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On Locally Finite Groups
and the Centralizers of Automorphisms (*).

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1. – Introduction.

Let $G$ be a group admitting an action of a group $A$. We denote by $C_G(A)$ the set $C_G(A) = \{x \in G; x^a = x \text{ for any } a \in A\}$, the centralizer of $A$ in $G$ (the fixed-point group). This paper deals with the situation when $A$ is an elementary abelian $p$-group, and $G$ is a (locally) finite $p'$-group. Let $A^\#$ denote the set of non-identity elements of $A$. Assume that $G$ is finite and that $C_G(a)$ is nilpotent for any $a \in A^\#$. J. N. Ward showed that if $A$ has rank at least 2 then $G$ is metanilpotent [9], and that if $A$ has rank at least 3 then $G$ is nilpotent [10]. Later the author found some extensions of these results to infinite groups [7, 8]. In this paper we obtain a sufficient condition for the group $G$ to be nilpotent of bounded class.

**Theorem A.** – Let $p$ be a prime, $G$ a locally finite $p'$-group acted on by an elementary abelian group $A$ of order $p^2$. Assume that there exists a positive integer $m$ such that $[C_G(a), C_G(b), \ldots, C_G(b)] = 1$ for any $a, b \in A^\#$. Then $G$ is nilpotent and the class of $G$ is bounded by a function depending only on $p$ and $m$.

As an immediate consequence of the above theorem we obtain

**Corollary B.** – Let $p$ be a prime, $G$ a locally finite $p'$-group acted on by an elementary abelian group $A$ of order $p^2$. Assume that there exists a positi-
There exists an integer \( m \) such that \( \langle C_G(a), C_G(b) \rangle \), the subgroup of \( G \) generated by \( C_G(a) \) and \( C_G(b) \), is nilpotent of class at most \( m \) for any \( a, b \in A^\# \). Then \( G \) is nilpotent and the class of \( G \) is bounded by a function depending only on \( p \) and \( m \).

The proof of the above results uses the associated Lie rings. In particular we will need the following proposition which may have some independent interest.

**Proposition C.** Let \( L \) be a Lie ring such that \( L = pL \). Let \( A \) be an elementary abelian group of order \( p^2 \) acting on \( L \) by automorphisms. Assume that there exists a positive integer \( m \) such that \( [C_L(a), C_L(b), \ldots, C_L(b)] = 0 \) for any \( a, b \in A^\# \). Then \( L \) is nilpotent and the class of \( L \) is bounded by a function depending only on \( p \) and \( m \).

2. Preliminaries.

The next lemma is well-known (see [2, 6.2.2, 6.2.4] for the proof).

**Lemma 2.1.** Let \( A \) be a finite \( p \)-group acting on a finite \( p^* \)-group \( G \).

1. If \( N \) is an \( A \)-invariant normal subgroup of \( G \) then \( C_{G/N}(A) = C_G(A)N/N \);

2. If \( A \) is an elementary abelian group of order \( p^2 \) then \( G = \langle C_G(a); a \in A^\# \rangle \).

Similar facts (with basically the same proof) hold for Lie rings.

**Lemma 2.2.** Let \( A \) be a finite \( p \)-group acting on a Lie ring \( L \).

1. If \( N \) is an \( A \)-invariant ideal of \( L \) such that \( pN = N \) then \( C_{L/N}(A) = (C_L(A) + N)/N \);

2. If \( A \) is an elementary abelian group of order \( p^2 \), and if \( pL = L \), then \( L = \sum_{a \in A^\#} C_G(a) \).

A well-known theorem of Kreknin [6] says that if a Lie ring \( L \) admits a fixed-point-free automorphism of finite order \( n \) then \( L \) is soluble and the derived length of \( L \) is bounded by a function of \( n \). We will require the following extension of this result [5].

**Theorem 2.3.** Let a Lie ring \( L \) admit an automorphism \( \phi \) of finite order \( n \) such that \( [L, C_L(\phi), \ldots, C_L(\phi)] = 0 \). Assume that \( nL = L \). Then \( L \) is soluble with derived length at most \( (m + 1)^{n-1} + \log_2 m \).
We will also require a Lie-theoretic analogue of the famous criterion of Ph. Hall for a group to be nilpotent [3]: if \( G \) is a group having a normal subgroup \( N \) such that both \( N \) and \( G/N' \) are nilpotent then \( G \) is nilpotent and the class of \( G \) is bounded in terms of the classes of \( N \) and \( G/N' \). The corresponding Lie-theoretic result was established in [1].

**Theorem 2.4.** – If a Lie ring \( L \) has an ideal \( N \) such that both \( N \) and \( L/N' \) are nilpotent then \( L \) is nilpotent and the class of \( L \) is bounded in terms of the classes of \( N \) and \( L/N' \).

3. – Main results.

Our first goal is to establish Proposition C. It will be convenient to start with the case where \( L \) is metabelian.

**Lemma 3.1.** – Let \( L \) be a metabelian Lie ring such that \( L = pL \). Let \( A \) be an elementary abelian group of order \( p^2 \) acting on \( L \) by automorphisms. Assume that there exists a positive integer \( m \) such that \([C_L(a), C_L(b), \ldots, C_L(b)] = 0\) for any \( a, b \in A \). Then \( L \) is nilpotent and the class of \( L \) is at most \((p + 1)(m + 1)\).

**Proof.** – Let \( A_1, \ldots, A_{p+1} \) be the cyclic subgroups of \( A \), and for \( i = 1, 2, \ldots, p + 1 \) we set \( C_i = C_L(A_i) \). Let \( M \) be the commutator subring of \( L \), \( M_i = C_i \cap M \), and \( N_i = M + C_i \). Lemma 2 tells us that \( M = \sum_j M_j \) and \( L = \sum_j C_j \). We observe that the \( N_i \) are ideals and, since \( L = \sum_j N_j \), it is sufficient to show that each \( N_i \) is nilpotent of class at most \( m + 1 \). Let \( \gamma_k(N_i) \) stand for the \( k \)-th term of the lower central series of \( N_i \). We have

\[
\gamma_{m+2}(N_i) = [N_i, \ldots, N_i] \leq [M, C_i, \ldots, C_i] = \left[ \sum_j M_j, C_i, \ldots, C_i \right] = \sum_j [M_j, C_i, \ldots, C_i] = 0
\]

as \([M_j, C_i, \ldots, C_i] = 0\) for any \( i, j \). The lemma follows.

**Proposition C.** – Let \( L \) be a Lie ring such that \( L = pL \). Let \( A \) be an elementary abelian group of order \( p^2 \) acting on \( L \) by automorphisms. Assume that there exists a positive integer \( m \) such that \([C_L(a),
\[ C_L(b), \ldots, C_L(b) \] = 0 for any \( a, b \in A^\# \). Then \( L \) is nilpotent and the class of \( L \) is bounded by a function depending only on \( p \) and \( m \).

**Proof.** – Let \( C_j \) have the same meaning as in the proof of Lemma 3.1. Since \( L = \sum_j C_j \), it follows that \( [L, \underbrace{C_i, \ldots, C_j}_{m}] = 0 \) for any \( i \). Indeed,

\[
[L, \underbrace{C_i, \ldots, C_i}_{m}] = \left[ \sum_j C_j, \underbrace{C_i, \ldots, C_i}_{m} \right] = \sum_j [C_j, \underbrace{C_i, \ldots, C_i}_{m}] = 0.
\]

Now Theorem 2.3 tells us that \( L \) is soluble and the derived length \( d \) of \( L \) is at most \((m + 1)^{p - 1} + \log_2 m\). We will use induction on \( d \) to show that \( L \) is nilpotent and that the nilpotency class of \( L \) is bounded by a function of \( d, m, p \).

If \( d = 2 \) then \( L \) is metabelian and the required result follows from Lemma 3.1. Assume \( d \geq 3 \) and let \( M \) be the metabelian term of the derived series of \( L \). The inductive hypothesis is that \( L/M' \) is nilpotent and has nilpotency class bounded in terms of \( d, m, p \). By Lemma 3.1 \( M \) is nilpotent of class at most \((p + 1)(m + 1)\). Thus, Theorem 2.4 implies that \( L \) is nilpotent of class bounded by a function of \( d, m, p \).

**Lemma 3.2.** – Assume the hypothesis of Theorem A and let \( G \) be finite. Then \( G \) is nilpotent.

**Proof.** – Assume that \( G \) is a counterexample whose order is as small as possible. Let \( A_1, \ldots, A_{p + 1} \) be the cyclic subgroups of \( A \). For any \( A \)-invariant subgroup \( H \) of \( G \) we let \( H_i \) denote \( C_{H_i}(A_i) \). Since each \( G_i \) is nilpotent, it follows that \( G \) is soluble [11]. Let \( F = F(G) \) be the Fitting subgroup of \( G \). If \( F \) is not abelian \( G/F' \) is nilpotent by the inductive hypothesis and so the Ph. Hall Criterion cited in the paragraph preceding Theorem 2.4 shows that \( G \) is nilpotent, a contradiction. Hence \( F \) is abelian and so, by Lemma 2.1, \( F = \prod_j F_j \).

Since the order of \( G \) is as small as possible, the quotient \( G/F \) is nilpotent. It follows that any subgroup of \( G \) containing \( F \) is subnormal. Since \( F \) is generated by all subnormal nilpotent subgroups, it follows that no subgroup properly containing \( F \) is nilpotent. Hence any such \( A \)-invariant subgroup provides a counterexample to the lemma and, using the minimality of \( |G| \), we conclude
that $G/F$ is abelian and $A$ acts irreducibly on $G/F$. By Lemma 2.1 $G/F$ is generated by the centralizers of $A_i$. These are all $A$-invariant and so some $A_k$ acts on $G/F$ trivially. Lemma 2.1 now shows that $G = FG_k$. Then we have

$$[G, \ldots, G] \leq [F, G_k, \ldots, G_k] = \prod_{j} [F_j, G_k, \ldots, G_k] = \prod_{j} [F_j, G_k, \ldots, G_k] = 1.$$ 

Thus, $G$ is nilpotent. □

Now we are ready to conclude the proof of Theorem A.

**Theorem A.** – Let $p$ be a prime, $G$ a locally finite $p'$-group acted on by an elementary abelian group $A$ of order $p^2$. Assume that there exists a positive integer $m$ such that $[C_G(a), C_G(b), \ldots, C_G(b)] = 1$ for any $a, b \in A^\#$. Then $G$ is nilpotent and the class of $G$ is bounded by a function depending only on $p$ and $m$.

**Proof.** – The usual inverse limit argument along the lines of [4] reduces the theorem to the case where $G$ is finite. So we assume that $G$ is finite and hence, by the previous lemma, nilpotent. The construction associating a Lie ring $L(G)$ with any nilpotent group $G$ is well-known. Let $\gamma_k$ denote the $k$th term of the lower central series of $G$. Set $L_k = \gamma_k/\gamma_{k+1}$ and view $L_k$ as an additive abelian group. Then $L(G) = \bigoplus_k L_k$. If $x \in \gamma_i$, $y \in \gamma_j$ then, for corresponding elements $x\gamma_i+1$, $y\gamma_j+1$ of $L(G)$, we set $[x\gamma_i+1, y\gamma_j+1] = [x, y]_{i+j+1}$. Thus, we obtain a product operation on the set $\cup_k L_k$. This can be uniquely extended by linearity on the additive abelian group $L(G)$ and, equipped with the product, $L(G)$ becomes a Lie ring. The Lie ring has the same nilpotency class as the group from which it was constructed. In our situation the group $A$ acts naturally on each quotient $\gamma_k/\gamma_{k+1}$ and this action extends uniquely to an action by automorphisms on the Lie ring $L(G)$. Lemma 2.1 shows that if $a \in A$ then $C_L(a)$ is the direct sum of the quotients $C_{\gamma_k}(a)\gamma_{k+1}/\gamma_{k+1}$ and, since $[C_G(a), C_G(b), \ldots, C_G(b)] = 1$ for any $a, b \in A^\#$, it follows that $[C_L(a), C_L(b), \ldots, C_L(b)] = 0$. Finally, we note that $L(G)$ is finite and has the same order as $G$. Therefore $pL(G) = L(G)$ and, by Proposition C, the nilpotency class of $L(G)$ (the same as of $G$) is bounded by a function depending only on $p$ and $m$.  


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