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Homogenization of Some Nonlinear Problems with Specific Dependence Upon Coordinates.

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Sunto. – *Questo articolo considera una successione di equazioni differenziali a derivate parziali non lineari in forma di divergenza del tipo*

$$-\operatorname{div}(Q^\varepsilon G(x, N^\varepsilon \nabla u)) = f^\varepsilon,$$

in un dominio limitato Ω dello spazio n-dimensionale; $Q^\varepsilon = Q^\varepsilon(x)$ e $N^\varepsilon = N^\varepsilon(x)$ sono matrici con coefficenti limitati, N^ε è invertibile e la sua matrice inversa R^ε ha anche coefficenti limitati. La non linearità è dovuta alla funzione $G = G(x, \xi)$; la condizione di crescita, la monotonicità e le ipotesi di coercitività sono modellate sul p-Laplaciano, $1 < p < \infty$, ed assicurano l'esistenza di una soluzione $u^\varepsilon \in W_0^{1,p}(\Omega)$ di ciascuna equazione, per ogni fissata $f^\varepsilon \in W^{-1,p'}(\Omega)$. Si ipotizza una dipendenza specifica della matrice dei coefficenti dalle coordinate: $Q^\varepsilon(x) = (q_{i,j}^\varepsilon(x_i'))$ e $R^\varepsilon(x) = (r_{i,j}^\varepsilon(x_i))$, dove il punto arbitrario di Ω è denominato $x = (x_i, x_i')$, con x_i reale e x_i' nello spazio $(n-1)$-dimensionale. Essenzialmente il risultato principale è il seguente. Supponiamo la seguente convergenza: per i coefficenti, $Q^\varepsilon \rightharpoonup Q$, $R^\varepsilon \rightharpoonup R$, rispetto alla topologia debole; per i termini di sorgente, $f^\varepsilon \to f$, rispetto alla topologia forte di $W_0^{-1,p'}(\Omega)$; e per le soluzioni $u^\varepsilon \rightharpoonup u$, rispetto alla topologia debole di $W_0^{1,p}(\Omega)$; allora u è soluzione dell'equazione limite*

$$-\operatorname{div}(QG(x, N\nabla u)) = f.$$

Si dimostra anche un risultato di tipo correttore e vengono date applicazioni del risultato ottenuto.

1. – Introduction.

We consider nonlinear monotone equations such as

$$\begin{cases} -\operatorname{div}(Q^\varepsilon G(x, N^\varepsilon \nabla u^\varepsilon)) = f^\varepsilon, \\ u^\varepsilon \in W_0^{1,p}(\Omega), \end{cases} \qquad (E^\varepsilon)$$

where Ω is a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$, $W_0^{1,p}(\Omega)$ is the usual Sobolev space, f^ε belongs to its dual $W^{-1,p'}(\Omega)$, where p' is the conjugate of p. (Here and in the following, we write Q^ε and N^ε instead of $Q^\varepsilon(x)$ and $N^\varepsilon(x)$.)

The function $G : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ is a Carathéodory function, that is $G(x, \xi)$ is measurable with respect to x and continuous with respect to ξ. We assume the existence of $\beta > 0$ and g in $L^{p'}(\Omega)$ such that

$$\text{(1.1)} \qquad \text{a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n, \ \|G(x, \xi)\| \leqslant \beta \|\xi\|^{p-1} + g(x).$$

(For convenience, we denote $\|\xi\|$ the euclidian norm of ξ in $\mathbb{R}^n$ or $\mathbb{R}^m$, independently of the dimension.)

The matrices Q^ε and N^ε have L^∞-coefficients, N^ε is invertible and $R^\varepsilon = (N^\varepsilon)^{-1}$ also has L^∞-coefficients. Moreover we assume that Q^ε and R^ε have the following specific dependence upon coordinates:

$$Q^\varepsilon : \qquad \Omega \to \mathbb{R}^{n \times m}$$
$$Q^\varepsilon(x) = \left(q_{ij}^\varepsilon(x_i')\right)\begin{cases} i = 1, \dots, n, \\ j = 1, \dots, m, \end{cases}$$

$$N^\varepsilon : \qquad \Omega \to \mathbb{R}^{n \times n}$$
$$R^\varepsilon(x) = (N^\varepsilon(x))^{-1} = \left(r_{ij}^\varepsilon(x_i)\right)\begin{cases} i = 1, \dots, n, \\ j = 1, \dots, n, \end{cases}$$

where the generic point in Ω is denoted by $x = (x_i, x_i')$, $x_i \in \mathbb{R}$, $x_i' \in \mathbb{R}^{n-1}$. We suppose that the matrices Q^ε, N^ε and the function G are related by the two following conditions:

$$\text{(1.2)} \quad \text{a.e. } x \in \Omega, \ \forall \xi, \ \eta \in \mathbb{R}^n, \ (Q^\varepsilon G(x, N^\varepsilon \xi) - Q^\varepsilon G(x, N^\varepsilon \eta), \xi - \eta) \geqslant 0$$

(here (.,.) denotes the scalar product in $\mathbb{R}^n$ but we shall use the same notation for the scalar product in $\mathbb{R}^m$),

$$\text{(1.3)} \qquad \begin{aligned} &\exists \alpha > 0, \ \exists h \in L^1(\Omega), \text{a.e. } x \in \Omega, \ \forall \xi \in \mathbb{R}^n, \\ &(Q^\varepsilon G(x, N^\varepsilon \xi), \xi) \geqslant \alpha \|\xi\|^p - h(x). \end{aligned}$$

(Examples will be given in Section 5.)

The above assumptions imply that for every ε, the equation (E^ε) admits one solution at least in $W_0^{1,p}(\Omega)$ denoted u^ε. The aim of this paper is to prove, under natural convergence assumptions on f^ε, Q^ε and R^ε, that u^ε converges weakly (up to extraction of a subsequence) to a solution of a similar equation (E). We also give a corrector type result which asserts that $\nabla u^\varepsilon - T^\varepsilon \nabla u$ converges strongly to zero in $L^p(\Omega)$ for a proper matrix T^ε. We indicate several applications, among which we consider the case of Q^ε being the cofactor matrix of R^ε. Finally we study the example of the p-Laplacian.

This paper was announced in a short note [1]. It is also the natural follow up of [6], which deals with the linear case, and of [2] and [4], which concern some particular cases of nonlinearities.

2. – Existence of solutions of (E^{ε}).

THEOREM 2.1. – *Under the previous assumptions,* (E^{ε}) *has (at least) one solution* u^{ε}.

PROOF. – For convenience, we forget the subscript ε, which is fixed here, we do not write explicitly the dependence in x and we denote by C any constant. Let $\mathcal{A}: W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be defined by:

$$\mathcal{A}u = -\operatorname{div}(QG(N\nabla u)),$$

or

$$\langle \mathcal{A}u, v\rangle = \int_{\Omega} (QG(N\nabla u), \nabla v)\, dx .$$

We shall prove that for all f in $W^{-1,p'}(\Omega)$, there exists u in $W_0^{1,p}(\Omega)$ such that $\langle \mathcal{A}u, v\rangle = \langle f, v\rangle$, for all v in $W_0^{1,p}(\Omega)$. This is a simple application of the theory of monotone operators. We have to prove that:

1) $\mathcal{A}$ is bounded and continuous from $W_0^{1,p}(\Omega)$ (with strong topology) to weak-$W^{-1,p'}(\Omega)$,

2) $\displaystyle\lim_{\|v\| \to +\infty} \frac{\langle \mathcal{A}v, v\rangle}{\|v\|_{W_0^{1,p}(\Omega)}} = +\infty$,

3) For all u, v in $W_0^{1,p}(\Omega)$, $\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \geqslant 0$.

Proof of 1)

We have

$$|\langle \mathcal{A}u, v\rangle| = \left|\int_{\Omega} (QG(N\nabla u), \nabla v)\, dx\right|$$
$$\leqslant \int_{\Omega} \|QG(N\nabla u)\|\,\|\nabla v\|\, dx .$$

Using inequality (1.1), since the coefficients of Q and N are bounded,

$$|\langle \mathcal{A}u, v\rangle| \leqslant C\int_{\Omega} (\|\nabla u\|^{p-1} + g)\|\nabla v\|\, dx .$$

Applying Hölder inequality yields

$$|\langle \mathcal{A}u, v\rangle| \leqslant C(\|\nabla u\|^{p-1}_{L^p(\Omega)^n} + \|g\|_{L^{p'}(\Omega)})\|\nabla v\|_{L^p(\Omega)^n}$$
$$\leqslant C(\|u\|^{p-1}_{W_0^{1,p}(\Omega)} + \|g\|_{L^{p'}(\Omega)})\|v\|_{W_0^{1,p}(\Omega)}.$$

Hence $\mathcal{A}u$ belongs to $W^{-1,p'}(\Omega)$ and

$$\|\mathcal{A}u\|_{W^{-1,p'}(\Omega)} \leq C(\|u\|^{p-1}_{W_0^{1,p}(\Omega)} + \|g\|_{L^{p'}(\Omega)}),$$

$\mathcal{A}$ is bounded.

Let us check that $\mathcal{A}$ is continuous from strong-$W_0^{1,p}(\Omega)$ into weak-$W^{-1,p'}(\Omega)$. We have to prove that if u_k tends to u strongly in $W_0^{1,p}(\Omega)$, then $\langle \mathcal{A}u_k, v\rangle$ tends to $\langle \mathcal{A}u, v\rangle$, for any v in $W_0^{1,p}(\Omega)$. Actually

$$\langle \mathcal{A}u_k, v\rangle = \int_\Omega (QG(N\nabla u_k), \nabla v)\, dx\,,$$

$N\nabla u_k$ tends to $N\nabla u$ strongly in $L^p(\Omega)^n$, when k goes to infinity. Since (1.1) is satisfied by the Carathéodory function G, the map $U \to G(.\,, U(.))$ is continuous from $L^p(\Omega)^n$ to $L^{p'}(\Omega)^m$. (This is a classical consequence of Lebesgue's theorem.) Thus $QG(N\nabla u_k)$ tends to $QG(N\nabla u)$ strongly in $L^{p'}(\Omega)^m$ and

$$\lim_{k\to\infty} \langle \mathcal{A}u_k, v\rangle = \langle \mathcal{A}u, v\rangle.$$

Proof of 2)

Because of (1.3),

$$\begin{aligned}\langle \mathcal{A}v, v\rangle &= \int_\Omega (QG(N\nabla v), \nabla v)dx\\ &\geq \int_\Omega (\alpha\|\nabla v\|^p - h)\, dx\\ &\geq \alpha\|v\|^p_{W_0^{1,p}(\Omega)} - \int_\Omega h dx\,,\end{aligned}$$

which proves 2).

Proof of 3)

It is obvious by making $\xi = \nabla u$ and $\eta = \nabla v$ in (1.2).

3. – Limit problem of (E^ε).

THEOREM 3.1. – *We now make the following additional hypotheses*:

$$f^\varepsilon \to f \text{ strongly in } W^{-1,p'}(\Omega), \tag{3.1}$$

$$\begin{cases} Q^\varepsilon \rightharpoonup Q \text{ weakly}\star \text{ in } L^\infty(\Omega)^{n\times m}, \\ R^\varepsilon = (N^\varepsilon)^{-1} \rightharpoonup R \text{ weakly}\star \text{ in } L^\infty(\Omega)^{n\times n}, \end{cases} \tag{3.2}$$

(3.3) R *is invertible with* $N = R^{-1} \in L^{\infty}(\Omega)^{n \times n}$,

(3.4) $\{N^{\varepsilon}\}_{\varepsilon}$ *is bounded in* $L^{\infty}(\Omega)^{n \times n}$;

then any sequence of solutions u^{ε} *of* (E^{ε}) *is bounded in* $W_0^{1,p}(\Omega)$. *Moreover if* $u^{\varepsilon'}$ *tends to* u *weakly in* $W_0^{1,p}(\Omega)$ *for some subsequence* ε' *of* ε, *then* u *is a solution of*

$$\begin{cases} -\operatorname{div}(QG(x, N\nabla u)) = f, \\ u \in W_0^{1,p}(\Omega). \end{cases} \qquad (E)$$

REMARK. – Because of the specific dependence on the coordinates that we assume, it follows that (see [6]):

$$S^{\varepsilon} = {}^t Q^{\varepsilon} R^{\varepsilon} \rightharpoonup S = {}^t QR \text{ weakly}\star \text{ in } L^{\infty}(\Omega)^{m \times n}.$$

As an immediate consequence of (1.2) and the above, one can take the limit in

$$\forall \varphi \in \mathcal{D}(\Omega),\ \varphi \geq 0, \int_{\Omega} (G(x, \xi) - G(x, \eta), {}^t Q^{\varepsilon}(x) R^{\varepsilon}(x)(\xi - \eta))\, \varphi(x)\, dx \geq 0,$$

so that (1.2) is satisfied with Q^{ε}, N^{ε} replaced by Q, N.

Moreover (1.3) can be written as

$$\begin{gathered} \exists \alpha > 0,\ \exists h \in L^1(\Omega), \text{a.e. } x \in \Omega,\ \forall \eta \in \mathbb{R}^n, \\ (G(x, \eta), S^{\varepsilon}(x)\, \eta) \geq \alpha \|R^{\varepsilon}(x)\, \eta\|^p - h(x), \end{gathered}$$

and we can take the limit in the same way as above. Thus Q and N satisfy (1.3).

It follows that the limit equation (E) verifies the same conditions as the equations (E^{ε}) and (E) has at least one solution.

PROOF OF THEOREM 3.1. – We have (see the end of Section 2)

$$\begin{aligned} \alpha \|u^{\varepsilon}\|^p_{W_0^{1,p}(\Omega)} - \int_{\Omega} h dx &\leq \langle \mathcal{A}^{\varepsilon} u^{\varepsilon}, u^{\varepsilon} \rangle \\ &= \langle f^{\varepsilon}, u^{\varepsilon} \rangle \\ &\leq C \|u^{\varepsilon}\|_{W_0^{1,p}(\Omega)}, \end{aligned}$$

by (3.1), which implies that u^{ε} is bounded in $W_0^{1,p}(\Omega)$.

It remains to prove that if $u^{\varepsilon'}$ tends to u weakly in $W_0^{1,p}(\Omega)$, then u is a solution of (E).

By compactness $u^{\varepsilon'}$ tends to u strongly in $L^p(\Omega)$. Let us write ε instead of

ε'. Let $v \in W_0^{1,p}(\Omega)$. Making $\xi = \nabla u^\varepsilon$ and $\eta = R^\varepsilon N \nabla v$ in (1.2), we obtain

$$\begin{aligned} 0 &\leqslant F^\varepsilon(v) \\ &= \int_\Omega (Q^\varepsilon G(N^\varepsilon \nabla u^\varepsilon) - Q^\varepsilon G(N\nabla v), \nabla u^\varepsilon - R^\varepsilon N \nabla v)\, dx \\ &= A - B - C + D \end{aligned}$$

where

$$\begin{aligned} A &= \int_\Omega (Q^\varepsilon G(N^\varepsilon \nabla u^\varepsilon), \nabla u^\varepsilon)\, dx\,, \\ B &= \int_\Omega (Q^\varepsilon G(N^\varepsilon \nabla u^\varepsilon), R^\varepsilon N \nabla v)\, dx, \\ C &= \int_\Omega (Q^\varepsilon G(N \nabla v), \nabla u^\varepsilon)\, dx\,, \\ D &= \int_\Omega (Q^\varepsilon G(N \nabla v), R^\varepsilon N \nabla v)\, dx\,. \end{aligned}$$

We shall take the limit in each term separately.

FIRST TERM A:

$$A = \langle \mathcal{A}^\varepsilon u^\varepsilon, u^\varepsilon \rangle = \langle f^\varepsilon, u^\varepsilon \rangle \underset{\varepsilon \to 0}{\longrightarrow} \langle f, u \rangle$$

by (3.1) and the weak convergence of u^ε in $W_0^{1,p}(\Omega)$.

THIRD TERM C. – Let $P^\varepsilon = {}^t Q^\varepsilon$, $P = {}^t Q$. Since $G(N\nabla v)$ belongs to $L^{p'}(\Omega)^m$, it suffices to show that $P^\varepsilon \nabla u^\varepsilon$ tends to $P \nabla u$ weakly in $L^p(\Omega)^m$ or that $P^\varepsilon \nabla u^\varepsilon$ tends to $P\nabla u$ in $\mathcal{D}'(\Omega)^m$ (since $P^\varepsilon \nabla u^\varepsilon$ is bounded in $L^p(\Omega)^m$). Note that

$$\begin{aligned} (P^\varepsilon \nabla u^\varepsilon)_i &= \sum_j p_{ij}^\varepsilon(x_j') \frac{\partial u^\varepsilon}{\partial x_j} \\ &= \sum_j \frac{\partial}{\partial x_j} (p_{ij}^\varepsilon(x_j')\, u^\varepsilon)\,. \end{aligned}$$

We already know that u^ε tends to u strongly in $L^p(\Omega)$ and p_{ij}^ε tends to p_{ij} weakly$\star$ in $L^\infty(\Omega)$, thus $p_{ij}^\varepsilon u^\varepsilon$ tends to $p_{ij} u$ in $\mathcal{D}'(\Omega)$. It follows that

$$\begin{aligned} C &= \int_\Omega (G(N\nabla v), P^\varepsilon \nabla u^\varepsilon)\, dx \\ &\underset{\varepsilon \to 0}{\longrightarrow} \int_\Omega (G(N \nabla v), P \nabla u)\, dx = \int_\Omega (QG(N\nabla v), \nabla u)\, dx\,. \end{aligned}$$

FOURTH TERM D. – Recall that $S^{\varepsilon} = P^{\varepsilon} R^{\varepsilon} \rightharpoonup S = PR$ weakly$\star$ in $L^{\infty}(\Omega)^{m \times n}$. Then $P^{\varepsilon}\, R^{\varepsilon}\, N\nabla v$ tends to $P\nabla v$ weakly in $L^{p}(\Omega)^{m}$, so that

$$D = \int_{\Omega} (G(N\nabla v),\, P^{\varepsilon}\, R^{\varepsilon}\, N\nabla v)\, dx$$

$$\xrightarrow[\varepsilon \to 0]{} \int_{\Omega} (G(N\nabla v),\, P\nabla v)\, dx = \int_{\Omega} (QG(N\nabla v),\, \nabla v)\, dx\,.$$

SECOND TERM B. – Let $\sigma^{\varepsilon} = Q^{\varepsilon} G(N^{\varepsilon} \nabla u^{\varepsilon})$. The sequence $\{\sigma_{\varepsilon}\}_{\varepsilon}$ is bounded in $L^{p'}(\Omega)^{n}$, thus there exits σ in $L^{p'}(\Omega)^{n}$ such that (up to a subsequence) σ^{ε} tends to σ weakly in $L^{p'}(\Omega)^{n}$. Furthermore $-\operatorname{div} \sigma^{\varepsilon} = f^{\varepsilon}$ in $\mathcal{D}'(\Omega)$ passes to the limit, so that $-\operatorname{div} \sigma = f$. Let $M^{\varepsilon} = {}^{t}R^{\varepsilon}$, $M = {}^{t}R$, we shall prove that

$$M^{\varepsilon} \sigma^{\varepsilon} \xrightarrow[\varepsilon \to 0]{} M\sigma \ \text{ in } \ \mathcal{D}'(\Omega)^{n}. \tag{3.5}$$

Admitting (3.5) for a while, as $M^{\varepsilon} \sigma^{\varepsilon}$ is bounded in $L^{p'}(\Omega)^{n}$, we deduce that $M^{\varepsilon} \sigma^{\varepsilon}$ tends to $M\sigma$ weakly in $L^{p'}(\Omega)^{n}$, so that

$$B = \int_{\Omega} (\sigma^{\varepsilon},\, R^{\varepsilon} N\nabla v)\, dx = \int_{\Omega} (M^{\varepsilon} \sigma^{\varepsilon},\, N\nabla v)\, dx$$

$$\xrightarrow[\varepsilon \to 0]{} \int_{\Omega} (M\sigma,\, N\nabla v)\, dx = \int_{\Omega} (\sigma,\, \nabla v)\, dx = \langle f,\, v \rangle.$$

It remains to prove (3.5). Let $\mathcal{C} = \prod_{j=1}^{n}]\underline{x}_j, \overline{x}_j[$ be the smallest cube containing Ω. Clearly M^{ε} and M are defined in $\mathcal{C}$; moreover M^{ε} tends to M weakly$\star$ in $L^{\infty}(\mathcal{C})^{n \times n}$. Now let us consider

$$v_{ij}^{\varepsilon}(x) = \int_{\underline{x}_j}^{x_j} m_{ij}^{\varepsilon}(t)\, dt\,, \qquad v_{ij}(x) = \int_{\underline{x}_j}^{x_j} m_{ij}(t)\, dt\,.$$

In Ω, $\partial v_{ij}^{\varepsilon}/\partial x_k = m_{ij}^{\varepsilon}$ if $k = j$ and $\partial v_{ij}^{\varepsilon}/\partial x_k = 0$ otherwise. The same result holds for v_{ij}. Moreover it is clear that v_{ij}^{ε} tends to v_{ij} weakly in $W^{1,p}(\Omega)$. As σ^{ε} tends to σ weakly in $L^{p'}(\Omega)^{n}$ and $\operatorname{div} \sigma^{\varepsilon} = -f^{\varepsilon}$ tends to $\operatorname{div} \sigma = -f$ strongly in $W^{-1,p'}(\Omega)$ by assumption, then by compensated compactness (see [7]),

$$m_{ij}^{\varepsilon} \sigma_j^{\varepsilon} = (\nabla v_{ij}^{\varepsilon},\, \sigma^{\varepsilon}) \xrightarrow[\varepsilon \to 0]{} (\nabla v_{ij},\, \sigma) = m_{ij} \sigma_j \ \text{ in } \ \mathcal{D}'(\Omega)\,.$$

Summing over j, we obtain (3.5), component by component.

END OF PROOF. – By taking the limit in $F^{\varepsilon}(v) \geqslant 0$, we get

$$F(v) = \int_{\Omega} (QG(N\nabla v),\, \nabla v - \nabla u)\, dx - \langle f,\, v - u \rangle \geqslant 0\,.$$

A standard argument shows that

$$\int_\Omega (QG(N\nabla u), \nabla w)\, dx = \langle f, w\rangle ,$$

for all w in $W_0^{1,p}(\Omega)$ so that u is a solution of (E).

4. – Correctors.

As already noticed, (1.2) can be written

(4.1) a.e. $x \in \Omega$, $\forall \xi, \eta \in \mathbb{R}^n$, $(Q^\varepsilon G(x, \xi) - Q^\varepsilon G(x, \eta), R^\varepsilon \xi - R^\varepsilon \eta) \geqslant 0$.

We are going to prove that the result of the previous section can be improved, under a stronger assumption.

THEOREM 4.1. – *We now replace* (1.2) *by the stronger condition*

$$\begin{aligned} &\exists C > 0, \text{ a.e. } x \in \Omega, \ \forall \xi, \eta \in \mathbb{R}^n, \\ &\quad (Q^\varepsilon(x)\, G(x, \xi) - Q^\varepsilon(x)\, G(x, \eta), R^\varepsilon(x)\, \xi - R^\varepsilon(x)\, \eta) \\ &\qquad \geqslant \begin{cases} C\|\xi - \eta\|^p & \text{if } p \geqslant 2\,, \\ C\dfrac{\|\xi - \eta\|^2}{(\|\xi\| + \|\eta\|)^{2-p}} & \text{if } 1 < p < 2\,. \end{cases} \end{aligned} \tag{4.2}$$

Then (E^ε) *and* (E) *have unique solutions* u^ε *and* u *respectively,* u^ε *tends to* u *weakly in* $W_0^{1,p}(\Omega)$ (*for the whole sequence*). *Moreover* $v^\varepsilon = \nabla u^\varepsilon - R^\varepsilon N \nabla u$ *tends to zero strongly in* $L^p(\Omega)^n$.

PROOF. – Assumption (4.2) implies that $\langle \mathcal{A}^\varepsilon u - \mathcal{A}^\varepsilon v, u - v\rangle$ is positive if $u \neq v$, thus (E^ε) has at most one solution. As already done for (1.2), one can take the limit in (4.2) and obtain the uniqueness of solution of (E) in the same way.

It remains to prove that v^ε tends to zero in $L^p(\Omega)^n$. This will be obtained from the convergence $F^\varepsilon(u) \xrightarrow[\varepsilon \to 0]{} F(u) = 0$ that has been proved in the previous section. We consider separately the two cases $p \geqslant 2$ and $1 < p < 2$.

Case $p \geqslant 2$: Using (4.2), we have

$$\begin{aligned} F^\varepsilon(u) &= \int_\Omega (Q^\varepsilon G(N^\varepsilon \nabla u^\varepsilon) - Q^\varepsilon G(N\nabla u), \nabla u^\varepsilon - R^\varepsilon N \nabla u)\, dx \\ &\geqslant C \int_\Omega \|N^\varepsilon \nabla u^\varepsilon - N\nabla u\|^p\, dx \\ &= C\int_\Omega \|N^\varepsilon v^\varepsilon\|^p\, dx\,. \end{aligned}$$

Since $(N^\varepsilon)^{-1} = R^\varepsilon$ is bounded in $L^\infty(\Omega)^{n\times n}$, we deduce that $F^\varepsilon(u) \geqslant C\|v^\varepsilon\|^p_{L^p(\Omega)^n}$, which ends the proof in this case.

Case $1<p<2$: As above, it is sufficient to show that $N^\varepsilon\nabla u^\varepsilon - N\nabla u$ tends to zero when $\varepsilon\to 0$, strongly in $L^p(\Omega)^n$. Let $w^\varepsilon = N^\varepsilon\nabla u^\varepsilon$. Note that

$$\int_\Omega \|w^\varepsilon - N\nabla u\|^p\,dx = \int_\Omega A^\varepsilon B^\varepsilon\,dx\,,$$

where

$$A^\varepsilon = \frac{\|w^\varepsilon - N\nabla u\|^p}{(\|w^\varepsilon\| + \|N\nabla u\|)^{(2-p)\,p/2}}\,,$$

$$B^\varepsilon = (\|w^\varepsilon\| + \|N\nabla u\|)^{(2-p)\,p/2}.$$

By Hölder inequality,

$$\int_\Omega \|w^\varepsilon - N\nabla u\|^p\,dx \leqslant \Big(\int_\Omega (A^\varepsilon)^{2/p}\,dx\Big)^{p/2}\Big(\int_\Omega (B^\varepsilon)^{2/(2-p)}\,dx\Big)^{(2-p)/2}.$$

We shall prove that

1) $\|A^\varepsilon\|_{L^{2/p}(\Omega)}$ tends to zero when $\varepsilon\to 0$,

2) B^ε is bounded in $L^{2/(2-p)}(\Omega)$.

1) Using inequality (4.2),

$$\int_\Omega (A^\varepsilon)^{2/p}\,dx = \int_\Omega \frac{\|w^\varepsilon - N\nabla u\|^2}{(\|w^\varepsilon\| + \|N\nabla u\|)^{2-p}}\,dx \leqslant \frac{1}{C}F^\varepsilon(u)\,,$$

which tends to zero.

2) Since ∇u^ε is bounded in $L^p(\Omega)^n$ and since N^ε is bounded in $L^\infty(\Omega)^{n\times n}$, w^ε is bounded in $L^p(\Omega)^n$ and

$$\int_\Omega (B^\varepsilon)^{2/(p-2)}\,dx = \int_\Omega (\|w^\varepsilon\| + \|N\nabla u\|)^p\,dx$$

is bounded, which ends the proof.

5. – Applications.

5.1. – *The linear case.*

Taking $n=m$, $G(x,\xi)\equiv\xi$, $p=2$ and $f^\varepsilon\equiv f$, equation (E^ε) becomes

$$\begin{cases} -\operatorname{div}(Q^\varepsilon N^\varepsilon\nabla u^\varepsilon) = f^\varepsilon\,, \\ u^\varepsilon\in H^1_0(\Omega) \end{cases}$$

and Theorem 3.1 implies that $Q^\varepsilon N^\varepsilon$ H-converges to QN. It follows from the general theory of H-convergence (see [7]) that ${}^t(Q^\varepsilon N^\varepsilon) = {}^t(N^\varepsilon){}^t(Q^\varepsilon)$ H-converges to ${}^tN\,{}^tQ$: we recover the linear case studied (under slightly different assumptions) by S. Fabre and J. Mossino in [6].

5.2. – *The case of diagonal matrices.*

Let us assume that Q^ε and N^ε are diagonal, then

$$n = m, \quad N^\varepsilon = \mathrm{diag}\,(n_i^\varepsilon(x_i)), \quad Q^\varepsilon = \mathrm{diag}\,(q_i^\varepsilon(x_i'))\,,$$

and let us assume that n_i^ε and q_i^ε satisfy

$$\underline{n} \leqslant n_i^\varepsilon(x_i) \leqslant \overline{n}\,, \quad \underline{q} \leqslant q_i^\varepsilon(x_i') \leqslant \overline{q}\,,$$

for some positive numbers $\underline{n}, \overline{n}, \underline{q}, \overline{q}$.

Let $G(x, \xi) = (G_i(\xi_i))_{1 \leqslant i \leqslant n}$ be continuous monotone nondecreasing functions. We assume that there exist positive constants α and β such that for any real t,

$$|G_i(t)| \leqslant \beta |t|^{p-1}, \quad G_i(t)\, t \geqslant \alpha |t|^p.$$

Condition (1.2) is satisfied since

$$(Q^\varepsilon G(x, \xi) - Q^\varepsilon G(x, \eta), R^\varepsilon \xi - R^\varepsilon \eta) = \sum_{i=1}^n \frac{q_i^\varepsilon}{n_i^\varepsilon}(G_i(\xi_i) - G_i(\eta_i))(\xi_i - \eta_i) \geqslant 0\,.$$

Condition (1.3) also holds since

$$(Q^\varepsilon G(x, N^\varepsilon \xi), \xi) = \sum_{i=1}^n \frac{q_i^\varepsilon(x_i')}{n_i^\varepsilon(x_i)} G_i(n_i^\varepsilon(x_i)\,\xi_i)\, n_i^\varepsilon(x_i)\,\xi_i$$

$$\geqslant \alpha \underline{q}\,\underline{n}^{p-1} \sum_{i=1}^n |\xi_i|^p.$$

We recover the result of R. Dufour in [2]: the limit equation of

$$\begin{cases} -\displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}(q_i^\varepsilon(x_i')\, G_i(n_i^\varepsilon(x_i)\, \nabla u^\varepsilon)) = f^\varepsilon, \\ u^\varepsilon \in W_0^{1,p}(\Omega) \end{cases} \qquad (E^\varepsilon)$$

is the equation (E) obtained by deleting ε, as soon as $f^\varepsilon \to f$ strongly in $W^{-1,p'}(\Omega)$ and for any i, $\dfrac{1}{n_i^\varepsilon} \rightharpoonup \dfrac{1}{n_i}$, $q_i^\varepsilon \rightharpoonup q_i$ in weak*-$L^\infty(\Omega)$. Moreover $\dfrac{\partial u^\varepsilon}{\partial x_i} - \dfrac{n_i}{n_i^\varepsilon}\dfrac{\partial u}{\partial x_i}$ tends to zero strongly in $L^p(\Omega)$ for any i, as soon as, for any i,

G_i satisfies the strong monotonicity condition

$$(G_i(t)-G_i(t'))(t-t') \geqslant \begin{cases} C|t-t'|^p & \text{if } p \geqslant 2\,, \\ C\dfrac{|t-t'|^2}{(|t|+|t'|)^{2-p}} & \text{if } 1<p<2\,. \end{cases}$$

5.3. – *The cofactor matrix case.*

Consider a sequence of matrices $R^\varepsilon = (r_{ij}^\varepsilon(x_i))$ bounded in $L^\infty(\Omega)^{n\times n}$ such that $\operatorname{Det} R^\varepsilon(x) \geqslant \delta$, for some positive number δ which does not depend on ε and x. Then let Q^ε be the cofactor matrix of R^ε:

$$m=n\,,\qquad Q^\varepsilon = \operatorname{Cof} R^\varepsilon = \frac{1}{\operatorname{Det} N^\varepsilon}\,{}^t(N^\varepsilon) = (\operatorname{Det} R^\varepsilon)^t(N^\varepsilon)\,.$$

It is easy to check that $q_{ij}^\varepsilon(x) = q_{ij}^\varepsilon(x_i')$ and that Q^ε and N^ε are bounded in $L^\infty(\Omega)^{n\times n}$.

We assume that G satisfies (1.1) as before and that it is monotone and coercive in the following sense

$$\text{a.e. } x\in\Omega,\ \forall\xi,\ \eta\in\mathbb{R}^n,\ (G(x,\xi)-G(x,\eta),\ \xi-\eta)\geqslant 0\,, \tag{5.1}$$

$$\text{a.e. } x\in\Omega,\ \forall\xi\in\mathbb{R}^n,\ (G(x,\xi),\xi)\geqslant\gamma\|\xi\|^p. \tag{5.2}$$

Then clearly (1.2) and (1.3) hold.

Hence we can apply Theorem 3.1 and get the following result.

THEOREM 5.1. – *Assume that* $R^\varepsilon = (r_{ij}^\varepsilon(x_i))$ *and that* $\operatorname{Det}(R^\varepsilon(x))\geqslant\delta$ *for some positive* δ. *Assume* (1.1), (5.1), (5.2) *and that when* ε *tends to zero,*

$$f^\varepsilon\to f \text{ strongly in } W^{-1,p'}(\Omega),\qquad R^\varepsilon\rightharpoonup R \text{ weakly-}\star \text{ in } L^\infty(\Omega)^{n\times n}.$$

Then $Q^\varepsilon = \operatorname{Cof} R^\varepsilon \rightharpoonup Q = \operatorname{Cof} R$ *weakly-*$\star$ *in* $L^\infty(\Omega)^{n\times n}$. *Moreover, up to a subsequence, any sequence of solutions* u^ε *of*

$$\begin{cases} -\operatorname{div}(\operatorname{Cof} R^\varepsilon(x)\,G(x,(R^\varepsilon(x))^{-1}\nabla u^\varepsilon)) = f^\varepsilon, \\ u^\varepsilon\in W_0^{1,p}(\Omega) \end{cases} \tag{E^ε}$$

converges to a solution of

$$\begin{cases} -\operatorname{div}(\operatorname{Cof} R(x)\,G(x,(R(x))^{-1}\nabla u)) = f\,, \\ u\in W_0^{1,p}(\Omega)\,. \end{cases} \tag{E}$$

PROOF. – The second assertion being a direct corollary of Theorem 3.1, let us check the first one. It is sufficient to note that each coefficient q_{ij}^ε of Q^ε is a sum of terms which are products of functions of separate variables and that, by assumption, each term of such products converges weakly* in $L^\infty(\Omega)$. Then $Q^\varepsilon = \mathrm{Cof}\, R^\varepsilon \rightharpoonup Q = \mathrm{Cof}\, R$ is a consequence of Lemma 1 in [6]. By the same argument, $\mathrm{Det}\, R^\varepsilon \rightharpoonup \mathrm{Det}\, R$ weakly* in $L^\infty(\Omega)$, so that $\mathrm{Det}\,(R(x)) \geqslant \delta$, R is invertible and R^{-1} has L^∞-coefficients.

REMARKS.

- Instead of (5.1), let us assume that

$$
\begin{aligned}
&\exists C>0\,,\text{a.e. } x\in\Omega\,,\ \ \forall\xi,\eta\in\mathbb{R}^n,\\
&(G(x,\xi)-G(x,\eta),\ (\xi-\eta))\\
&\qquad\geqslant\begin{cases} C\|\xi-\eta\|^p & \text{if } p\geqslant 2\\[4pt] C\dfrac{\|\xi-\eta\|^2}{(\|\xi\|+\|\eta\|)^{2-p}} & \text{if } 1<p<2\,.\end{cases}
\end{aligned}
\tag{5.3}
$$

Then clearly (4.2) is satisfied with $C\delta$ in place of C. As already noted, the limit form (when ε tends to zero) of (4.2) also holds. In this case the equations (E^ε) and (E) have unique solutions u^ε and u respectively. Moreover the whole sequence u^ε tends to u and $\nabla u^\varepsilon - R^\varepsilon R^{-1}\nabla u$ tends to zero, for the same topologies as before.

- Now assume that G satisfies (1.1), (5.1) and (5.2) and assume moreover that $\partial G_i/\partial\xi_j = \partial G_j/\partial\xi_i$ for any i, j. Then defining

$$\mathcal{G}(x,\xi) = \int_0^1 G(x, t\xi)\,\xi\,dt\,,$$

one has $\partial\mathcal{G}/\partial\xi_i = G_i$ and the following minimization problem

$$\mathrm{Inf}\left\{\int_\Omega \mathrm{Det}\,(R^\varepsilon(x))\,\mathcal{G}(x, N^\varepsilon(x)\,\nabla v)\,dx - \langle f^\varepsilon, v\rangle\right\} \quad (\mathcal{P}^\varepsilon)$$

is well-posed. Its Euler equation is (E^ε) and it characterizes the solutions of $(\mathcal{P}^\varepsilon)$.

- In the linear case our results apply to

$$-\mathrm{div}\,((\mathrm{Det}\,R^\varepsilon)^t(N^\varepsilon)\,N^\varepsilon\,\nabla u^\varepsilon) = f^\varepsilon,$$

which is the Euler equation of

$$\mathrm{Inf}\left\{\frac{1}{2}\int_\Omega (\mathrm{Det}\,R^\varepsilon(x))\|N^\varepsilon(x)\,\nabla v\|^2\,dx - \langle f^\varepsilon, v\rangle\right\}. \quad (\mathcal{P}^\varepsilon)$$

5.4. – *The case of G in matrix form and Q^ε in vector form.*

In this subsection we show that we can apply the general result to the equation

$$\begin{cases} -\operatorname{div}(H(x, N^\varepsilon \nabla u^\varepsilon)\, V^\varepsilon(x)) = f^\varepsilon, \\ u^\varepsilon \in W_0^{1,p}(\Omega), \end{cases} \qquad (\widetilde{E}^\varepsilon)$$

where

- N^ε and f^ε are the same as before,
- $H : \Omega \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a square matrix, with zero coefficients on its diagonal:

$$H_{ii} \equiv 0\,, \tag{5.4}$$

H_{ij} is a Carathéodory function such that

$$|H_{ij}(x, \xi)| \leqslant \beta \|\xi\|^{p-1} + g(x)\,, \tag{5.5}$$

with β and g as in Section 1,

- $V^\varepsilon \colon \Omega \to \mathbb{R}^n$ is a vector valued function with L^∞ components such that

$$v_i^\varepsilon(x) = v_i^\varepsilon(x_i)\,, \tag{5.6}$$

- H, N^ε, V^ε are related by

$$(H(x, N^\varepsilon \xi)\, V^\varepsilon - H(x, N^\varepsilon \eta)\, V^\varepsilon, \xi - \eta) \geqslant 0\,, \tag{5.7}$$

$$(H(x, N^\varepsilon \xi)\, V^\varepsilon, \xi) \geqslant \alpha \|\xi\|^p - h(x)\,, \tag{5.8}$$

with α and h as before.

We are going to show that $(\widetilde{E}^\varepsilon)$ is a particular form of (E^ε) in which $m = n^2$. Let us suppose that j is a double index in the coefficients q_{ij}^ε of the matrix Q^ε as well as in the components G_j of the nonlinear function G:

$$j = kl\,, \qquad k \in \{1, \ldots, n\}\,, \qquad l \in \{1, \ldots, n\}\,,$$

$$q_{ij}^\varepsilon = q_{ikl}^\varepsilon(x_i')\,, \qquad G_j = G_{kl}(x, \xi)\,.$$

We also assume that G_{kl} vanishes for $k = l$, that for any $k \neq i$, $q_{ikl}^\varepsilon = 0$ and that for any $i \neq l$, q_{iil}^ε is independent of i and, as a function of x, depends only on the coordinate x_l. In this case, we can set

$$H_{kl}(x, \xi) \equiv G_{kl}(x, \xi) \equiv G_j(x, \xi)\,,$$

$$v_l^\varepsilon(x) \equiv v_l^\varepsilon(x_l) \equiv q_{iil}^\varepsilon(x_l) \qquad \text{for any } i \neq l$$

and it is easy to show that

$$Q^\varepsilon G(x,\, \xi) \equiv H(x,\, \xi)\, V^\varepsilon$$

and that (5.5), (5.7), (5.8) are nothing but rewriting (1.1), (1.2), (1.3).

As a consequence of Theorem 3.1, the following result holds.

THEOREM 5.2. – *Besides the hypotheses* (5.4) *to* (5.8), *let us assume that*

$$\begin{aligned} f^\varepsilon &\to f \text{ strongly in } W^{-1,\,p'}(\Omega), \\ V^\varepsilon &\rightharpoonup V \text{ weakly}\star \text{ in } L^\infty(\Omega)^n, \\ R^\varepsilon = (N^\varepsilon)^{-1} &\rightharpoonup R \text{ weakly}\star \text{ in } L^\infty(\Omega)^{n\times n}. \end{aligned}$$

If moreover R is invertible with inverse $N = R^{-1}$ in $L^\infty(\Omega)^{n\times n}$ and if the sequence N^ε is bounded in $L^\infty(\Omega)^{n\times n}$, then any sequence of solutions u^ε to $(\widetilde{E}^\varepsilon)$ is bounded in $W_0^{1,\,p}(\Omega)$. Moreover if $u^{\varepsilon'} \rightharpoonup u$ weakly in $W_0^{1,\,p}(\Omega)$ for some subsequence ε' of ε, then u is a solution of

$$\begin{cases} -\operatorname{div}\,(H(x,\, N\nabla u)\, V) = f, \\ u \in W_0^{1,\,p}(\Omega). \end{cases} \qquad (\widetilde{E})$$

REMARKS.

- The corrector result applies if

$$\exists C > 0, \text{a.e. } x \in \Omega\,,\ \forall \xi,\, \eta \in \mathbb{R}^n,$$
$$(H(x,\, \xi)\, V^\varepsilon(x) - H(x,\, \eta)\, V^\varepsilon(x),\ R^\varepsilon(x)\, \xi - R^\varepsilon(x)\, \eta)$$
$$\geqslant \begin{cases} C\|\xi - \eta\|^p & \text{if } p \geqslant 2\,, \\ C\dfrac{\|\xi - \eta\|^2}{(\|\xi\| + \|\eta\|)^{2-p}} & \text{if } 1 < p < 2\,. \end{cases}$$

- A particular example is obtained with $N^\varepsilon = \operatorname{diag}(n_i^\varepsilon(x_i))$, $\underline{n} \leqslant n_i^\varepsilon(x_i) \leqslant \overline{n}$, $\underline{v} \leqslant v_i^\varepsilon(x_i) \leqslant \overline{v}$, $H_{ij}(x,\, \xi) \equiv H_{ij}(x,\, \xi_i)$, H_{ij} monotone nondecreasing in $\mathbb{R}$,

$$|H_{ij}(x,\, t)| \leqslant \beta |t|^{p-1} + g(x)\,,$$

$$H_{ij}(x,\, t)\, t \geqslant \alpha |t|^p - h(x)$$

and the corrector result applies if

$$(H_{ij}(t) - H_{ij}(t'))(t - t') \geqslant \begin{cases} C|t - t'|^p & \text{if } p \geqslant 2\,, \\ C\dfrac{|t - t'|^2}{(|t| + |t'|)^{2-p}} & \text{if } 1 < p < 2\,. \end{cases}$$

• The case $N^\varepsilon =$ identity but H replaced by H^ε was studied by O. Khoumri in [4].

5.5. – *The case of the p-laplacian.*

In [6] and in Section 5.1, we have considered the linear case, which corresponds to $p = 2$ and $G(x, \xi) \equiv \xi$. In this section we study the case $G(x, \xi) \equiv \|\xi\|^{p-2}\xi$, for $1 \leqslant p \leqslant \infty$. Then (1.1) is trivial. Let us look for natural conditions on $S^\varepsilon = {}^tQ^\varepsilon R^\varepsilon$ which imply that assumptions (1.2) and (1.3) are satisfied.

Since the coefficients of Q^ε and R^ε are bounded,

$$\exists \delta^\varepsilon > 0, \text{a.e } x \in \Omega,\ \forall \xi \in \mathbb{R}^n,\ \|S^\varepsilon(x)\,\xi\| \leqslant \delta^\varepsilon \|\xi\|\,. \tag{5.9}$$

(Actually as Q^ε and R^ε are uniformly bounded, (5.9) holds with a larger δ independent of ε.) Now let us assume that S^ε is coercive, uniformly in x,

$$\exists \gamma^\varepsilon > 0, \text{a.e } x \in \Omega,\ \forall \xi \in \mathbb{R}^n,\ (S^\varepsilon(x)\,\xi, \xi) \geqslant \gamma^\varepsilon \|\xi\|^2. \tag{5.10}$$

Then it is clear that $\gamma^\varepsilon \leqslant \delta^\varepsilon$ and the following result holds

PROPOSITION. – 5.3. – *Assume* (5.9) *and* (5.10), *with*

$$\frac{\gamma^\varepsilon}{\delta^\varepsilon} \geqslant \frac{|p-2|}{p}\,. \tag{5.11}$$

Then $G(x, \xi) \equiv \|\xi\|^{p-2}\xi$ *satisfies condition* (1.2) *and furthermore,*

$$a.e\ x \in \Omega,\ \forall \xi \neq \eta \in \mathbb{R}^n,\ (G(\xi) - G(\eta), S^\varepsilon(x)(\xi - \eta)) > 0\,. \tag{5.12}$$

PROOF. – This proof, as well as the proof of Proposition 5.5 is inspired by [3]. Of course one can assume $p \neq 2$. Let $\xi \neq \eta$ be two vectors of $\mathbb{R}^n$. There exist z, w in $\mathbb{R}^n$ and $\lambda \neq \mu$ in $\mathbb{R}$ such that

$$\xi = z + \lambda w\,, \quad \eta = z + \mu w\,, \quad \|w\| = 1\,, \quad (z, w) = 0\,.$$

(Remark that w is a unit vector on the line defined by ξ, η and z is the orthogonal projection of 0 on this line.) Then we can write

$$(G(\xi) - G(\eta), S^\varepsilon(\xi - \eta)) = (\lambda - \mu)[k(\lambda) - k(\mu)]\,,$$

where

$$k(t) = (\|z\|^2 + t^2)^{(p-2)/2}[t(S^\varepsilon w, w) + (S^\varepsilon w, z)].$$

We have to check that k is strictly increasing. If $z = 0$ this is obvious, so we may assume $z \neq 0$. Setting $a = \|z\| > 0$, $b = (S^\varepsilon w, w) > 0$ (by (5.10))) and $c = (S^\varepsilon w, z)$, we obtain

$$k(t) = (a^2 + t^2)^{(p-2)/2}(bt + c)$$

and an easy computation shows that k is strictly increasing if and only if

$$\Delta = c^2(p-2)^2 - 4a^2 b^2(p-1) \leqslant 0\,.$$

But one can write

$$S^\varepsilon w = (S^\varepsilon w, w)\, w + \theta z'\,,$$

with $(z', w) = 0$ and $\|z'\| = 1$. Using (5.9) and (5.10), we deduce

$$(\delta^\varepsilon)^2 \geqslant \|S^\varepsilon w\|^2 = b^2 + \theta^2, \quad (\gamma^\varepsilon)^2 \leqslant b^2$$

and hence

$$c^2 = (S^\varepsilon w, z)^2 = \theta^2 (z, z')^2 \leqslant \theta^2 a^2 \leqslant a^2((\delta^\varepsilon)^2 - b^2)\,,$$

$$\Delta \leqslant a^2((\delta^\varepsilon)^2(p-2)^2 - b^2 p^2) \leqslant a^2((\delta^\varepsilon)^2(p-2)^2 - (\gamma^\varepsilon)^2 p^2)$$

and (5.12) is satisfied if $(\delta^\varepsilon)^2(p-2)^2 - (\gamma^\varepsilon)^2 p^2 \leqslant 0$, that is (5.11).

PROPOSITION 5.4. – *Under condition* (5.10) *and if the sequence* $\{R^\varepsilon\}_\varepsilon$ *is bounded, then* (1.3) *holds for* $G(x, \xi) = \|\xi\|^{p-2}\xi$, *if the sequence* $\{1/\gamma^\varepsilon\}_\varepsilon$ *is bounded.*

PROOF. – We can rewrite (1.3)

$$\exists \alpha > 0,\ \exists h \in L^1(\Omega), \text{ a.e } x \in \Omega,\ \forall \xi \in \mathbb{R}^n,$$

$$(G(x, \xi), S^\varepsilon(x)\,\xi) \geqslant \alpha \|R^\varepsilon(x)\,\xi\|^p - h(x)$$

and in the present case, using (5.10) and the boundedness of R^ε and $1/\gamma^\varepsilon$,

$$(G(x, \xi), S^\varepsilon(x)\,\xi) = \|\xi\|^{p-2}(S^\varepsilon \xi, \xi) \geqslant$$

$$\gamma^\varepsilon \|\xi\|^p \geqslant \frac{\gamma^\varepsilon}{\|R^\varepsilon(x)\|^p}\|R^\varepsilon(x)\,\xi\|^p \geqslant \alpha \|R^\varepsilon(x)\,\xi\|^p,$$

which ends the proof.

PROPOSITION 5.5. – *Let* $\delta \geqslant 0$ *be such that*

$$a.e \ x \in \Omega, \ \forall \xi \in \mathbb{R}^n, \ \forall \varepsilon > 0, \ \|S^\varepsilon(x)\,\xi\| \leqslant \delta \|\xi\| .$$

Assume that (5.10) *holds uniformly in* ε, *or equivalently that there exists* γ, $0 < \gamma \leqslant \delta$, *such that*

$$a.e \ x \in \Omega, \ \forall \xi \in \mathbb{R}^n, \ \forall \varepsilon > 0, \ (S^\varepsilon(x)\xi, \ \xi) \geqslant \gamma \|\xi\|^2.$$

Then the reinforced condition (4.2) *holds for* $G(x, \xi) = \|\xi\|^{p-2}\xi$, *if*

$$\frac{\gamma}{\delta} > \frac{|p-2|}{p} .$$

PROOF. – We refine the proof of Proposition 5.3, to which the reader is refered. We have

$$(G(\xi) - G(\eta), \ S^\varepsilon(\xi - \eta)) = (\lambda - \mu)[k(\lambda) - k(\mu)],$$

$$k'(t) = (a^2 + t^2)^{(p-4)/2}\pi(t),$$

$$\pi(t) = b(p-1)\,t^2 + c(p-2)\,t + ba^2.$$

First we prove that for any t, if $\delta|p-2| < \gamma p$,

$$\pi(t) \geqslant r(a^2 + t^2), \tag{5.13}$$

with r depending on δ, γ and p only. Actually,

$$\pi(t) - r(a^2 + t^2) = [b(p-1) - r]\,t^2 + c(p-2)\,t + (b - r)\,a^2$$

$$\geqslant [\gamma(p-1) - r]\,t^2 + c(p-2)\,t + (\gamma - r)\,a^2 = \tilde{\pi}(t).$$

We are going to find r such that $\tilde{\pi}(t) \geqslant 0$, for any t. We assume that $r < \gamma(p-1)$, so that the first coefficient of $\tilde{\pi}$ is positive. The discriminant of $\tilde{\pi}$ is

$$\tilde{\Delta} = c^2(p-2)^2 - 4a^2(\gamma - r)[\gamma(p-1) - r]$$

and since $c^2 \leqslant a^2(\delta^2 - b^2)$,

$$\tilde{\Delta} \leqslant a^2[-4r^2 + 4\gamma p r + \varrho],$$

where $\varrho = (\delta^2 - \gamma^2)(p-2)^2 - 4\gamma^2(p-1) < 0$ for $\delta|p-2| < \gamma p$. It follows that for r small enough, r depending on δ, γ and p only, $\tilde{\Delta} \leqslant 0$ and $\tilde{\pi}(t) \geqslant 0$ for any t.

We deduce from (5.13) that

$$k'(t) \geqslant r(a^2+t^2)^{(p-2)/2}.$$

After perhaps exchanging ξ and η and replacing w by $-w$, we may assume that $|\mu| \leqslant \lambda$. We consider the two cases $1<p<2$ et $p \geqslant 2$ separately.

- Case $1<p<2$: For all t in $[\mu, \lambda]$,

$$(a^2+t^2)^{1/2} \leqslant (a^2+\mu^2)^{1/2}+(a^2+\lambda^2)^{1/2}.$$

Since $p<2$ and $\lambda \geqslant \mu$, it follows that

$$(k(\lambda)-k(\mu))(\lambda-\mu) = (\lambda-\mu)\int_\mu^\lambda k'(t)\,dt$$

$$\geqslant r(\lambda-\mu)^2((a^2+\mu^2)^{1/2}+(a^2+\lambda^2)^{1/2})^{p-2},$$

which can be rewritten

$$(G(\xi)-G(\eta), S^\varepsilon(\xi-\eta)) \geqslant r\frac{\|\xi-\eta\|^2}{(\|\xi\|+\|\eta\|)^{2-p}}.$$

- Case $p \geqslant 2$: Note that

$$k(\lambda)-k(\mu) \geqslant r\int_\mu^\lambda |t|^{p-2}\,dt.$$

If $\mu \geqslant 0$,

$$k(\lambda)-k(\mu) \geqslant r\int_\mu^\lambda (t-\mu)^{p-2}\,dt = \frac{r}{p-1}(\lambda-\mu)^{p-1}.$$

If $\mu \leqslant 0$, then $\lambda \geqslant -\mu$ and $2\lambda \geqslant \lambda-\mu$, so that

$$k(\lambda)-k(\mu) \geqslant r\int_0^\lambda t^{p-2}\,dt = \frac{r}{p-1}\lambda^{p-1} \geqslant \frac{r}{2^{p-1}(p-1)}(\lambda-\mu)^{p-1}.$$

In any case, if $p \geqslant 2$, we obtain

$$(G(\xi)-G(\eta), S^\varepsilon(\xi-\eta)) \geqslant \frac{r}{2^{p-1}(p-1)}\|\xi-\eta\|^{p-1}.$$

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