1]}$ or $l^* = \chi_{[R_1, R_0]}$, with
- $R_1 < R_0 < R$ or $R < R_1 < R_0$, or finally $R_1 < R < R_0$.

We can easily prove that the three first cases cannot occur. The last one leads to solve a non linear equation $(u(R_1) = u(R_0))$ which has a unique solution (R_1, R_0) satisfying the optimality conditions and the constraint $\Pi(R_0^2 - R_1^2) = C$. So, in this case Ω_0 has two connected components separated by the ring $\Omega_1 = \chi_{[R_1, R_0]}$. Now if we take $f = \varepsilon\chi_{[0,R]} + a\chi_{[R,1]}$, ε small, the minimizer l_ε^*, closed to l^*, is relaxed. Indeed, when $\varepsilon = 0$, the internal part of Ω_0 $([0, R_1])$, (on which u is constant) becomes, as soon as $\varepsilon \neq 0$, a relaxed part

Ω_*. By continuity arguments with respect to the perturbation ε, we can prove that l^* is relaxed in a ball $B(0, R^*)$, equal to 1 in a ring $[R^*, R_1]$, $R_1 < R$ and null on the rest of the unit ball.

Let us now come back to the case f non increasing, where we can give a complete answer:

THEOREM 4.1. – *Assume that Ω is the unit ball and $f = f(r)$ is a radially symmetric non increasing function in $L^2(\Omega)$. Then the solution is unique and it is the characteristic function of a radially symmetric domain.* ■

PROOF. – We are going to prove the following steps:

(i) There exists a classical solution $l^* = \chi_{\Omega_1}$, with $\Omega_1 = B(0, R_1)$.

(ii) There does not exist any other radially symmetric solution.

(iii) Conclusion.

According to the above discussion, the point (i) will be proved if the unique solution of

$$\text{(16)} \qquad \begin{cases} -u'' - \dfrac{1}{r}u' + u = f(r) & \text{in }]0, R_1[, \\ -u'' - \dfrac{1}{r}u' = f(r) & \text{in }]R_1, 1[, \\ u,\ u' \text{ continuous for } r = R_1, \\ u'(0) = 0,\ u(1) = 0\,. \end{cases}$$

satisfies $\sup_{\Omega_0} u = \inf_{\Omega_1} u$. More precisely, we are proving that u is non increasing as soon as f is non increasing. First of all let us prove that $u'(R_1) \leqslant 0$ (u is in $H^2(0,1)$, so u' is continuous). Multiplying the equation by r and integrating on $(0, R_1)$ yield

$$\int_0^{R_1} r(u-f)\,dr = \int_0^{R_1} (ru'' + u')\,dr = R_1 u'(R_1)\,.$$

Now, by maximum principle, we know that $u \leqslant u_0$ in Ω where u_0 is the solution of $-\Delta u_0 = f$ in Ω, $u_0 = 0$ on $\partial\Omega$. So

$$\text{(17)} \qquad R_1 u'(R_1) \leqslant \int_0^{R_1} r(u_0 - f)\,dx\,.$$

Now, since f is radially symmetric we can write an explicit expression for u_0 that is $u_0(r) = -\ln r \int_0^r tf(t)\,dt - \int_r^1 t \ln t f(t)\,dt$. Therefore a straightforward computation gives

$$\int_0^{R_1} t(u_0(t) - f(t))\,dt = (R_1^2/4 - (R_1^2/2)\ln R_1 - 1)\int_0^{R_1} tf(t)\,dt - \tag{18}$$

$$(1/4)\int_0^{R_1} t^3 f(t)\,dt - (R_1^2/2)\int_{R_1}^1 t\ln tf(t)\,dt\,.$$

Since f is non increasing, the last term in (18) is estimated from above by

$$-R_1^2/2\int_{R_1}^1 t\ln tf(t)\,dt \leqslant (R_1^2/2)\,f(R_1)\int_{R_1}^1 (-t\ln t)\,dt \leqslant (R_1^2/2e)\,f(R_1)\,.$$

Moreover, in $]0, 1[$

$$R_1^2/4 - (R_1^2/2)\ln R_1 - 1 \leqslant -3/4 \quad \text{and} \quad \int_0^{R_1} tf(t)\,dt \leqslant (R_1^2/2)\,f(R_1)\,,$$

and the first term in the right hand side of (18) is less than $-(3/8)\,R_1^2 f(R_1)$. So, finally

$$4R_1 u'(R_1) \leqslant 2(e^{-1} - 3/4)\,R_1^2 f(R_1) - \int_0^{R_1} t^3 f(t)\,dt \leqslant 0\,.$$

To conclude to the non increasingness of u, we use one more time the maximum principle. Let us assume that f is regular (*e.g.* C^1) and for a general f we argue thanks to a density argument. Differentiating the equation on $]0, r_1[$ yields:

$$-\frac{d^2}{dr^2}u' - \frac{1}{r}\frac{d}{dr}u' + \left(1 + \frac{1}{r^2}\right)u' = f'(r) \leqslant 0 \quad \text{in }]0, R_1[\,.$$

So u' must have its maximum on the boundary. Now $u'(0) = 0$ and $u'(R_1) \leqslant 0$ hence u' remains non positive on $]0, R_1[$. We use the same argument on $]R_1, 1[$ since $u'(1) \leqslant 0$.

Proof of (ii). Let us assume, for a contradiction, that there exists a radially symmetric solution with Ω^* non empty. We claim that, in this case, Ω^* is necessarily a ring with an outer boundary common with Ω_0. Indeed, if it was not the case we would have a component of Ω^* with an outer boundary common with Ω_1. But the fact that u necessarily decreases in each component of Ω_1 would contradict the optimality conditions given in the theorem 2.2. So if Ω^* is not empty, there exists R^* such that the sphere of radius R^* is a common boundary to Ω^* and Ω_0. Therefore, on $]R^*, 1[$, u is solution of

$$-u'' - \frac{1}{r}u' = f(r) \quad \text{in }]R^*, 1[, \; u'(R^*) = 0, \; u(1) = 0\,,$$

and is given by $u(r) = -\ln r \int_{R^*}^{r} tf(t)\,dt - \int_{r}^{1} t \ln tf(t)\,dt$, with

$$c^* = u(R^*) = -\int_{R^*}^{1} t \ln tf(t)\,dt\,. \tag{19}$$

Now, since f is non increasing, we would have from (19) $c^* \leqslant f(R^*)\,e^{-1} \leqslant f(t)\,e^{-1}$ for all $t < R^*$. But in Ω^* we must have $(f/c^*) < 1$ a.e. and this is a contradiction with the above inequality.

(iii) Conclusion. Any relaxed solution provides a radially symmetric relaxed solution and that is incompatible with the point (ii). This implies uniqueness of the minimizer according to the convexity of J. ∎

REMARK 4.2. – In the case where f is not decreasing, we give an example of minimizer with $\Omega_1 = \emptyset$. Let $0 < C < 1/2$ fix the volume constraint and α be a positive number. Take $R = \sqrt{2C}$, $f = \alpha\chi_{[0,R[} + \chi_{[R,1[}$. The couple (u, l) defined by:

$$u(t) = \alpha \quad \text{in } [0, R[$$

$$-u'' - \frac{1}{r}u' = f \quad \text{in} \quad [R, 1[, \; u'(0) = 0, \; u(1) = 0$$

$$u(R) = \alpha$$

$$l(t) = t\chi_{[0,R]}(t)$$

satisfies the theorem 2.2 with l totally relaxed. ∎

5. – Some numerical results.

Let us observe that the functional J being convex, we could use any classical minimization algorithm to solve the above minimization problem. Nevertheless, we propose another algorithm based on an intensive use of the optimality conditions given in the theorem 2.2. This algorithm turns out to be very efficient and derives from the following observation.

PROPOSITION 5.1. – *Let $E(l, \alpha) = (E_1(l, \alpha), E_2(l, \alpha))$ be the functional defined on $\tilde{L}_C \times \mathbb{R}_+$ by:*

$$E_1(l, \alpha) = \int_{\{u<\alpha\}} l^2\,dx + \int_{\{u=\alpha\}\cap\{0<f<\alpha\}} \left(l - \frac{f}{\alpha}\right)^2 dx + \int_{\{u>\alpha\}\cup\{u=f=\alpha\}} (l-1)^2\,dx$$

and

$$E_2(l, \alpha) = \left| C - \int_{\{u \geq \alpha\}} l(x)\, dx \right| .$$

Then, l^* is a minimizer of J if and only if $E(l^*, c^*) = (0, 0)$ (where c^* is defined by (12)).

PROOF. – Clearly the couple (l^*, c^*) where l^* minimizes J and c^* is defined as in the theorem 2.2 is a solution of $E(l, \alpha) = (0, 0)$. Conversely, it is easy to check that a point (l, α) where E vanishes satisfies the optimality conditions (i) and (ii) of the theorem 2.2 and the volume constraint. ■

Since the functionals $E_i(l, \alpha)$ are nonnegative, their roots are also their minimizers and (3) is equivalent to:

(20) $$\mathrm{Opt}(l, \alpha): \inf_{(l, \alpha) \in \tilde{L}_C \times \mathbb{R}^+} E_i(l, \alpha), \ i = 1, 2 .$$

Let us now describe how we approximate (3) and (20). We take $\Omega =]0, 1[\times]0, 1[$ and we denote by $P_h(L)$ the approximation of $P(l)$ by a Finite Element Method or a Finite Difference Method. In any case, we will denote by u_i the approximate value of u at i^{th} node ($i = 1 \dots N$, we will denote by I the discrete set $I = \{1, \dots, N\}$). Then, using a trapezoidal quadrature rule, the functional J can be approximated as follows:

(21) $$J_h(L) = h^2 \sum_{i \in I} f_i u_i ,$$

where $U = \{u_1 \dots u_N\}$, $F = \{f_1 \dots f_N\}$ and $L = \{l_1 \dots l_N\} \in \tilde{L}_{Ch} = \{L \in \mathbb{R}^N,\ 0 \leq l_i \leq 1 \ \ \forall i \in I,\ h^2 \sum_{i \in I} l_i = C\}$ are the approximations of u, f and l respectively.

As in the continuous case, the functional J_h is convex and has a minimizer on the compact convex set $\tilde{L}_{Ch}$.

We are in a simple case where the gradient of the discretized problem corresponds to an approximation of the continuous gradient, so the approximation of the optimality condition associated to J_h reads:

If $L^* \in \tilde{L}_{Ch}$ minimizes J_h and if $U^* = \{u_1^* . u_N^*\}$ solves $P_h(L^*)$, then

(22) $$\langle J_h'(L^*), H \rangle = -h^2 \sum_{i \in I} h_i u_i^{*2} \geq 0$$

for every $H = \{h_1 \dots h_N\}$ in $T_h'(L^*)$ the tangent cone to $\tilde{L}_{Ch}$ at the point L^*.

The constraints being linear, it is easy to describe the tangent cone in this case. Introducing the discrete version of the definition 2.1:

$$I_0 = \{i \in I,\ l_i = 0\} \quad I^* = \{i \in I,\ 0 < l_i < 1\} \quad I_1 = \{i \in I,\ l_i = 1\}$$

yields immediately

$$T_h'(L^*) = \Big\{H = (h_1, \ldots, h_N),\ h_i \geq 0 \text{ for } i \in I_0,\ h_i \leq 0 \text{ for } i \in I_1, \text{ and } \sum_{i\in I} h_i = 0\Big\}.$$

As an easy consequence (mimic the proof of Theorem 2.2), we can give

THEOREM 5.1. – *Let $L^* = (l_1^*, \ldots, l_N^*) \in \widetilde{L}_{Ch}$ and $U^* = (u_1^*, \ldots, u_N^*)$ be the solution of $P_h(L^*)$. Let I_0, I^*, I_1 be the partition of Ω associated with L^*.*
Then L^* minimizes J_h if and only if:

(i) $u_i^* = c^*$, $\forall i \in I^*$ (*as soon as* $I^* \neq \emptyset$)

(ii) $\forall (i, j, k) \in I_0 \times I^* \times I_1$, *we have* $u_i^* \leq u_j^* \leq u_k^*$.

In order to approximate (20) we have to introduce:

$$I_0(\alpha) = \{i \in I,\ u_i \leq \alpha\}, \tag{23}$$

$$I^*(\alpha) = \{i \in I,\ u_i = \alpha,\ f_i < \alpha\}, \tag{24}$$

$$I_1(\alpha) = \{i \in I,\ u_i \geq \alpha\} \cup \{i \in I,\ u_i = \alpha = f_i\}, \tag{25}$$

and we consider:

$$E_{1h}(L, \alpha) = h^2 \left(\sum_{I_0(\alpha)} l_i^2 + \sum_{I^*(\alpha)} \left(l_i - \frac{f_i}{\alpha} \right)^2 + \sum_{I_1(\alpha)} (l_i - 1)^2 \right), \tag{26}$$

$$E_{2h}(L, \alpha) = \left| C - h^2 \sum_{I^*(\alpha) \cup I_1(\alpha)} l_i \right|. \tag{27}$$

Notice that (L^*, c^*) such that $E_i(L^*, c^*) = 0$ satisfies the theorem 5.1 with $I_0(c^*) = I_0$, $I^*(c^*) = I^*$ and $I_1(c^*) = I_1$. Therefore we have

PROPOSITION 5.2. – *Let $E_h(L, \alpha) = (E_{1h}(L, \alpha), E_{2h}(L, \alpha))$ be the functional defined on $\widetilde{L}_{Ch} \times \mathbb{R}_+$ by* (26)(27).
Then (L, α) minimizes E_h if and only if L minimizes J_h with $\alpha = c^*$. ■

Before describing the steps of our algorithm, let us remark that the sets $I_0(\alpha)$, $I^*(\alpha)$ and $I_1(\alpha)$ are not really adapted to a numerical treatment and have to be replaced by the following ones (with $\varepsilon > 0$ fixed):

$$I_0(\alpha, \varepsilon) = \{i \in I,\ u_i < \alpha(1 - \varepsilon)\} \tag{28}$$

$$I^*(\alpha, \varepsilon) = \left\{ i \in I,\ \left| \frac{u_i}{\alpha} - 1 \right| < \varepsilon,\ f_i < \alpha(1 - \varepsilon) \right\} \tag{29}$$

$$I_1(\alpha, \varepsilon) = \{i \in I,\ u_i > \alpha(1 + \varepsilon)\} \cup \left\{ i \in I,\ \left| \frac{u_i}{\alpha} - 1 \right| < \varepsilon,\ \left| \frac{f_i}{\alpha} - 1 \right| < \varepsilon \right\}. \tag{30}$$

Then, the level line c^* associated to the optimal L^* is defined by

$$\left| c^* - \max_{I_0(c^*,\, \varepsilon)} u_i^* \right| < \varepsilon\,, \qquad \left| c^* - \min_{I_1(c^*,\, \varepsilon)} u_i^* \right| < \varepsilon\,. \tag{31}$$

So, for $\varrho > 0$ well chosen, the algorithm is:

Step 0: (initialization) $L_0 \in \widetilde{L}_{Ch}$. $K = 0$.
(Loop on K: while $K \leqslant Q$)
Step 1: resolution of $P_h(L^K)$.
Step 2: computation of α^K which minimizes $\alpha \mapsto E_{2h}(L^K, \alpha)$.
Step 3: computation of the sets $I_0^K(\alpha^K, \varepsilon)$, $I^{*K}(\alpha^K, \varepsilon)$ and $I_1^K(\alpha^K, \varepsilon)$.
Step 4: computation of $\nabla E_{1h}(L^K, \alpha^K)$ and $L_{K+1} = L^K + \varrho \nabla E_{1h}(L^K, \alpha^K)$.
(End of loop).

If $\mathrm{Card}\,(I^{*Q}(\alpha^Q, \varepsilon)) > 0$, $l_i^{Q+1} = 0$ for $i \in I_0^Q(\alpha^Q, \varepsilon)$ and $l_i^{Q+1} = f_i / (\alpha^{Q+1})$ for $i \in I^{*Q}(\alpha^Q, \varepsilon)$, and $l_i^{Q+1} = 1$ for $i \in I_1^Q(\alpha^Q, \varepsilon)$,

$$\alpha^{Q+1} = \frac{h^2 \sum\limits_{I^{*Q}(\alpha^Q,\, \varepsilon)} f_i}{C - h^2\, \mathrm{Card}\,(I_1^Q(\alpha^Q, \varepsilon))}\,.$$

If $\mathrm{Card}\,(I^{*Q}(\alpha^Q, \varepsilon)) = 0$ $l_i^{Q+1} = 0$ for $i \in I_0^Q(\alpha^Q, \varepsilon)$ and $l_i^{Q+1} = 1$ for $i \in I_1^Q(\alpha^Q, \varepsilon)$.

REMARK 5.1. – The number Q is generally small. Moreover this number, together with the sets $I_0^Q(\alpha^Q, \varepsilon)$ and $I_1^Q(\alpha^Q, \varepsilon)$, does not depend on L_0. Finally, the solution of $P_h(L^{Q+1})$ satisfies the optimality conditions of the theorem 5.1 (where the point (i) is replaced by the condition $|u_i - \alpha^{Q+1}| < \varepsilon$). ■

REMARK 5.2. – Our method is very well adapted to computation of a classical solution. This solution may coexist with a relaxed one. Indeed, let us consider the one dimensional case and give the following example. Take $C = 7/4$, $\Omega =]0, 6[$ and f given by:

$$f(x) = \begin{cases} 1/2 & \text{in }]0, 2[\cup]4, 6[\\ 1 & \text{in } [5/2, 7/2] \\ x - 3/2 & \text{in } [2, 5/2] \\ -x + 9/2 & \text{in } [7/2, 4] \end{cases}$$

The function l defined by

$$l(x) = \begin{cases} 0 & \text{in }]0, 2[\cup]4, 6[\\ f(x) & \text{in } \quad [2, 4] \end{cases}$$

belongs to $\widetilde{L}_C$ and $P(l)$ is satisfied by

$$u(x) = \begin{cases} -x^2/4 + x & \text{in }]0, 2[\\ 1 & \text{in } [2, 4] \\ -1/4(x-6)^2 - (x-6) & \text{in } [4, 6]. \end{cases}$$

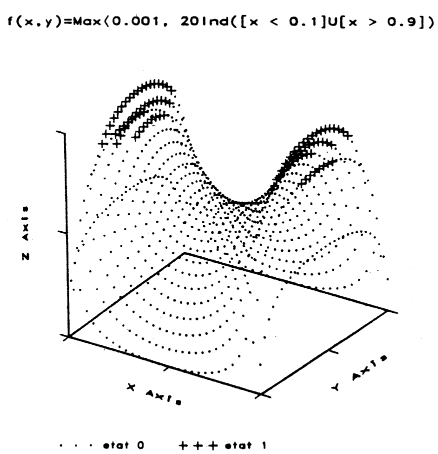
This solution satisfies the optimality conditions, *i.e.* l minimizes J. Nevertheless, l is not a classical solution since $l \in]0, 1[$ in the intervals $[2, 5/2[\cup]7/2, 4]$.

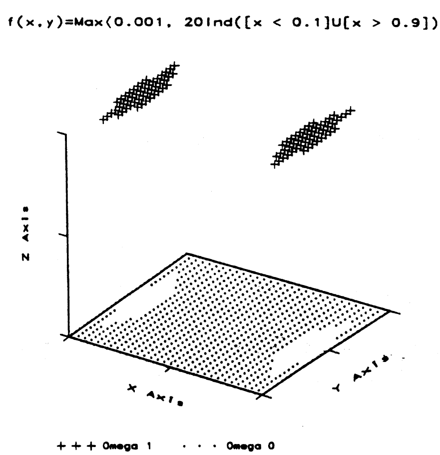
Now looking for a classical solution leads to solve

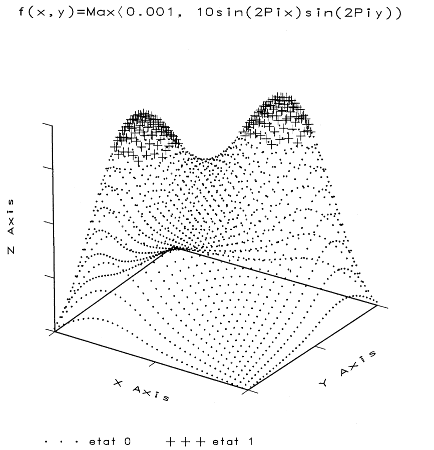
$$\min \{J(\omega),\ \omega =]\alpha, \alpha + 7/4[,\ \alpha \in]0, 6 - 7/4[\}$$

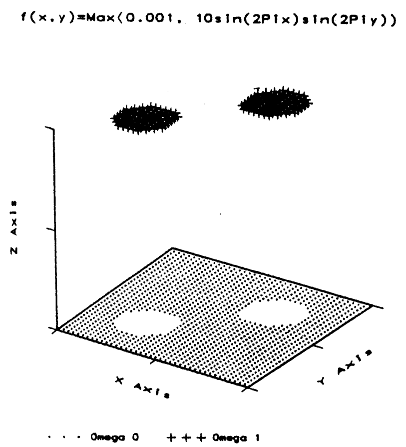
Using the software Maple we obtain $\alpha = 17/8$ and $J(]17/8, 31/8[) = J(l) = 37/12$, *i.e.* the subset $]17/8, 31/8[$ provides an optimal domain. In such a case, our algorithm may converge to the classical solution and ignore the relaxed ones (what could be considered as a good thing!).

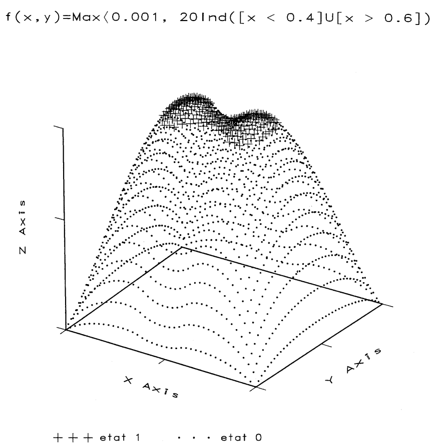
The following figures show some examples of minimizers obtained thanks to this procedure. For each second member f we give the solution of $P(l^*)$ («etat0», «etat1») associated to the optimal domain l^* («Omega0», «Omega1»).

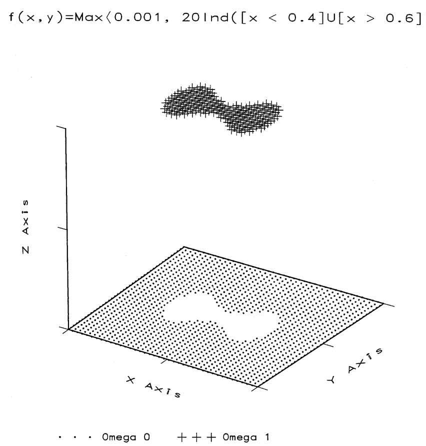


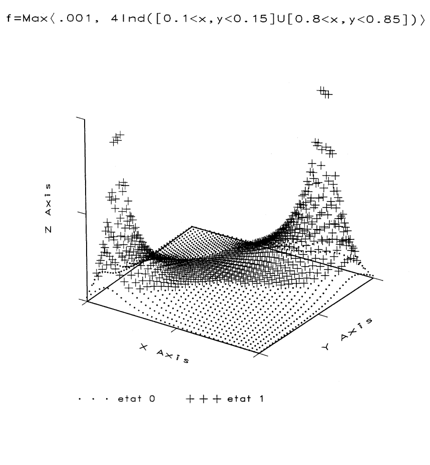


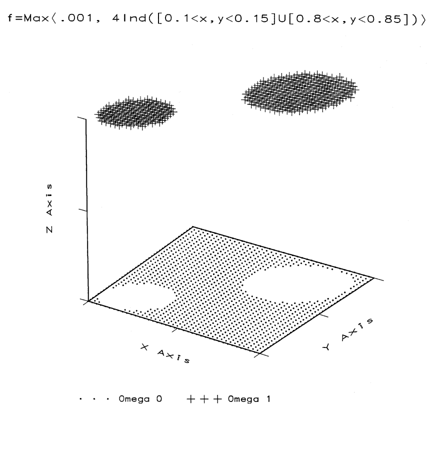

f(x,y)=Max(0.001, 10sin(2Pix)sin(2Piy))
Z Axis
X Axis
Y Axis
· · · etat 0 +++ etat 1


f(x,y)=Max(0.001, 10sin(2Pix)sin(2Piy))
Z Axis
X Axis
Y Axis
· · · Omega 0 +++ Omega 1


f(x,y)=Max(0.001, 20Ind([x < 0.4]U[x > 0.6])
Z Axis
X Axis
Y Axis
+++ etat 1 · · · etat 0


f(x,y)=Max(0.001, 20Ind([x < 0.4]U[x > 0.6])
Z Axis
X Axis
Y Axis
· · · Omega 0 +++ Omega 1


f=Max(.001, 4Ind([0.1<x,y<0.15]U[0.8<x,y<0.85]))
Z Axis
X Axis
Y Axis
· · · etat 0 +++ etat 1


f=Max(.001, 4Ind([0.1<x,y<0.15]U[0.8<x,y<0.85]))
Z Axis
X Axis
Y Axis
· · · Omega 0 +++ Omega 1

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Antoine Henrot: Ecole des Mines and Institut Elie Cartan, UMR CNRS 7502
and INRIA, BP 239, 54506 Vandoeuvre-les-Nancy Cedex, France
E-mail: henrot@iecn.u-nancy.fr

Hervé Maillot: Laboratoire d'Analyse Numérique
Université de Paris 6, 75252 Paris Cedex 05 France
E-mail: maillot@ann.jussieu.fr

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