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The Rank of the Multiplication Map for Sections of Bundles on Curves.

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Sunto. – *Sia X una curva liscia di genere $g \geq 2$ ed A, B fasci coerenti su X. Sia $\mu_{A,B}$: $H^0(X, A) \otimes H^0(X, B) \to H^0(X, A \otimes B)$ l'applicazione di moltiplicazione. Qui si dimostra che $\mu_{A,B}$ ha rango massimo se $A \cong \omega_X$ e B è un fibrato stabile generico su X. Diamo un'interpretazione geometrica dell'eventuale non-surgettività di $\mu_{A,B}$ quando A, B sono fibrati in rette generati da sezioni globali e* $\deg(A) + \deg(B) \geq 3g - 1$. *Studiamo anche il caso* $\dim(\mathrm{Coker}(\mu_{A,B})) \geq 2$.

Introduction.

Let X be a smooth connected projective curve of genus $g \geq 2$ defined over an algebraically closed field $\boldsymbol{K}$ and A, B coherent sheaves on X; $\mu_{A,B}$: $H^0(X, A) \otimes H^0(X, B) \to H^0(X, A \otimes B)$ will denote the multiplication map. Set $\omega := \omega_X$. For several pairs (A, B) the rank of $\mu_{A,B}$ has a geometric meaning (see e.g. [Bu], [E], [EKS], [G], [GL] and [Re]). For instance if $A = B \in Pic(X)$ is very ample and $h^1(X, A) = 0$ the map $\mu_{A,A}$ is surjective if and only if the corresponding complete embedding is projectively normal; furthermore, $\mathrm{Ker}(\mu_{\omega,\omega})$ is the domain of the classical Wahl (or Gaussian) map. As obvious from [G], 4.a.1 and 4.e.4, [EKS], Th. 1, and [Bu] the case $A \cong \omega$ is on the border of the known results on the surjectivity of $\mu_{\omega,B}$ for vector bundles B with large slope. In section one we study the rank of $\mu_{\omega,B}$ when B is a general stable bundle on X and prove that $\mu_{\omega,B}$ has maximal rank. For all integers r, d with $r > 0$ $M(X; r, d)$ will denote the scheme of all rank r stable vector bundles on X with degree d. It is well-known that $M(X; r, d)$ is a smooth irreducible variety of dimension $r^2(g-1)+1$. The aim of section one is the proof of the following result.

THEOREM 0.1. – *Assume* $\mathrm{char}(\boldsymbol{K}) = 0$. *Let X be a smooth projective curve of genus $g \geq 2$ and r, d positive integers. Fix a general $E \in M(X; r, d)$. If $d \geq rg + r$ the multiplication map $\mu_{\omega,E}$ is surjective. If $d < rg + r$ the map $\mu_{\omega,E}$ is injective.*

In section two we give a geometric interpretation of the non-surjectivity of $\mu_{L,M}$ for spanned line bundles L, M on X with $\deg(M) + \deg(L) \geq 3g-1$. We prove the following result.

THEOREM 0.2. – *Let X be a smooth projective curve of genus g and L, M spanned line bundles on X such that* $\deg(M) + \deg(L) \geq 3g-1$. *The map $\mu_{L,M}$ is not surjective if and only if there exists an effective divisor $D \subset X$, $D \neq 0$, with $h^0(X, L(-D)) \geq 2$, $\deg(D) \geq 2(h^0(X,M) - h^0(X, M(-D)) + 2(h^0(X,L) - h^0(X,L(-D))$ and such that the map $\sigma_D \circ \mu_{L,M}: H^0(X,L) \otimes H^0(X,M) \to H^0(X, L\otimes M|D) \cong L\otimes M|D$ is not surjective. Furthermore, if $\mu_{L,M}$ is not surjective there is such D with $h^0(X, L(-D)) + h^1(X, M(D)) \geq h^0(X,L) + h^1(X,M)$ and $4 \leq 2(\deg(D)) \leq \deg(L) + \deg(M) + 2 - g$.*

Notice that the inequality $\deg(M) + \deg(L) \geq 3g-1$ in the statement of Theorem 0.2 is always satisfied if $M \cong L^{\otimes t}$ with $t \geq 2$ and $h^1(X,L) = 0$. Then we study the case $\dim(\mathrm{Coker}(\mu_{L,M})) \geq 2$ and prove the following result.

PROPOSITION 0.3. – *Fix integers g, b with $g \geq 4$ and $b \geq 2$. Let X be a smooth projective curve of genus g and L, M very ample line bundles on X such that* $\deg(M) + \deg(L) \geq 3g-1$ *and* $\dim(\mathrm{Coker}(\mu_{L,M})) = b$. *Then there exists an effective divisor $D \subset X$, $D \neq 0$, with $h^0(X, L(-D)) \geq 2$, $\deg(D) \geq 2(h^0(X,M) - h^0(X,M(-D))) + 2(h^0(X,L) - h^0(X, L(-D)))$ and such that the map $\sigma_D \circ \mu_{L,M}: H^0(X,L) \otimes H^0(X,M) \to H^0(X, L\otimes M|D) \cong L \otimes M|D$ is not surjective. Furthermore, if $\mu_{L,M}$ is not surjective there is such D with $h^0(X, L(-D)) + h^1(X, M(D)) \geq h^0(X,L) + h^1(X,M)$, $2(\deg(D)) \leq \deg(L) + \deg(M) + 2 - g$ and $h^0(X, \boldsymbol{O}_X(D)) + \varepsilon(D) \geq b$, where $\varepsilon(D) := \dim(\mathrm{Coker}(\sigma_D \circ \mu_{L,M}))$.*

The proofs of 0.2 and 0.3 are just small modifications of the proof of [GL], Th. 3.

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1. – Proof of 0.1.

Let C be a one-dimensional projective locally Cohen-Macaulay scheme. We will use the notation $\mu_{A,B}$ even for sheaves A, B on C. If $L \in \mathrm{Pic}(C)$ and L is spanned, $h_L: C \to \boldsymbol{P}(H^0(C,L))$ will denote the associated morphism.

We need the following well-known generalization of a lemma of Castelnuovo.

LEMMA 1.1. – *Let C be a one-dimensional projective locally Cohen-Macaulay scheme with $h^0(C, \boldsymbol{O}_C) = 1$ and $R \in \mathrm{Pic}(C)$ with R spanned and $h^0(C,R) = 2$. Then the multiplication map $\mu_{\omega,R}: H^0(C,\omega_C) \otimes H^0(C,R) \to H^0(C, \omega_C \otimes R)$ is surjective.*

PROOF. – A choice of a basis of $H^0(C, R)$ induces an exact sequence

$$0 \to \omega_C \otimes R^* \to \omega_C \oplus \omega_C \to \omega_C \otimes R \to 0 \tag{1}$$

Since $h^0(C, \boldsymbol{O}_C) = 1$, we have $h^1(C, \omega_C) = 1$ by duality ([AK]). Since $h^1(C, \omega_C \otimes R^*) = h^0(C, R) = 2 = 2(h^1(C, \omega_C))$ and $h^1(C, \omega_C \otimes R) = h^0(C, R^*) = 0$ (duality and the assumption $h^0(C, \boldsymbol{O}_C) = 1$), we obtain that in the long cohomology exact sequence induced by (1) the map $H^1(C, \omega_C \otimes R^*) \to H^1(C, \omega_C) \oplus H^1(C, \omega_C)$ is an isomorphim. Thus the multiplication map $H^0(C, \omega_C) \oplus H^0(C, \omega_C) = H^0(C, \omega_C) \otimes H^0(C, R) \to H^0(C, \omega_C \otimes R)$ is surjective, as wanted.

LEMMA 1.2. – *Assume* $\operatorname{char}(\boldsymbol{K}) = 0$*. Let* C *be an integral projective curve with* $C \neq \boldsymbol{P}^1$ *and* $R \in \operatorname{Pic}(C)$*,* R *spanned and with* h_R *birational. Then the multiplication map* $\mu_{\omega, R}: H^0(C, \omega_C) \otimes H^0(C, R) \to H^0(C, \omega_C \otimes R)$ *is surjective.*

PROOF. – If $x := h^0(C, R) - 2 \geq 0$, 1.2 is a particular case of 1.1. Assume $x > 0$ and take x general points $P_1, \ldots, P_x$ of C_{reg}. Thus $h^0(C, R(-P_1 - \ldots - P_x)) = 2$. Since h_R is birational and $\operatorname{char}(\boldsymbol{K}) = 0$, the line bundle $L := R(-P_1 - \ldots - P_x)$ is spanned by $H^0(C, L)$ (trisecant lemma). Hence we may apply the case $x = 0$ and obtain the surjectivity of $\mu_{\omega, L}$. Use $P_1 + \ldots + P_x$ to see $R(-P_1 - \ldots - P_x)$ (resp. $\omega_C \otimes R(-P_1 - \ldots - P_x)$) as a subsheaf of R (resp. $\omega_C \otimes R$). With these identifications it is easy to check that $\dim(\operatorname{Im}(\mu_{\omega, R})) \geq \dim(\operatorname{Im}(\mu_{\omega, L})) + x$; here we use $h^0(C, \omega_C \otimes L) \neq 0$, i.e. $C \neq \boldsymbol{P}^1$. Since $\deg(L) > 0$ and L, R are locally free, we have $h^0(C, \omega_C \otimes L) = \deg(L) + p_a(C) - 1 = h^0(C, \omega_C \otimes R) - x$ (even if C is not Gorenstein). Hence $\mu_{\omega, R}$ must be surjective.

LEMMA 1.3. – *Let* C *be a Cohen-Macaulay one-dimensional projective scheme with* $h^0(C, \boldsymbol{O}_C) = 1$*. Let* E *be a rank* r *vector bundle on* C *spanned by its global sections. Assume that* E *has no trivial factor. Let* F *be the kernel of the evaluation map* $ev_E: H^0(C, E) \otimes \boldsymbol{O}_C \to E$*. We have* $\dim(\operatorname{Coker}(\mu_{\omega, E})) = h^0(C, F^*) - h^0(C, E)$*.*

PROOF. – The definition of F gives the following exact sequence

$$0 \to F \to H^0(X, E) \otimes \boldsymbol{O}_C \to E \to 0 \tag{2}$$

Since E has no trivial factors, we have $0 = h^0(C, E^*) = h^1(C, \omega \otimes E)$ by duality on Cohen-Macaulay schemes ([AK]). Moreover by duality the assumption $h^0(C, \boldsymbol{O}_C) = 1$ is equivalent to $h^1(C, \omega) = 1$. Therefore after tensoring (2) by ω we deduce that $\dim(\mathrm{Coker}(\mu_{\omega, E})) = h^1(C, F \otimes \omega) - h^0(C, E) = h^0(C, F^*) - h^0(C, E)$, the last equality being again given by duality.

PROOF OF 0.1. – For a general $E \in M(X; r, d)$ we have $h^0(X, E) = 0$ if $d \leq r(g-1)$ and $h^0(X, E) = d + r(1-g)$ if $d \geq r(g-1)$, i.e. either $h^0(X, E) = 0$ or $h^1(X, E) = 0$ ([La] or [Su] or, in arbitrary characteristic, [BR], Lemma 1.2). Hence a general $E \in M(X; r, d)$ is spanned only if $d \geq rg + 1$. It is known and easy to check that if $d \geq rg + 1$ a general $E \in M(X; r, d)$ is spanned. Since $\omega \otimes E$ is a general element of $M(X; r, d + r(2g-2))$ and $d > 0$, the bundle $\omega \otimes E$ is spanned and we have $h^1(X, \omega \otimes E) = 0$ and $h^0(X, \omega \otimes E) = d + r(g-1) > 0$. Hence if $\mu_{\omega, E}$ is surjective the bundle E must be spanned and hence $d = \deg(E) \geq rg + 1$. First assume $d \geq rg + 1$ and hence E spanned. We obtain an exact sequence (2). Since $h^0(X, E) = d + r(1-g)$, we have $\mathrm{rank}(F) = d - rg$. Since $\deg(F^*) = d$ we have $h^0(X, F^*) \geq \chi(F^*) = d + (d - rg)(1-g)$ (Riemann-Roch). Hence $h^0(X, F^*) - h^0(X, E) \geq (g-1)(rg + r - d)$ and we have equality if and only if $h^0(X, F^*) = \chi(F^*)$ for general E. Thus $h^0(X, F^*) > h^0(X, E)$ if $d < rg + r$. Thus by 1.3 to prove 0.1 for an integer $d \geq gr + 1$ it is sufficient to check that $h^0(X, F^*) = \max\{\chi(F^*), h^0(X, E)\}$ for general E. First assume $d \geq rg + r$. Take $r-1$ general line bundles L_i, $1 \leq i \leq r-1$, with $\deg(L_i) = g+1$. Thus $h^0(X, L_i) = 2$, $h^1(X, L_i) = 0$, L_i is spanned and if $1 \leq i \leq r-1$ we have exact sequence

$$0 \to L_i^* \to \boldsymbol{O}_X^{\oplus 2} \to L_i \to 0 \tag{3}$$

Take a general $M \in \mathrm{Pic}(X)$ with $\deg(M) = d - (r-1)(g+1)$. Since $\deg(M) \geq g+1$ we have $h^1(X, M) = 0$, $h^0(X, M) = d - rg - r + 2$ and M is spanned. Call T the kernel of the evaluation map $H^0(X, M) \otimes \boldsymbol{O}_X \to M$. Thus we have an exact sequence

$$0 \to M^* \to H^0(X, M)^* \otimes \boldsymbol{O}_X \to T^* \to 0 \tag{4}$$

Tensoring (4) with ω we obtain $h^0(X, T^*) = h^0(X, M) + \dim(\mathrm{Coker}(\mu_{\omega, M}))$. Hence by 1.2 (at least if $\mathrm{char}(\boldsymbol{K}) = 0$) for general M we have $h^0(X, T^*) = h^0(X, M)$; in positive characteristic we look at the proof of 1.2 and work in the following way; we start with a spanned $L \in \mathrm{Pic}(X)$ with $\deg(L) = g-1$ and $h^1(X, L) = 0$; take x general points $P_1, \ldots, P_x$ of x and set $R := L(P_1 + \ldots + P_x)$; then we conclude as in the last part of the proof of 1.2. Set $G := M \oplus (\oplus_{1 \leq i \leq r-1} L_i)$. G is spanned, $h^0(X, G) = d + r(1-g)$ and $h^1(X, G) = 0$. Set $N := T \oplus (\oplus_{1 \leq i \leq r-1} L_i^*)$. Thus N is the kernel of the evaluation map $H^0(X, G) \otimes \boldsymbol{O}_X \to G$ and $h^0(X, N^*) = h^0(X, G)$. Since $h^1(X, G) = 0$ and G is a flat limit of a flat family of stable vector bundles with constant cohomology ([NR], Prop. 2.6), we obtain that for general $E \in M(X; r, d)$ we have $h^0(X, F^*) = h^0(X, E)$, concluding the proof for $d \geq rg + r$. This part of the proof part could have been proved using [BR], Th. 2.1. Now assume $d < gr + r$.

Since E is general, it is easy to check that the natural map $H^0(X, E) \otimes \boldsymbol{O}_X \to E$ is an inclusion of sheaves (and even an embedding of bundles if $d \leqslant gr + r - 2$, but we do not need it); of course, $h^0(X, E) = 0$ if $d \leqslant r(g-1)$. Thus the injectivity of $\mu_{\omega, E}$ follows from the fact that the induced map $H^0(X, E) \otimes \omega \to E \otimes \omega$ is injective.

2. – Proofs of 0.2 and 0.3.

Let X be a smooth curve and L, M spanned line bundles on X. We want to find geometric restrictions for the nonsurjectivity of $\mu_{L, M}$ and we want to show that, under suitable assumptions, $\mu_{L, M}$ is surjective if there is no such geometric restriction. A tautological restriction is the existence of an effective divisor D of X such that the restriction map $\sigma_D \colon H^0(X, L \otimes M) \to H^0(X, L \otimes M | D) \cong L \otimes M | D$ is surjective, but $\sigma_D \circ \mu_{L, M}$ is not surjective. If however $h^0(X, L(-D)) > 0$, this condition is not quite stupid for the following reasons. First consider the case $h^0(X, L(-D)) = 1$. If $L = M$, $h^1(X, L) = 0$ and L is very ample, $h_L(X)$ is not projectively normal if and only if none of its hyperplane sections is arithmetically Cohen-Macaulay or if and only if there is one such hyperplane section, say $h_L(D)$, which is not arithmetically Cohen-Macaulay; hence in the range of degrees we are interested in $\mu_{L, L}$ is not surjective if and only if there is $D \in |L|$ such that $\sigma_D \circ \mu_{L, M}$ is not surjective. The case $h^0(X, L(-D)) \geqslant 2$ is more interesting; for instance in the range of integers for $\deg(L)$ we are interested in (i.e. when all maps $\mu_{L, L^{\oplus t}}$, $t \geqslant 2$, are surjective) if L is very ample and $h_L(X)$ has a quadrisecant line (case $\deg(D) = 4$, $h^0(X, L(-D)) = h^0(X, L) - 2$) then $\mu_{L, L}$ cannot be surjective. For a very interesting converse in the case $M = L$ and L very ample, see [GL], Th. 3. Following very, very closely the proof of [GL], Th. 3, we will prove Theorem 0.2.

PROOF OF. – 0.2. – Assume that $\mu_{L, M}$ is not surjective, i.e. that its transpose $\mu^*_{L, M} \colon H^0(X, L \otimes M)^* \to H^0(X, L)^* \otimes H^0(X, M)^*$ is not injective. Take $\boldsymbol{e} \in \operatorname{Ker}(\mu^*_{L, M})$, $\boldsymbol{e} \neq 0$. Since $H^0(X, L \otimes M)^* \cong \operatorname{Ext}^1(X; L, \omega \otimes M^*)$, $\boldsymbol{e}$ represents a non-trivial extension

$$0 \to \omega \otimes M^* \to E \to L \to 0 \tag{5}$$

E is a rank 2 vector bundle on X. Let A be a rank 1 subbundle of E with maximal degree. By a theorem of C. Segre and M. Nagata ([N]) we have $\deg(E/A) - \deg(A) \geqslant g$. Since $\deg(E) = 2g - 2 + \deg(L) - \deg(M)$ and $\deg(L) + \deg(M) \geqslant 3g - 1$, we have $\deg(A) > 2g - 2 - \deg(M) = \deg(\omega \oplus M^*)$. Thus the inclusion $A \to E$ induces a non-zero map $\alpha_D \colon A \to L$, i.e. there is an effective divisor D with $A \cong L(-D)$. The inequality $\deg(E/A) - \deg(A) \geq g$ is equivalent to the inequality $2(\deg(D)) \leq \deg(L) + \deg(M) + 2 - g$. Now we will check that $\sigma_D \circ \mu_{L,M}$ is not surjective. The effective divisor D induces an inclusion $H^0(X, L(-D)) \subseteq H^0(X, L)$. Set $W_D := H^0(X, L)/H^0(X, L(-D))$. Evaluation on D yields a homomorphism $\varrho_D \colon H^0(X, M) \otimes W_D \to L \otimes M|D$ with $\mathrm{Im}(\varrho_D) = \mathrm{Im}(\sigma_D \circ \mu_{L,M})$. Consider the following diagram of exact sequences

$$
\tag{6}
\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
H^0(X, L \otimes M|D)^* & \xrightarrow{\varrho_D^*} & H^0(X,M)^* \otimes W_D^* \\
\downarrow & & \downarrow \\
H^0(X, L \otimes M)^* & \xrightarrow{\mu_{L,M}^*} & H^0(X,M)^* \otimes H^0(X,L)^* \\
\downarrow & & \downarrow \\
H^0(X, L \otimes M(-D))^* & \longrightarrow & H^0(X,M)^* \otimes H^0(X,L(-D))^* \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

By construction α_D lifts to a homomorphism $\beta_D \colon L(-D) \to E$ and hence (5) induces the trivial extension of $L(-D)$ by $\omega \otimes M^*$. Thus $\boldsymbol{e} \in \mathrm{Ker}(\mu_{L,M}^*)$ maps to zero in $H^0(X, L \otimes M(-D))^* \cong \mathrm{Ext}^1(X; L(-D), \omega \otimes M^*)$. By the diagram (6), there thus exists a non-zero element $\boldsymbol{f} \in \mathrm{Ker}(\varrho_D^*)$ mapping to $\boldsymbol{e}$. Hence ϱ_D is not surjective. Thus $\sigma_D \circ \mu_{L,M}$ is not surjective. Recall that for any non-degenerate pairing $\gamma : V \otimes W \to T$ between finite dimensional vector spaces over $\boldsymbol{K}$ we have $\dim(\mathrm{Im}(\gamma)) \geq \dim(V) + \dim(W) - 1$; hence $\dim(T) \geq \dim(V) + \dim(W)$ if γ is not surjective. Since $h^0(X, M) - h^0(X, M(-D))$ (resp. $h^0(X, L) - h^0(X, L(-D))$) is the dimension of the image of the restriction map $H^0(X, M) \to M|D$ (resp. $H^0(X, L) \to L|D$) and $\varrho_D \circ \mu_{L,M}$ is not surjective, we obtain $\deg(D) \geq 2(h^0(X, M) - h^0(X, M(-D)) + 2(h^0(X, L) - h^0(X, L(-D))$. Since $\boldsymbol{e} \in \mathrm{Ker}(\mu_{L,M}^*)$, the exact sequence (5) is exact on global sections. Since (5) is exact on global sections, we have $h^0(X, \omega \otimes M^*) + h^0(X, L) = h^0(X, E) \leq h^0(X, A) + h^0(X, E/A)$, i.e.

$$h^0(X, L(-D)) + h^1(X, M(D)) \geq h^0(X, L) + h^1(X, M).$$

Since (5) does not split, the inclusion $A \to L$ is not an isomorphism, i.e. $D \neq 0$. Assume M and L spanned; thus for every $P \in X$ we have $h^0(X, L(-P)) = h^0(X, L) - 1$ and $h^0(X, M(-P)) = h^0(X, M) - 1$; the last equality implies $h^0(X, \omega \otimes M^*(P)) = h^0(X, \omega \otimes M^*)$ for every $P \in X$; hence we have $\deg(D) \geq 2$.

REMARK 2.1. – The inequality $\deg(D) \geq 2(h^0(X, M) - h^0(X, M(-D)) + 2(h^0(X, L) - h^0(X, L(-D))$ in the statement of 0.2 is the translation in our setting of the inequality $\deg(D) \geq 2n + 2$ in the statement of [GL], Th. 3. This inequality implies $\deg(D) \geqslant 3$ if L or M is very ample and $\deg(D) \geqslant 4$ if both L and M are very ample.

PROOF OF 0.3. – By 0.2 only the last assertion of 0.3 must be proved. Look at the proof of 0.2. For every $\boldsymbol{e} \in \mathrm{Ker}(\mu^{*}_{L,M})$ with $\boldsymbol{e} \neq 0$ we obtained an effective divisor, $D(\boldsymbol{e})$, satisfying the thesis of 0.2 and such that $\sigma_{D(\boldsymbol{e})} \circ \mu_{L,M}$ contains a class corresponding to the dual of $\boldsymbol{e}$. We have $D(\lambda\boldsymbol{e}) = D(\boldsymbol{e})$ if $\lambda \in (\boldsymbol{K}\setminus\{0\})$. Thus we obtain a rational map, γ, from $\boldsymbol{P}^{b-1}$ to a symmetric power $S^x(X)$, $x := \deg(D(\boldsymbol{e}))$ for general $\boldsymbol{e}$. Since $\mathrm{Pic}^0(X)$ is an Abelian variety, all the divisors $D(\boldsymbol{e})$ are linearly equivalent, but some of them may coincide. Since $b-1 = \dim(\mathrm{Im}(\gamma)) + \dim(\gamma^{-1}(u))$, where u is a general element of $\mathrm{Im}(\gamma)$, we conclude.

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