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Algebraic Properties of Decorated Splitting Obstruction Groups.

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Sunto. – *In questo articolo si riassumono le definizioni e le principali proprietà dei gruppi di ostruzione con decorazione di tipo LS e LP. Si stabiliscono nuove relazioni fra questi gruppi e si descrivono le proprietà delle mappe naturali fra differenti gruppi con decorazione. Si costruiscono varie successioni spettrali, contenenti questi gruppi con decorazione, e si studiano la loro connessione con le successioni spettrali in K-teoria per certe estensioni quadratiche di antistrutture. Infine, si introduce il concetto di diagramma geometrico di gruppi e si calcolano esplicitamente i gruppi di ostruzione per un diagramma formato da 2-gruppi finiti.*

1. – Introduction.

Wall [43] introduced surgery obstruction groups $L_*^s(\mathbb{Z}[\pi])$ as bordism groups of normal maps between manifolds with fundamental group π. The superscript s means that this group is the obstruction group to surgery, up to simple homotopy equivalence. In a natural way, other L-groups were defined in geometry as $L_*^h(\mathbb{Z}[\pi])$ (obstructions to surgery, up to homotopy equivalence) and projective Novikov groups $L_*^p(\mathbb{Z}[\pi])$ [36]. It is possible to study all these groups systematically from an algebraic point of view by using the concept of decorated L-groups (see [21], [45], and [47]). The algebraic methods give us good techniques for computing these groups, and for describing relations between L-groups with different decorations. The most effective method for computing L-groups was developed by Wall in series of papers on the classification of Hermitian forms (see [47]). In particular, several deep results

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about L_n^Y-groups were obtained. These groups coincide with the groups L_n' for n even, whereas for n odd we must factor out the subgroup $\mathbb{Z}_2$ of L_n^Y generated by the automorphism τ defined in Section 2 (see [47]). It follows from [45] that the groups L_*' are intermediate in the sense of Cappell. In addition, for the groups of odd order we have an isomorphism $L_*' \cong L_*^s$. Further developments of the Wall methods and their applications to computation of natural maps in L-theory and Browder-Livesay groups were done in [29], [31], and [33].

We have an analogous situation with splitting obstruction groups LS_* and groups LP_* of obstructions to surgery on manifold pairs. In simple cases, these groups coincide with Wall groups of the ring with antistructure, so the decorated groups can be defined in a natural way as shown in [17], [38] and [43]. However, in general, the definition is more complicated. Some results on decorated LS- and LP-groups, for the case which generalizes that of one-sided submanifolds, were obtained in [28] and [34].

In the present paper, we extend these results, and obtain new relations between decorated LS- and LP-groups. We describe then new properties of natural maps between different decorated groups. In particular, we study surgery spectral sequences involving these decorated groups and their relations with the spectral sequences in K-theory for twisted quadratic extensions of antistructures [10] (see also [18]). Furthermore, we introduce the groups $LS_*^Y(F)$ and $LP_*^Y(F)$, where F is a geometric diagram of groups (see [20], [23], and [32]), and describe their properties. In the case of finite 2-groups (see [46] and [47]) we obtain sufficiently complete results for computing the groups LS_*^Y and LP_*^Y and their natural maps.

For general references on algebraic K- and L-theory see [11], [36], [38], [39] and [45]. Concerning connections between algebraic geometry and algebraic K-theory see for example [12] and [37]. For surgery theory on compact manifolds we refer to [27] and [43]. Basic concepts and definitions concerning homotopy and homology can be found in monographs [4] and [42].

2. – Surgery obstruction groups.

Let X^n be a closed connected n-dimensional manifold in the category $\mathbb{H}$ ($\mathbb{H} = \mathrm{TOP}$, PL or DIFF) with fundamental group $\pi_1 = \pi_1(X)$, and orientation character $w : \pi_1 \to \{\pm 1\}$. In higher dimensions the problem of determining the homotopy type and the cobordism class of X was successfully reduced to the determination of normal invariants and surgery obstruction groups. This reduction works also in dimension four, provided π_1 is *good* in the sense of [14] (see also [15], [16], and [24]). In recent years, many extensions of surgery obstruction groups have been successfully introduced to study various geometric problems in the theory of compact manifolds (as for example the splitting problem treated in the next section) (see [13], [14], [17], [27], [31], [40], [41], and [43]).

Let $\mathcal{S}_n^h(X)$ be the set of h-cobordism classes in the category $\mathbb{H}$ of orientation preserving homotopy equivalences $h : M \to X$, where M is a closed connected n-manifold in $\mathbb{H}$. Two such maps are said to be *equivalent* if there is an h-cobordism W between them, with a map from W to X extending the ones given on the boundary.

The main tool to describe the set $\mathcal{S}_n^h(X)$ is the surgery exact sequence (see for example [14], [24], [40], and [43]).

THEOREM 2.1. – *Let X^n be a closed connected n-manifold in the category* $\mathbb{H}$ ($\mathbb{H} =$ TOP, PL, *or* DIFF) *with fundamental group $\pi_1 = \pi_1(X)$, and orientation character $w : \pi_1 \to \{\pm 1\}$. Then the surgery sequence*

$$\begin{aligned} \dots \longrightarrow \mathcal{S}_{n+1}^h(X \times I, \partial(X \times I)) &\longrightarrow [\Sigma X, G/\mathbb{H}] \xrightarrow{\sigma_{n+1}} L_{n+1}^h(\pi_1, w) \\ \xrightarrow{\omega_n} \quad \mathcal{S}_n^h(X) &\xrightarrow{\eta_n} [X, G/\mathbb{H}] \xrightarrow{\sigma_n} L_n^h(\pi_1, w) \end{aligned}$$

is exact, for every $n \geqslant 5$. If $n = 4$, then the sequence is exact provided, π_1 is good in the sense of [14].

There is a version for simple structures obtained by replacing L^h and $\mathcal{S}^h$ with L^s and $\mathcal{S}^s$. The superscript s means that the elements of the group L^s are the obstructions to surgery, up to simple homotopy equivalence. The definition of $\mathcal{S}^s$ uses the concept of s-cobordism which replaces that of h-cobordism.

Without the assumption that π_1 is good, we can still obtain interesting results on the stable classification of closed connected orientable 4-manifolds, i.e. a classification modulo connected sum with factors homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$. Examples of groups which are not good are given by the free groups of rank greater than one, and by the fundamental groups of the aspherical surfaces of genus at least two. The classification of the homotopy type and the determination of the s-cobordism class of a closed connected orientable 4-manifold with fundamental group lying in such classes of groups were given in [6], [7], [8] and [9].

From now on, we write $L_n(\pi, w)$ for $L_n^s(\pi, w)$, and $L_n(\pi)$ if moreover the orientation character w is trivial. Finally, recall that the obstruction groups are periodic of period four, i.e. $L_m \cong L_{m+4}$.

Subsequently, Wall [44] [45] defined the groups $L_*(R)$ of a ring R with antistructure. These groups were described by Ranicki as the algebraic cobordism groups of R-module chain complexes with Poincaré duality (see for example [38]).

An *antistructure* [44] is a triple (R, α, u) where R is a ring with unity 1, $u \in R^*$ is an invertible element, and $\alpha : R \to R$ is an antiautomorphism such that $\alpha(u) = u^{-1}$, and $\alpha^2(x) = uxu^{-1}$ for any $x \in R$.

Let X be a subgroup in $K_1(R)$ which is invariant with respect to the involution T induced by α. The groups $L_n^X(R, \alpha, u)$, n (mod 4), were defined in [44], [45], and [47]. These groups are called the *Wall surgery obstruction groups with decoration* (or briefly, *decorated L-groups*). They depend only on the antistructure (R, α, u), on the invariant subgroup $X \subset K_1(R)$, and on the natural number n (mod 4).

Suppose, for example, that R is the group ring $\mathbb{Z}[\pi_1]$ with standard involution $^{-} : \Sigma a_g g \to \Sigma a_g w(g) g^{-1}$, where $\pi_1 = \pi_1(X)$ is the fundamental group of a manifold X with orientation character w, $a_g \in \mathbb{Z}$, and $g \in \pi_1$. Denote by U the subgroup of $K_1(\mathbb{Z}[\pi_1])$ generated by the images of the elements $\pm g$ from the group π_1. Setting $K = K_1(\mathbb{Z}[\pi_1])$, $X = SK_1(\mathbb{Z}[\pi_1]) = \mathrm{Ker}\,(K_1(\mathbb{Z}[\pi_1]) \to K_1(\mathbb{Q}[\pi_1]))$, and $Y = X + U$, we have isomorphisms (see [47])

$$L_{2n}^s(\pi_1) \cong L_{2n}^U(\mathbb{Z}[\pi_1]), \qquad L_{2n}'(\pi_1) \cong L_{2n}^Y(\mathbb{Z}[\pi_1]), \qquad L_{2n}^h(\pi_1) \cong L_{2n}^K(\mathbb{Z}[\pi_1]),$$

while, in odd dimensions, the right-hand sides would be factorized by the subgroup ($\cong \mathbb{Z}_2$) generated by the class of the automorphism $\tau = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$.

It is useful to recall that there are decorated Wall groups whose decorations lie in $K_0(R)$ or in $\widetilde{K}_i(R)$ for any $i = 0, 1$. These cases can be treated analogously without additional problems (as a reference see [21]).

For two L_*-groups with different decorations $X \subset Y \subset K_1(R)$ there exists the following exact sequence

$$\dots \to L_n^X(R) \to L_n^Y(R) \to H^n(Y/X) \to L_{n-1}^X(R) \to \dots \tag{2.1}$$

called the *Rothenberg exact sequence* (see [21] and [45]). Here $H^*(Y/X)$ is the Tate cohomology defined for any group A with an involution $x \to \overline{x}$. Recall that

$$H^0(A) = \{a = \overline{a} : a \in A\}/\{b\overline{b} : b \in A\}$$

and

$$H^1(A) = \{a = \overline{a}^{-1} : a \in A\}/\{b\overline{b}^{-1} : b \in A\}.$$

A *morphism of antistructures* $f : (R, \alpha, u) \to (R', \alpha', u')$ is a ring homomorphism $f : R \to R'$ such that $f(u) = u'$, and $\alpha' \circ f = f \circ \alpha$.

Consider two antistructures (R, α, u) and (R', α', u') with invariant subgroups X and X', respectively. Let $f : R \to R'$ be a morphism of antistructures such that $f_*(X) \subset X'$. The relative groups $L_n^{X, X'}(f)$ were also defined in [38] and [43], which fit in the following exact sequence

$$\dots \to L_n^X(R) \to L_n^{X'}(R') \to L_n^{X, X'}(f) \to L_{n-1}^X(R) \to \dots \tag{2.2}$$

In the sequel, we shall simply write $L_n(f)$ instead of $L_n^{X, X'}(f)$, if this will not lead to any confusion.

We recall now the definition of the quadratic extension of an antistructure [39]. Let (R, α, u) be an antistructure, and let (ϱ, a) be a structure on the ring R, i.e. $\varrho : R \to R$ is an automorphism and $a \in R^*$ is a unity such that $\varrho(a) = a$, and $\varrho^2(x) = axa^{-1}$, for all $x \in R$. The *quadratic extension* of an antistructure (R, α, u) with respect to a structure (ϱ, a) is defined as the antistructure (S, α, u), where $S = R[t]/(t^2 - a)$, $tx = \varrho(x)t$, $\alpha(t)\, t \in R$, and $\alpha^2(t) = utu^{-1}$.

Denote by i the natural morphism $(R, \alpha, u) \to (S, \alpha, u)$ of antistructures and by γ the automorphism on the ring S over R given by setting $\gamma(x + yt) = x - yt$, for any x and $y \in R$. Observe that the map i is the natural inclusion of the rings. This map defines the transfer map $i^!\colon K_1(S) \to K_1(R)$ which is given on modules by the restriction of the S-action onto $R \subset S$ (see for details [21] and [39]). Let $Y \subset K_1(S)$ and $X \subset K_1(R)$ be subgroups such that $i^!(Y) \subset X$. Then there exist relative groups $L_n^{Y, X}(i^!)$ (where the index n is taken mod 4) of the transfer map $i^!$ which fit in the following exact sequence

$$(2.3)\quad \ldots \to L_n^Y(S, \alpha, u) \xrightarrow{i^!} L_n^X(R, \alpha, u) \to L_n^{Y, X}(i^!) \to L_{n-1}^Y(S, \alpha, u) \to \ldots$$

Sometimes we shall simply write $L_n(i^!)$ if it is clear from the context what decorations are currently considered. The automorphism ϱ extends to the ring S by the formula $\varrho(x + yt) = t(x + yt)t^{-1}$, for every x and $y \in R$. So we obtain another quadratic extension of antistructures $\tilde{i}\colon (R, \tilde{\alpha}, \tilde{u}) \to (S, \tilde{\alpha}, \tilde{u})$. The morphism $\tilde{i}$ coincides with i as map of rings.

3. – Splitting problem.

Let $f : M \to Y$ be a simple homotopy equivalence between PL or smooth n-manifolds, and let $X \subset Y$ be a submanifold of codimension q. We say that the map f *splits along the submanifold* X if it is homotopic to a map g such that g is transversal to X, and the restrictions

$$g|_N\colon N \to X, \qquad g|_{M \setminus N}\colon M \setminus N \to Y \setminus X$$

are simple homotopy equivalences. In particular, $g^{-1}(X) = N \subset M$ is a codimension q submanifold. There exists a group $LS_{n-q}(F)$ of obstructions for splitting that depends functorially on the pushout square F of fundamental groups with orientation

$$(3.1)\qquad F = \begin{pmatrix} \pi_1(\partial U) & \to & \pi_1(Y \setminus X) \\ \downarrow & & \downarrow \\ \pi_1(X) & \to & \pi_1(Y) \end{pmatrix},$$

where U is a tubular neighborhood of X in Y, and on dimension $n-q$ (mod 4) (see [38] and [43]).

Let $f: M \to Y$ be only a normal map of degree 1. In this case, the groups $LP_{n-q}(F)$ of obstructions to surgery of manifold pairs were defined in [38] and [43]. The LP_n-groups depend functorially only on the square F and on the natural number n (mod 4).

If X is an one-sided submanifold of Y and the horizontal maps in the square F are isomorphisms, then the groups $LS_*(F)$ coincide with the Browder-Livesay groups $LN_*(\pi_1(Y\backslash X) \to \pi_1(Y))$ (see [3], [5], [17], [22] and [26]). Such pairs of manifolds are called *Browder-Livesay pairs*. We also have an isomorphism $LP_n(F) \cong L_{n+1}(i^!)$, where

$$i^!: L_{n+1}(\pi_1(X)) \to L_{n+1}(\pi_1(\partial U))$$

is a transfer map.

For any index 2 inclusion $\pi \to G$ of groups with orientation, the Browder-Livesay groups $LN_*(\pi \to G)$ are the L-groups of the ring with antistructure $(\mathbb{Z}[\pi], \alpha, u)$, where $\alpha(x) = t\overline{x}t^{-1}$, $u = -w(t)t^2$, $t \in G\backslash\pi$, and the bar denotes the standard involution of the ring $\mathbb{Z}[\pi]$.

The groups LS_* and LP_* are closely related to the Wall surgery obstruction groups $L_*(\pi_1(Y))$ and $L_*(\pi_1(X))$. A deep result, proved by Wall in [43], illustrates this connection by means of the following diagrams

$$\begin{array}{ccccccccccc}
\to L_{n+q+1}(C) & & \to & & L_{n+q+1}(D) & & \xrightarrow{\Theta} & & LS_n(F) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & LP_{n+1}(F) & & & & L_{n+q+1}(C \to D) & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to LS_{n+1}(F) & & \to & & L_{n+1}(B) & & \to & & L_{n+q}(C) & & \to
\end{array}$$

and

$$\begin{array}{ccccccccccc}
\to LN_{n-q}(A \to B) & & \to & & L_{n-q}(B) & & \to & & L_n(C \to D) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
& & LS_{n-q}(F) & & & & L_n(A \to B) & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
\to L_{n+1}(C \to D) & & \to & & L_{n+1}(F) & & \to & & LN_{n-q-1}(A \to B) & & \to
\end{array}$$

where $A = \pi_1(\partial U)$, $B = \pi_1(X)$, $C = \pi_1(Y\backslash X)$, and $D = \pi_1(Y)$.

Properties of maps in these diagrams are important for various geometric applications. For example, let Y be a closed manifold of dim $Y = n + q \geqslant 6$, such that there is an isomorphism of oriented groups $\pi_1(Y) \cong D$ (the fundamental group of the manifold is equipped by the orientation). Consider a submanifold $X \subset Y$ of codimension q. The Wall group $L_{n+q+1}(D)$ acts on the set $\mathcal{S}^s_{n+q}(Y)$ of homotopic triangulations of the manifold Y. Applying this action to the trivial triangulation and then taking the obstruction for splitting along the submanifold X yields the map $\Theta : L_{n+q+1}(D) \to LS_n(F)$. Hence the map Θ in the diagram is the composition of maps

$$L^s_{n+q+1}(D) \xrightarrow{w_{n+q}} \mathcal{S}^s_{n+q}(Y) \to LS_n(F),$$

where the first one arises from the Sullivan-Wall exact sequence described in Theorem 2.1. From this description it follows that if $\Theta(x) \neq 0$ for an element $x \in L^s_{n+q+1}(D)$, then x acts nontrivially on the set $\mathcal{S}^s_{n+q}(Y)$.

The following result was proved by Cappell and Shaneson in [5].

THEOREM 3.1. – *Let $C \to D$ be an inclusion of index 2 between oriented groups. If $\Theta(x) \neq 0$ for an element $x \in L_n(D)$, then x can not be realized by normal maps of closed manifolds, and it acts nontrivially on the set of homotopic triangulations $\mathcal{S}^s_n(Y)$ of any closed connected manifold Y with fundamental group $\pi_1(Y)$ isomorphic to D.*

A natural extension of the concept of Browder-Livesay pair of manifolds is given by a pair for which the horizontal maps in square (3.1) are epimorphisms. Such squares were studied in [1], [25], [28], [31], [34] and [35]. In this situation, square (3.1) is called a *geometric diagram.* A geometric example of such a square is obtained by setting $X = \mathbb{R}P^1 \times M \subset Y = \mathbb{R}P^2 \times M$, where M is a compact connected m-manifold with fundamental group π.

In particular, the groups LS_* and LP_* can be defined for the square of antistructures (see [28], [31], and [34]) in which the horizontal maps are epimorphisms and the vertical maps are quadratic extensions of antistructures [39]. Such squares of antistructures are a natural generalization of geometric diagrams of groups. For example the square of such type arises from a geometric diagram F by passing to the square of group rings with the standard involutions. The natural way to obtain such squares of antistructures is based on the concept of a *geometric antistructure* (see [20], [22], [23], and [32]). This is a group π with an additional structure which gives a possibility to introduce an antistructure on the group ring $R[\pi]$ in a natural way.

4. – LS_*- and LP_*-groups and K-theory.

In this section we study squares of antistructures which generalize the geometric diagrams of groups described in Section 3. In this case we introduce decorated LS_*- and LP_*-groups and obtain new relations among these groups and decorated L_*-groups. The definition of these groups was given in [28] and [34], where some preliminary results about them were obtained.

Let (R, α, u) and (P, β, v) be antistructures with structures (ϱ, a) and

(ϱ', a'), respectively. Consider the commutative diagram of antistructures

$$F = \begin{pmatrix} (R, \alpha, u) \xrightarrow{f} (P, \beta, v) \\ \downarrow^{i} \qquad\qquad \downarrow^{j} \\ (S, \alpha, u) \xrightarrow{g} (Q, \beta, v) \end{pmatrix} = \begin{pmatrix} R & \to & P \\ \downarrow & & \downarrow \\ S & \to & Q \end{pmatrix}, \tag{4.1}$$

where the horizontal maps are epimorphisms of the rings and the pair of vertical maps gives a quadratic extension of the morphism f (see [28] and [29] for details). In particular, we have $g(t) = t'$ for $t^2 = a$ and $t'^2 = a'$. Diagram (4.1) with these properties is called a *geometric diagram* since it is the natural extension of the diagram arising from the splitting problem for the one-sided submanifold.

In this section we shall sometimes write for simplicity only the underlying rings and we shall not mention antistructures unless this would lead to some confusion.

Using the automorphism γ and the quadratic extensions $\tilde{i}$ and $\tilde{j}$ we obtain three additional geometric diagrams

$$\tilde{F} = \begin{pmatrix} (R, \tilde{\alpha}, \tilde{u}) \xrightarrow{f} (P, \tilde{\beta}, \tilde{v}) \\ \downarrow^{\tilde{i}} \qquad\qquad \downarrow^{\tilde{j}} \\ (S, \tilde{\alpha}, \tilde{u}) \xrightarrow{\tilde{g}} (Q, \tilde{\beta}, \tilde{v}) \end{pmatrix} = \begin{pmatrix} \tilde{R} & \to & \tilde{P} \\ \downarrow & & \downarrow \\ \tilde{S} & \to & \tilde{Q} \end{pmatrix}, \tag{4.2}$$

$$\tilde{F}_\gamma = \begin{pmatrix} (R, \tilde{\alpha}, \tilde{u}) \xrightarrow{f} (P, \tilde{\beta}, \tilde{v}) \\ \downarrow^{\tilde{i}_\gamma} \qquad\qquad \downarrow^{\tilde{j}_\gamma} \\ (S, \gamma\tilde{\alpha}, \tilde{u}) \xrightarrow{\tilde{g}} (Q, \gamma\tilde{\beta}, \tilde{v}) \end{pmatrix} = \begin{pmatrix} \tilde{R} & \to & \tilde{P} \\ \downarrow & & \downarrow \\ \tilde{S}_\gamma & \to & \tilde{Q}_\gamma \end{pmatrix}, \tag{4.3}$$

and

$$F_\gamma = \begin{pmatrix} (R, \alpha, u) \xrightarrow{f} (P, \beta, v) \\ \downarrow^{i_\gamma} \qquad\qquad \downarrow^{j_\gamma} \\ (S, \gamma\alpha, u) \xrightarrow{g_\gamma} (Q, \gamma\beta, v) \end{pmatrix} = \begin{pmatrix} R & \to & P \\ \downarrow & & \downarrow \\ S_\gamma & \to & Q_\gamma \end{pmatrix}. \tag{4.4}$$

Consider a commutative diagram of K_1-groups induced by the square F, i.e.

$$K_1(F) = \begin{pmatrix} K_1(R) \xrightarrow{f_*} K_1(P) \\ \downarrow^{i_*} \qquad\qquad \downarrow^{j_*} \\ K_1(S) \xrightarrow{g_*} K_1(Q) \end{pmatrix}.$$

Note that the group $K_1(R)$ is equipped with the involutions induced by the anti-automorphisms α, ϱ, and $\tilde{\alpha}$. We denote these involutions by T, Ω, and $\tilde{T}$, respectively. Then, from the definition of the anti-automorphism $\widetilde{\alpha}$, we have $\widetilde{T} = \Omega \circ T$. The group $K_1(S)$ is equipped with the involutions induced by α, γ, and $\widetilde{\alpha}$ which we denote by T, Γ, and $\widetilde{T}$, respectively. It is clear from the context how to recognize the group on which the involution acts. In the group $K_1(S)$, we have $\widetilde{T} = \Gamma \circ T$. The situation for the groups $K_1(P)$ and $K_1(Q)$ is similar so we can denote the corresponding involutions by the same letters.

Consider now the commutative square of subgroups of appropriate K_1-groups

$$k = \begin{pmatrix} X & \xrightarrow{f_*} & Z \\ \downarrow i_* & & \downarrow j_* \\ Y & \xrightarrow{g_*} & W \end{pmatrix} \subset \begin{pmatrix} K_1(R) & \xrightarrow{f_*} & K_1(P) \\ \downarrow i_* & & \downarrow j_* \\ K_1(S) & \xrightarrow{g_*} & K_1(Q) \end{pmatrix} = K_1(F)\,. \tag{4.5}$$

All maps in the square k are obtained as the corresponding restrictions of the maps from the square $K_1(F)$. In what follows we shall assume that all groups in the square k are T- and $\widetilde{T}$-invariant. In addition, we recall the list of conditions for the square k:

$$\begin{gathered} i_*(X) \subset Y\,, \quad j_*(Z) \subset W\,, \quad f_*(X) \subset Z\,, \quad g_*(Y) \subset W\,, \\ i^!(Y) \subset X\,, \qquad j^!(W) \subset Z \end{gathered} \tag{4.6}$$

when the decorated groups $LS_n^k(F)$ and $LP_n^k(F)$ are defined (see [28] and [34]).

Denote by

$$\Psi = \begin{pmatrix} (R,\alpha,u) & \xrightarrow{Id} & (R,\alpha,u) \\ \downarrow i & & \downarrow i \\ (S,\alpha,u) & \xrightarrow{Id} & (S,\alpha,u) \end{pmatrix}, \qquad \Phi = \begin{pmatrix} (P,\beta,v) & \xrightarrow{Id} & (P,\beta,v) \\ \downarrow j & & \downarrow j \\ (Q,\beta,v) & \xrightarrow{Id} & (Q,\beta,v) \end{pmatrix}$$

the geometric diagrams which we can construct by using geometric diagram (4.1). Then there exist natural maps of these diagrams

$$\Psi \xrightarrow{\sigma} F \xrightarrow{\varepsilon} \Phi\,. \tag{4.7}$$

The squares of decorations k_Ψ and k_Φ for the diagrams Ψ and Φ are defined in a natural way. The columns of the square k_Ψ are isomorphic to the left column of the square k. The columns of the square k_Φ are isomorphic to the right column of the square k. These squares evidently satisfy conditions (4.6). We now recall some basic facts concerning spectra in L-theory (see for example [11], [21], [28], [29], [34], [38], and [43]). For any antistructure (R, α, u) and subgroup $X \subset K_1(R)$ invariant under the involution induced by α, a simplicial Ω-spectrum $\mathbb{L}^X(R, \alpha, u)$ is defined. It is natural with respect to both transfer maps and maps induced by morphisms of antistructures. For such a spectrum, we have

$$\pi_q(\mathbb{L}^X(R, \alpha, u)) = L_q^X(R, \alpha, u).$$

In particular, for any quadratic extension $i:(R, \alpha, u) \to (S, \alpha, u)$ of antistructures and for any subgroups of decorations $X \subset K_1(R)$ and $Y \subset K_1(S)$ such that $i_*(X) \subset Y$ and $i^!(Y) \subset X$, exact sequences (2.2) and (2.3) are homotopy long exact sequences of the corresponding fibrations of spectra $\boldsymbol{i}_*$ and $\boldsymbol{i}^!$. The cofiber of $\boldsymbol{i}_*$ is the spectrum $\mathbb{L}^{X,Y}(i)$ and the cofiber of $\boldsymbol{i}^!$ is the spectrum $\mathbb{L}^{Y,X}(i^!)$. Taking the $\mathbb{L}$-spectra of square (4.2) yields the following homotopy commutative diagram of spectra which is infinite in horizontal and vertical directions

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & & \mathbb{L}^X(\tilde{R}) & \to & \mathbb{L}^Z(\tilde{P}) & \to & \mathbb{L}^{X,Z}(\tilde{f}) \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & & \mathbb{L}^Y(\tilde{S}) & \to & \mathbb{L}^W(\tilde{Q}) & \to & \mathbb{L}^{Y,W}(\tilde{g}) \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & & \mathbb{L}^{X,Y}(\tilde{i}) & \to & \mathbb{L}^{Z,W}(\tilde{j}) & \to & \mathbb{L}^k(\tilde{F}) \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array} \tag{4.8}$$

The spectrum $\mathbb{L}S^k(F)$ is defined as the homotopy cofiber of one of the following maps arising from diagram (4.8) (see [28] for more details)

$$\Omega^2\mathbb{L}^k(\tilde{F}) \to \mathbb{L}^X(\tilde{R}), \quad \Omega\mathbb{L}^Y(\tilde{S}) \to \Omega\mathbb{L}^{Z,W}(\tilde{j}), \quad \Omega\mathbb{L}^Z(\tilde{P}) \to \Omega\mathbb{L}^{Y,W}(\tilde{g}). \tag{4.9}$$

Observe that there are isomorphisms $\pi_q(\mathbb{L}S^k(F)) \cong LS_q^k(F)$ for any q. Analogously, by using square (4.3) we can construct the homotopy commutative diagram of spectra

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & & \mathbb{L}^Y(S_\gamma) & \to & \mathbb{L}^W(Q_\gamma) & \to & \mathbb{L}^{Y,W}(g_\gamma) \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & & \mathbb{L}^X(R) & \to & \mathbb{L}^Z(P) & \to & \mathbb{L}^{X,Z}(f) \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & & \mathbb{L}^{Y,X}(i_\gamma^!) & \to & \mathbb{L}^{W,Z}(j_\gamma^!) & \to & \mathbb{L}^k(F_\gamma^!) \to \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array} \tag{4.10}$$

The spectrum $\mathbb{L}P^k(F)$ is defined as the homotopy fiber of one of the following maps arising from diagram (4.10) (see [34] for more details)

$$\Omega\mathbb{L}^{X,Z}(f)\to\mathbb{L}^{Y,X}(i^!_\gamma),\quad \mathbb{L}^Y(S_\gamma)\to\mathbb{L}^Z(P),\quad \Omega\mathbb{L}^{W,Z}(j^!_\gamma)\to\mathbb{L}^{Y,W}(g_\gamma). \tag{4.11}$$

Observe that there are isomorphisms $\pi_q(\mathbb{L}P^k(F))\cong LP^k_q(F)$ for any q.

THEOREM 4.1. – *Let F be a geometric diagram of antistructures and let k be the square of T and $\tilde{T}$ invariant subgroups of $K_1(F)$ with properties* (4.6). *There exist groups $LS^k_n(F)$ and $LP^k_n(F)$, where n is defined* mod 4, *fitting in the following braids of exact sequences*:

$$\begin{array}{ccccccccccccc}
\to & & L^k_{n+1}(\tilde S) & & \to & & L^{Z,W}_{n+2}(j) & & \to & & L^k_{n+2}(F) & & \to \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & L^{X,Y}_{n+2}(i) & & & & LS^k_n(F) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L^k_{n+3}(F) & & \to & & L^X_n(\tilde R) & & \to & & L^Y_n(\tilde S) & & \to,
\end{array} \tag{4.12}$$

$$\begin{array}{ccccccccccccc}
\to & & L^Y_{n+1}(\tilde S) & & \to & & L^{Z,W}_{n+2}(j) & & \to & & L^Z_n(\tilde P) & & \to \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & L^W_{n+1}(\tilde Q) & & & & LS^k_n(F) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L^Z_{n+1}(\tilde P) & & \to & & L^{Y,W}_{n+1}(\tilde g) & & \to & & L^Y_n(\tilde S) & & \to,
\end{array} \tag{4.13}$$

$$\begin{array}{ccccccccccccc}
\to & & L^Z_{n+1}(\tilde P) & & \to & & L^{Y,W}_{n+1}(\tilde g) & & \to & & L^k_{n+2}(F) & & \to \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & L^{X,Z}_{n+1}(\tilde f) & & & & LS^k_n(F) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L^k_{n+3}(F) & & \to & & L^X_n(\tilde R) & & \to & & L^Z_n(\tilde P) & & \to,
\end{array} \tag{4.14}$$

$$\begin{array}{ccccccccccccc}
\to & & L^Y_{n+1}(S_\gamma) & & \to & & L^Z_{n+1}(P) & & \to & & L^{X,Z}_{n+1}(f) & & \to \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & L^X_{n+1}(R) & & & & LP^k_n(F) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L^{X,Z}_{n+2}(f) & & \to & & L^{Y,X}_{n+1}(i^!_\gamma) & & \to & & L^Y_n(S_\gamma) & & \to,
\end{array} \tag{4.15}$$

$$\begin{array}{ccccccccccccc}
\to & & L^{W,Z}_{n+2}(j^!_\gamma) & & \to & & L^{Y,W}_{n+1}(g_\gamma) & & \to & & L^Y_n(S_\gamma) & & \to \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & & & L^W_{n+1}(Q_\gamma) & & & & LP^k_n(F) & & & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L^Y_{n+1}(S_\gamma) & & \to & & L^Z_{n+1}(P) & & \to & & L^{W,Z}_{n+1}(j^!_\gamma) & & \to,
\end{array} \tag{4.16}$$

$$
\begin{array}{ccccccccccccc}
\to & & L^{X,Z}_{n+2}(f) & & \to & & L^{Y,X}_{n+1}(i^!_\gamma) & & \to & & L^{W,Z}_{n+1}(j^!_\gamma) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & L^k_{n+2}(F^!_\gamma) & & & & LP^k_n(F) & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L^{W,Z}_{n+2}(j^!_\gamma) & & \to & & L^{Y,W}_{n+1}(g_\gamma) & & \to & & L^{X,Z}_{n+1}(f) & & \to,
\end{array}
\tag{4.17}
$$

$$
\begin{array}{ccccccccccccc}
\to & & L^{Z}_{n+1}(P) & & \to & & L^{W}_{n+1}(Q) & & \to & & LS^k_{n-1}(F) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & LP^k_n(F) & & & & L^{Z,W}_{n+1}(j) & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & LS^k_n(F) & & \to & & L^{Y}_{n}(S_\gamma) & & \to & & L^{Z}_{n}(P) & & \to,
\end{array}
\tag{4.18}
$$

and

$$
\begin{array}{ccccccccccccc}
\to & & L^{Y,W}_{n+1}(g_\gamma) & & \to & & LP^k_n(F) & & \to & & L^{W}_{n+1}(Q) & & \to \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
& & & & LS^k_n(F) & & & & L^{W,Z}_{n+1}(j^!_\gamma) & & & & \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\to & & L^{W}_{n+2}(Q) & & \to & & L^{Z}_{n}(\widetilde{P}) & & \to & & L^{Y,W}_{n}(g_\gamma) & & \to.
\end{array}
\tag{4.19}
$$

In addition, there exists the following commutative diagram of abelian groups with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & L^X_n(\widetilde{R}) & \to & L^{Y,X}_{n+1}(i^!_\gamma) & \to & L^Y_{n+1}(S) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & LS^k_n(F) & \to & LP^k_n(F) & \to & L^W_{n+1}(Q) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & L^k_{n+2}(F) & \to & L^{X,Z}_{n+1}(f) & \to & L^{Y,W}_{n+1}(g) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
\tag{4.20}
$$

PROOF. – Let us consider a natural map of cofibrations

$$
\begin{array}{ccccc}
\Omega^2\mathbb{L}^k(\widetilde{F}) & \to & \Omega\mathbb{L}^{X,Y}(\tilde{i}) & \to & \Omega\mathbb{L}^{Z,W}(\tilde{j}) \\
\| & & \downarrow & & \downarrow \\
\Omega^2\mathbb{L}^k(\widetilde{F}) & \to & \mathbb{L}^X(\widetilde{R}) & \to & \mathbb{L}S^k(F)
\end{array}
$$

where the vertical map on the right side exists, in order to obtain a homotopy commutative diagram, as shown in [42]. It follows from the diagram that the right square is in fact a push-out since the fibers (and, hence, the cofibers) of the horizontal maps are naturally homotopy equivalent. This implies that the fibers and the cofibers of the vertical maps are naturally homotopy equivalent, too. Now diagram (4.12) follows by considering homotopy long exact sequences of the maps in the right square of the diagram above if we use the following homotopy equivalences of spectra $\mathbb{L}^{X,Y}(\tilde{i}) \simeq \Omega\mathbb{L}^{X,Y}(i)$, $\mathbb{L}^{Z,W}(\tilde{j}) \simeq \Omega\mathbb{L}^{Z,W}(j)$, and $\mathbb{L}^{k}(\tilde{F}) \simeq \Omega\mathbb{L}^{k}(F)$ (see [28] and [39]). Diagrams (4.13) and (4.14) can be obtained similarly.

Diagrams (4.15), (4.16) and (4.17) can be treated in a similar way by using definition (4.11) and diagram (4.10) (see [10], [28] and [34] for more details).

Let us consider the following homotopy commutative diagram in which the horizontal rows are fibrations

$$\begin{array}{ccccc}
\Omega\mathbb{L}^{Z}(\tilde{S}) & \to & \Omega\mathbb{L}^{Z,W}(\tilde{j}) & \to & \mathbb{L}S^{k}(F) \\
\simeq\downarrow & & \downarrow & & \downarrow \\
\Omega\mathbb{L}^{Z}(S_{\gamma}) & \to & \Omega\mathbb{L}^{Z}(P) & \to & \mathbb{L}P^{k}(F).
\end{array}$$

The vertical map in the left side is a homotopy equivalence, and the commutativity of the left square follows from [39]. The vertical map in the right side exists according to [42]. Hence the right square is a push-out. Taking now the homotopy long exact sequences of this square yields diagram (4.18).

The natural map $\varepsilon : F \to \Phi$ in (4.7) induces the homotopy commutative square of spectra

$$\begin{array}{ccc}
\mathbb{L}S^{k}(F) & \to & \mathbb{L}P^{k}(F) \\
\downarrow & & \downarrow \\
\mathbb{L}S^{k_{\Phi}}(\Phi) & \to & \mathbb{L}P^{k_{\Phi}}(\Phi).
\end{array}$$

There exist homotopy equivalences $\mathbb{L}S^{k_{\Phi}}(\Phi) \simeq L^{Z}(\tilde{P})$ and $\mathbb{L}P^{k_{\Phi}}(\Phi) \simeq \Omega\mathbb{L}^{W,Z}(j_{\gamma}^{!})$ (see [29] and [39]). The cofibers of the horizontal maps are homotopy equivalent to $\Omega\mathbb{L}^{W}(Q)$, as follows from diagram (4.18). Hence the above diagram is a push-out. A cofiber of any vertical map is homotopy equivalent to $\mathbb{L}^{Y,W}(\tilde{g})$ by diagram (4.14). Passing to the homotopy long exact sequences of all maps in the diagram above, we get diagram (4.19). Commutative diagram (4.20) can be obtained similarly to (4.19) if we consider the map $\sigma : \Psi \to F$ in (4.7). This completes the proof. ■

Let X be a subgroup of $K_1(\mathbb{Z}[\pi])$ which is invariant under the involution induced by α. Then we have $LN_*^X(\pi \to G) \cong L_*^X(\mathbb{Z}[\pi], \alpha, -w(t)\, t^2)$, where

$t \in G \backslash \pi$ (see [17], [21], and [47]). In the considered case the groups which play the role of the LN-groups are given by $LS_n^{k_\Psi}(\Psi)$. In this case, we have $L_n^{k_\Psi}(\Psi) \cong 0$ for any n (see for example [38]). It follows from diagram (4.14) that there are isomorphisms $LS_n^{k_\Psi}(\Psi) \cong L_n^X(\widetilde{R})$ for any n (see [28]).

PROPOSITION 4.2. – *Let Ψ be the geometric diagram of antistructures, considered above, with the square of decorations k_Ψ satisfying conditions $i_*(X) \subset Y$ and $i^!(Y) \subset X$. Then there are natural isomorphisms*

$$LP_n^{k_\Psi}(\Psi) \cong L_{n+1}(i_\gamma^!)$$

for all n.

PROOF. – Let us consider diagram (4.17) for the decorated square Ψ. Since the groups $L_*^{X,X}(\mathrm{Id})$ are trivial, the result follows from one of the exact sequences of this diagram. ■

We now consider diagram (4.18) for the square Ψ. Using the isomorphisms obtained above, we can write it in the following way (see also [29] and [39])

$$\begin{array}{ccccccccccccc}
\to & L_n^X(R) & & \to & & L_n^Y(S) & & \to & & L_{n-2}^X(\widetilde{R}) & & \to & \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \\
 & & & L_n^{Y,X}(i_\gamma^!) & & \downarrow \Gamma & & L_n^{X,Y}(i) & & & & & \qquad (4.21)\\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \\
\to & L_{n-1}^X(\widetilde{R}) & & \to & & L_{n-1}^Y(S_\gamma) & & \to & & L_{n-1}^X(R) & & \to. &
\end{array}$$

Let us consider the square Ψ_γ with the same decorations k_Ψ. Then we have $(\Psi_\gamma)_\gamma = \Psi$, $(i_\gamma)_\gamma = i$, and $(S_\gamma)_\gamma = S$.

PROPOSITION 4.3. – *For every n there exist isomorphisms*

$$LS_n^{k_\Psi}(\Psi_\gamma) \cong L_{n+2}^X(\widetilde{R}).$$

PROOF. – Applying $\widetilde{\ }$ to the map $i_\gamma : (R, \alpha, u) \to (S, \gamma \circ \alpha, u)$ we get a new antistructure on the ring R, that is

$$(R, \widetilde{\alpha}, -\widetilde{u}) = (R, \varrho \circ \gamma \circ \gamma \circ \alpha, -t\gamma(\alpha(t^{-1}))u) = (R, \varrho \circ \alpha, t\alpha(t^{-1})u)$$

since $\gamma|_R$ is the identity map. In the considered case, diagram (4.14) yields the isomorphisms

$$LS_n^{k_\Psi}(\Psi_\gamma) \cong L_n^X(R, \widetilde{\alpha}, -\widetilde{u}) \cong L_{n+2}^X(R, \widetilde{\alpha}, \widetilde{u})$$

for all n. ■

By using the isomorphisms obtained above, we can write diagram (4.18) for the decorated square Ψ_γ in the following form:

$$
\begin{array}{ccccccccccccc}
\to & L_n^X(R) & & \to & & L_n^Y(S_\gamma) & & \to & & L_n^X(\widetilde{R}) & & \to & \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \\
& & & L_n^{Y,X}(i^!) & & \downarrow \Gamma & & L_n^{X,Y}(i_\gamma) & & & & & \\
\searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \\
\to & L_{n+1}^X(\widetilde{R}) & & \to & & L_{n-1}^Y(S) & & \to & & L_{n-1}^X(R) & & \to . &
\end{array}
\tag{4.22}
$$

Diagrams (4.21) and (4.22) are realized on the spectra level, i.e. to obtain them it suffices to write only the central squares on the spectra level. This allows us to construct a spectral sequence which generalizes the surgery spectral sequence obtained in [18]. It is necessary to remark, here, that the map Γ in diagrams (4.21) and (4.22) denotes the isomorphism between the homology groups of the corresponding members in the top and the bottom chain complexes.

THEOREM 4.4. – *For the square Ψ with decorations k_Ψ there exists a spectral sequence which does not depend on decoration Y provided $i_*(X) \subset Y$ and $i^!(Y) \subset X$. The first member is given by*

$$E_1^{p,q} \cong L_{q+2}^X(\widetilde{R}) = L_{q+2}^X(R, \widetilde{\alpha}, \widetilde{u}).$$

For any p, the differential map $d_1^{p,q}\colon E_1^{p,q} \to E_1^{p+1,q}$ is the composition

$$L_{q+2}^X(\widetilde{R}) \to L_{q+1+(-1)^p}^Y(S_{(\gamma^{p+1})}) \to L_{q+2}^X(\widetilde{R}).$$

For even p, the first map lies in diagram (4.21), *and the second one lies in diagram* (4.22). *For odd p, the first map lies in diagram* (4.22), *and the second one lies in diagram* (4.21).

We remark that, for any even p, the higher differentials

$$d_k\colon E_k^{p,q} \cong L_{q+2}^X(\widetilde{R}) \to E_k^{p+k,q+k-1} \cong L_{q+k+1}^X(\widetilde{R})$$

are given by the composition of the following maps

$$
\begin{array}{ccc}
L_{q+2}^X(\widetilde{R}) \to & L_{q+2}^Y(S_\gamma) & \\
& \downarrow \Gamma & \\
& L_{q+1}^Y(S) & \\
& \downarrow \Gamma & \\
& L_q^Y(S_\gamma) & \\
& \vdots & \\
& * & \to L_{q+k+1}^X(\widetilde{R}).
\end{array}
$$

Here in the place of $*$ we have the group $L^Y_{q-k+3}(S_{\gamma^k})$. For any odd p, the higher differentials are described in a similiar way.

PROOF OF THEOREM 4.4. – Diagrams (4.21) and (4.22) are generated by the push-out squares of the spectra (see also [29]) which can be written in the following form

$$\begin{tikzcd}[column sep=small]
& \mathbb{L}^Y(S) \arrow[dr] & \\
\mathbb{L}^{Y,X}(i^!_\gamma) \arrow[ur] \arrow[dr] & & \mathbb{L}^{X,Y}(i) \\
& \Sigma\mathbb{L}^Y(S_\gamma) \arrow[ur] \arrow[dr] & \\
\Sigma\mathbb{L}^{Y,X}(i^!) \arrow[ur] \arrow[dr] & & \Sigma\mathbb{L}^{X,Y}(i_\gamma) \\
& \Sigma^2\mathbb{L}^Y(S) \arrow[ur] & \\
& \cdots &
\end{tikzcd}$$

where Σ is a functor which is defined on each spectrum $\mathbb{A} = \{\mathbb{A}_n\}$ by the formula $(\Sigma\mathbb{A})_n = \mathbb{A}_{n+1}$. Continuing with this procedure, we obtain the vertical column of the push-out squares. We can now apply a method, which is due to Hambleton and Kharshiladze [18] (see also [10]), to construct a further spectral sequence. The description of $E_1^{p,q}$ and the differentials are obtained similarly as in [10] and [18]. From this description it follows that $E_1^{p,q} = L^X_{q+2}(R, \tilde{\alpha}, \tilde{u})$ does not depend on the decoration Y. It remains only to prove the independence of the differentials in the spectral sequence from the decoration Y. Let us consider the commutative diagram

$$\begin{tikzcd}
L^X_{q+2}(\tilde{R}) \arrow[r] \arrow[d, "c"'] & L^Y_{q+2}(S_\gamma) \arrow[r] & L^X_{q+2}(\tilde{R}) \\
L^{i_*(X)}_{q+2}(S_\gamma) \arrow[r, equal] & L^{i_*(X)}_{q+2}(S_\gamma) \arrow[r, equal] \arrow[u] & L^{i_*(X)}_{q+2}(S_\gamma) \arrow[u, "\partial"']
\end{tikzcd}$$

in which the middle vertical map exists, since $i_*(X) \subset Y$ by (4.6). For the diagram above it follows that the differential $d_1^{p,q}$ is the composition $\partial \circ c$ for any p even, and that it is independent of $Y \subset K_1(S)$. The other cases can be treated similarly by using the functorial properties of diagrams (4.21) and (4.22). Thus the proof is completed. ■

REMARK 4.23. – The composite map $\varepsilon \circ \sigma : \Psi \to \Phi$ induces a natural map between the geometric diagrams equipped with decorations k_Ψ and k_Φ, respectively. This map gives further maps (of vertical columns) between the corresponding push-out squares. Considering the cofibers of all these maps yields vertical columns of «relative» push-out squares which define a relative spectral sequence of the map $\varepsilon \circ \sigma$. In particular, if $\Psi = \Phi$ and $k_\Psi \neq k_\Phi$, then we have a spectral sequence in K-theory for a twisted quadratic extension studied in [10].

Let

$$F' = \begin{pmatrix} (R', \alpha', u') \to (P', \beta', v') \\ \downarrow i' \qquad\qquad\qquad \downarrow \\ (S', \alpha', u') \to (Q', \beta', v') \end{pmatrix}$$

be a further geometric diagram of antistructures, and let

$$k' = \begin{pmatrix} X' \to Z' \\ \downarrow \qquad \downarrow \\ Y' \to W' \end{pmatrix}$$

be a square of decorations which satisfy conditions (4.6). Let $\mathcal{L}: F \to F'$ be a morphism such that the induced map $\mathcal{L}_*: k \to k'$ is defined, i.e. there are inclusions $\mathcal{L}_*(X) \subset X'$, $\mathcal{L}_*(Y) \subset Y'$, $\mathcal{L}_*(Z) \subset Z'$, and $\mathcal{L}_*(W') \subset W$, and the resulting diagram (having the form of a cube) is commutative.

THEOREM 4.5. – *Under these hypotheses, the relative groups $LS_n(\mathcal{L})$ and $LP_n(\mathcal{L})$ (n mod 4) are defined. These groups fit into the following long exact sequences*

$$\cdots \to LS_n^k(F) \to LS_n^{k'}(F') \to LS_n(\mathcal{L}) \to LS_{n-1}^k(F) \to \cdots \tag{4.24}$$

and

$$\cdots \to LP_n^k(F) \to LP_n^{k'}(F') \to LP_n(\mathcal{L}) \to LP_{n-1}^k(F) \to \dots \tag{4.25}$$

PROOF. – Let us denote by $\mathbb{L}P(\mathcal{L})$ the cofiber of the natural map $\mathbb{L}P^k(F) \to \mathbb{L}P^{k'}(F')$ (see [42]), and set $LP_n(\mathcal{L}) := \pi_n(\mathbb{L}P(\mathcal{L}))$. The homotopy long exact sequence of the obtained cofibration gives exact sequence (4.25). The case of LS-groups can be treated similarly (compare also with [28]). So the result is completely proved. ■

In particular, if $F = F'$ and $k \neq k'$, then exact sequences (4.24) and (4.25) represent the analogues of the Rothenberg exact sequence, where the relative groups are isomorphic to the Tate cohomology groups. This case was treated in [28] and [34].

So we can write for $\mathcal{L}: F \to F$ and $k \neq k'$ exact sequences (4.24) and (4.25) in the following form

$$\cdots \to LS_n^k(F) \to LS_n^{k'}(F) \to H^n(LS(k'/k)) \to LS_{n-1}^k(F) \to \dots \tag{4.26}$$

and

$$(4.27)\quad \cdots \to LP_n^k(F) \to LP_n^{k'}(F) \to H^n(LP(k'/k)) \to LP_{n-1}^k(F) \to \dots$$

where $H^n(LS(k'/k))$ and $H^n(LP(k'/k))$ are the Tate cohomology groups (compare with exact sequence (2.1)). For two different involutions T and $\widetilde{T}$, we shall write $H^*(A)$ if A is equipped with the involution T, and $H^*(\widetilde{A})$ if A is equipped with the involution $\widetilde{T}$.

If $f:(R, \alpha, u) \to (P, \beta, v)$ is a morphism between antistructures, and there exist two possible decorations $X \subset X' \subset K_1(R)$ and $Z \subset Z' \subset K_1(P)$, then we denote by $H^*(X'/X \to Z'/Z)$ the Tate cohomology groups which fit into the following exact sequences

$$\cdots \to L_n^{X,Z}(f) \to L_n^{X',Z'}(f) \to H^n(X'/X \to Z'/Z) \to \dots$$

and

$$\cdots \to H^n(X'/X) \to H^n(Z'/Z) \to H^n(X'/X \to Z'/Z) \to \dots$$

and so on, similarly for other morphisms (see [38] for details). In the following theorem we shall denote by $H^n(LS)$ the groups $H^n(LS(k'/k))$ and by $H^n(LP)$ the groups $H^n(LP(k'/k))$.

THEOREM 4.6. – *Under the above hypotheses, the Tate cohomology groups $H^n(LS)$ and $H^n(LP)$ fit into the following braids of exact sequences*:

$$(4.28)\quad \begin{array}{ccccccccc}
\to H^{n+1}(\widetilde{Y'/Y}) & & \to & & H^{n+2}(Z'/Z\to W'/W) & & \to & & H^{n+2}(k'/k)\to \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
 & & H^{n+2}(X'/X\to Y'/Y) & & & & H^n(LS) & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
\to H^{n+3}(k'/k) & & \to & & H^n(\widetilde{X'/X}) & & \to & & H^n(\widetilde{Y'/Y}) \to,
\end{array}$$

$$(4.29)\quad \begin{array}{ccccccccc}
\to H^{n+1}(\widetilde{Y'/Y}) & & \to & & H^{n+2}(Z'/Z\to W'/W) & & \to & & H^n(\widetilde{Z'/Z}) \to \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
 & & H^{n+1}(\widetilde{W'/W}) & & & & H^n(LS) & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
\to H^{n+1}(\widetilde{Z'/Z}) & & \to & & H^{n+1}(\widetilde{Y'/Y}\to \widetilde{W'/W}) & & \to & & H^n(\widetilde{Y'/Y}) \to,
\end{array}$$

$$(4.30)\quad \begin{array}{ccccccccc}
\to H^{n+1}(\widetilde{Z'/Z}) & & \to & & H^{n+1}(\widetilde{Y'/Y}\to \widetilde{W'/W}) & & \to & & H^{n+2}(k'/k)\to \\
\nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \\
 & & H^{n+1}(\widetilde{H'/X}\to \widetilde{Z'/Z}) & & & & H^n(LS) & & \\
\searrow & \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
\to H^{n+3}(k'/k) & & \to & & H^n(\widetilde{X'/X}) & & \to & & H^n(Z'/Z) \to,
\end{array}$$

(4.31)

$$
\begin{array}{ccccccccccc}
\to & H^{n+1}(\widetilde{Y'/Y}) & & \to & & H^{n+1}(Z'/Z) & & \to & & H^{n+1}(X'/X\to Z'/Z) & \to \\
& \nearrow \quad \searrow & & & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow & \\
& & & H^{n+1}(X'/X) & & & & H^{n}(LP) & & & \\
& \searrow \quad \nearrow & & & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & \\
\to & H^{n+2}(X'/X\to Z'/Z) & & \to & & H^{n+1}(\widetilde{Y'/Y}\to X'/X) & & \to & & H^{n}(Y'/Y) & \to,
\end{array}
$$

(4.32)

$$
\begin{array}{ccccccccccc}
\to & H^{n+2}(\widetilde{W'/W}\to Z'/Z) & & \to & & H^{n+1}(Y'/Y\to W'/W) & & \to & & H^{n}(\widetilde{Y'/Y}) & \to \\
& \nearrow \quad \searrow & & & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow & \\
& & & H^{n+1}(\widetilde{W'/W}) & & & & H^{n}(LP) & & & \\
& \searrow \quad \nearrow & & & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & \\
\to & H^{n+1}(\widetilde{Y'/Y}) & & \to & & H^{n+1}(Z'/Z) & & \to & & H^{n+1}(\widetilde{Y'/Y}\to Z'/Z) & \to,
\end{array}
$$

(4.33)

$$
\begin{array}{ccccccccccc}
\to & H^{n+2}(X'/X\to Z'/Z) & & \to & & H^{n+1}(\widetilde{Y'/Y}\to X'/X) & & \to & & H^{n+1}(\widetilde{W'/W}\to Z'/Z) & \to \\
& \nearrow \quad \searrow & & & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow & \\
& & & H^{n+2}((k'/k)^{!}_{\gamma}) & & & & H^{n}(LP) & & & \\
& \searrow \quad \nearrow & & & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & \\
\to & H^{n+2}(\widetilde{W'/W}\to Z'/Z) & & \to & & H^{n+1}(\widetilde{Y'/Y}\to \widetilde{W'/W}) & & \to & & H^{n+1}(X'/X\to Z'/Z) & \to,
\end{array}
$$

$$
(4.34) \qquad
\begin{array}{ccccccccccc}
\to & H^{n+1}(Z'/Z) & & \to & & H^{n+1}(W'/W) & & \to & & H^{n-1}(LS) & \to \\
& \nearrow \quad \searrow & & & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow & \\
& & & H^{n}(LP) & & & & H^{n+1}(Z'/Z\to W'/W) & & & \\
& \searrow \quad \nearrow & & & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & \\
\to & H^{n}(LS) & & \to & & H^{n}(\widetilde{Y'/Y}) & & \to & & H^{n}(Z'/Z) & \to,
\end{array}
$$

and

(4.35)

$$
\begin{array}{ccccccccccc}
\to & H^{n+1}(\widetilde{Y'/Y}\to \widetilde{W'/W}) & & \to & & H^{n}(LP) & & \to & & H^{n+1}(W'/W) & \to \\
& \nearrow \quad \searrow & & & & \nearrow \quad \searrow & & & & \nearrow \quad \searrow & \\
& & & H^{n}(LS) & & & & H^{n+1}(\widetilde{W'/W}\to Z'/Z) & & & \\
& \searrow \quad \nearrow & & & & \searrow \quad \nearrow & & & & \searrow \quad \nearrow & \\
\to & H^{n+2}(W'/W) & & \to & & H^{n}(\widetilde{Z'/Z}) & & \to & & H^{n}(\widetilde{Y'/Y}\to \widetilde{W'/W}) & \to.
\end{array}
$$

Furthermore, there exists the following commutative diagram of abelian groups with exact rows and columns

$$
\begin{array}{ccccccc}
\cdots \to & H^{n}(\widetilde{X'/X}) & \to & H^{n+1}(\widetilde{Y'/Y}\to X'/X) & \to & H^{n+1}(Y'/Y)\cdots \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \to & H^{n}(LS) & \to & H^{n}(LP) & \to & H^{n+1}(W'/W)\cdots \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \to & H^{n+2}(k'/k) & \to & H^{n+1}(X'/X\to Z'/Z) & \to & H^{n+1}(Y'/Y\to W'/W)\cdots
\end{array}
$$

PROOF. – We consider diagrams (4.12)-(4.20) on the spectra level for the squares F^k and $F^{k'}$. Here F^k (resp. $F^{k'}$) denotes the square F decorated by k (resp. k'). For each such diagram, it suffices to consider only the central square of spectra as done in the proof of Theorem 4.4. Now the map $\mathcal{L}: F^k \to F^{k'}$ induces the natural map between these push-out squares. The cofibers of the induced maps give rise to a new push-out square. The homotopy long exact sequences of the obtained squares yield diagrams (4.28)-(4.36) from the statement of the theorem. ■

We now describe the relative decorated LP_*-groups of the maps σ and ε from sequence (4.7). The corresponding result for decorated LS_*-groups can be found in [28], i.e. there are isomorphisms $LS_n(\varepsilon) \cong L_n^{Y,W}(\tilde{g})$ and $LS_n(\sigma) \cong L_{n+2}^k(F)$, where the subscript indices are taken mod 4.

THEOREM 4.7. – *Under these hypotheses, we have isomorphisms* $LP_n(\sigma) \cong L_{n+1}^{X,Z}(f)$ *and* $LP_n(\varepsilon) \cong L_n^{Y,W}(g_\gamma)$, *where the subscript indices are taken mod* 4.

PROOF. – The results follow from the commutative diagrams obtained from the maps of diagram (4.18) written for the squares Ψ, F, and Φ, respectively. ■

Let k_1 be the square of decorations k in which $Z = X$. Another interesting result can be obtained by considering the geometric diagram Ψ with the square of decorations k_1. Of course, we assume that all decorations satisfy conditions (4.6).

PROPOSITION 4.8. – *Under the hypotheses above, we have isomorphisms* $LS_n^{k_1}(\Psi) \cong L_n^X(\tilde{R})$ *and* $LP_n^{k_1}(\Psi) \cong L_{n+1}^{Y,X}(i_\gamma^!)$, *where the subscript indices are taken mod* 4.

PROOF. – The result for LP_*-groups follows directly from an exact sequence in diagram (4.15) since the groups $L_n^{X,X}(\mathrm{Id})$ are trivial. To obtain the result for LS_*-groups, we use diagram (4.14). ■

Let now consider the same square Ψ equipped with decorations $k = k_2$ (with $W = Y$) which satisfy conditions (4.6). Then we have the following result.

PROPOSITION 4.9. – *Under the hypotheses above, we have isomorphisms* $LS_n^{k_2}(\Psi) \cong L_n^Z(\tilde{R})$ *and* $LP_n^{k_2}(\Psi) \cong L_{n+1}^{Y,Z}(j_\gamma^!)$, *where the subscript indices are taken mod* 4.

PROOF. – The proof is similar to that of Proposition 4.8, and it can be obtained by considering diagrams (4.13) and (4.17). ■

5. – The groups LS^Y and LP^Y of geometric diagrams of groups.

In a series of fundamental papers on the classification of Hermitian forms, Wall developed some methods which produce deep results on computations of Wall and Browder-Livesay groups, and on descriptions of natural maps in L-theory (see [19], [23], [29], [31], [33], and [47]). For a finite group π, the group $L_*^K(\widehat{\mathbb{Z}}_2[\pi])$, where $K = K_1(\widehat{\mathbb{Z}}_2[\pi])$, was first computed. Recall, here, that U is the subgroup of $K_1(\mathbb{Z}[\pi])$ generated by the images of the elements $\pm g$ from the group π, $X = SK_1(\mathbb{Z}[\pi]) = \mathrm{Ker}\,(K_1(\mathbb{Z}[\pi]) \to K_1(\mathbb{Q}[\pi]))$, and $Y = X + U$. We shall denote by U, X, and Y the subgroups of $K = K_1(\widehat{\mathbb{Z}}_2[\pi])$ defined similarly to the case of $K_1(\mathbb{Z}[\pi])$ in Section 2 (see [47]). Then the Rothenberg exact sequence for the group ring $\widehat{\mathbb{Z}}_2[\pi]$ and decorations $Y \subset K$ allows us to compute the groups $L_*^Y(\widehat{\mathbb{Z}}_2[\pi])$. Moreover, for a finite 2-group π there exist isomorphisms $L_n^K(\widehat{\mathbb{Z}}_2[\pi]) \cong \mathbb{Z}_2$ for any $n \pmod 4$. At the final step of computation it was used the relative long exact sequence

$$\to L_n^Y(\mathbb{Z}[\pi]) \to L_n^Y(\widehat{\mathbb{Z}}_2[\pi]) \to L_n^Y(\mathbb{Z}[\pi] \to \widehat{\mathbb{Z}}_2[\pi]) \to L_{n-1}^Y(\mathbb{Z}[\pi]) \to, \tag{5.1}$$

where $L_n^Y(\mathbb{Z}[\pi] \to \widehat{\mathbb{Z}}_2[\pi]) \cong L_n^X(\mathbb{Z}[\pi] \to \widehat{\mathbb{Z}}_2[\pi])$ [47].

Hereafter we shall develop the Wall methods to introduce and study the decorated groups LS^Y and LP^Y of geometric diagrams of groups (see [20], [22], [23], and [32]). First we recall some necessary definitions. A *geometric antistructure* on a group π is a 4-tuple (π, w, θ, b), where $w : \pi \to \{\pm 1\}$ is an orientation homomorphism, $\theta \in \mathrm{Aut}(\pi)$, and $b \in \pi$ such that

$$\begin{aligned} &(i) \quad w\theta(g) = w(g), \text{ for every } g \in \pi, \\ &(ii) \quad \theta^2(g) = bgb^{-1}, \text{ for every } g \in \pi, \\ &(iii) \quad \theta(b) = b, \text{ and } w(b) = 1. \end{aligned} \tag{5.2}$$

For any ring R with involution and invertible element (orientation) $\varepsilon \in R^*$, such that $\bar{\varepsilon} = \varepsilon^{-1}$, we can define an associated antistructure $(R[\pi], \alpha, u)$. Here the anti-automorphism α is defined by

$$\alpha : \Sigma r_g g \to \Sigma \bar{r}_g w(g)\, \theta(g^{-1})$$

and we have $u = \varepsilon b$ as unit. In what follows we shall suppose that the involution on the ring R is trivial and $\varepsilon = \pm 1$. Whenever this does not lead to any confusion we shall denote the oriented geometric antistructure $(\pi, w, \theta, b, \varepsilon)$ by (π, ε). It is necessary to remark, here, that the group

π with orientation homomorphism $w : \pi \to \{\pm 1\}$ gives the geometric antistructure (π, w, θ, e), where θ is the identity map, and e is the unity element.

A *morphism of oriented geometric antistructures* $f : (\pi, \varepsilon) \to (\pi', \varepsilon')$ is defined as a homomorphism $f : \pi \to \pi'$ such that

$$(i) \qquad w' f(g) = w(g), \text{ for every } g \in \pi,$$

$$(ii) \qquad \theta' f(g) = f\theta(g), \text{ for every } g \in \pi,$$

$$(iii) \qquad f(b) = b', \text{ and } \varepsilon = \varepsilon' \circ f.$$

The map f induces the morphism of the associated antistructures which we denote by f, too. The inclusion $i : (\pi, w', \theta', b', \varepsilon') \to (G, w, \theta, b, \varepsilon)$ of index 2 of oriented geometric antistructures is defined in a natural way as a morphism for which the underlying map $i : \pi \to G$ is the inclusion of index 2. Then the group π is identified with a subgroup of G, and hence we have $b = b'$, $w' = w|_\pi$, $\varepsilon = \varepsilon'$. We can write such an inclusion as $(\pi \to G, w, \theta, b, \varepsilon)$. An element $t \in G \backslash \pi$ defines the quadratic extension of the associated antistructures $(R[\pi], \alpha, u) \to (R[G], \alpha, u)$ with respect to the structure (ϱ, a), where $u = \varepsilon b$, $\alpha(\Sigma r_g g) = \Sigma w(g)\, r_g\, \theta(g^{-1})$, $a = t^2$, $\varrho(\Sigma r_g g) = \Sigma r_g\, tgt^{-1}$, for any $r_g \in R$ and $g \in G$ (see [20] and [32] for details).

A *geometric diagram of groups* is a commutative square of oriented geometric antistructures

$$F = \begin{pmatrix} (\pi, w, \theta, b, \varepsilon) \to (\varrho, w', \theta', b', \varepsilon') \\ \downarrow \qquad\qquad\qquad \downarrow \\ (G, w, \theta, b, \varepsilon) \to (H, w', \theta', b', \varepsilon') \end{pmatrix} = \begin{pmatrix} \pi & \xrightarrow{f} & \varrho \\ \downarrow i & & \downarrow j \\ G & \xrightarrow{g} & H \end{pmatrix} \tag{5.3}$$

where the vertical maps are inclusions of index 2 of geometric antistructures, and the horizontal maps $\pi \to \varrho$ and $G \to H$ are epimorphisms.

In what follows we shall sometimes write π instead of $(\pi, w, \theta, b, \varepsilon)$ if this does not lead to any confusion. We shall use analogous agreements for the maps and squares of geometric antistructures.

For a ring T, the square F defines the *geometric diagram of antistructures*

$$\Phi = \begin{pmatrix} (R, \alpha, u) & \xrightarrow{f} & (P, \beta, v) \\ \downarrow i & & \downarrow j \\ (S, \alpha, u) & \xrightarrow{g} & (Q, \beta, v) \end{pmatrix} = \begin{pmatrix} R & \to & P \\ \downarrow & & \downarrow \\ S & \to & Q \end{pmatrix} \tag{5.4}$$

where all rings $R = T\pi$, $P = T\varrho$, $S = TG$, and $Q = TH$ are equipped with the associated antistructures. In diagram (5.4) the vertical maps are quadratic extensions of antistructures and the horizontal maps are epimorphisms of antistructures. Hence for square (4.5) of decorations $k \subset K_1(\Phi)$ satisfying conditions (4.6) we can define the decorated LS and LP-groups of the geometric diagram of groups in (5.3) as follows:

$$LS_n^k(F) \overset{\text{def}}{=} LS_n^k(\Phi) \quad \text{and} \quad LP_n^k(F) \overset{\text{def}}{=} LP_n^k(\Phi).$$

By definition, these groups satisfy all properties of the LS and LP-groups of the corresponding geometric diagram of antistructures described in Section 4.

For $T = \widehat{\mathbb{Z}}_2$ we denote by $\widehat{\mathbb{F}}_2$ the geometric diagram of associated antistructures shown in (5.4), and for $T = \mathbb{Z}$ we denote the corresponding geometric diagram by $\mathbb{F}$. Let K be the square of decorations

$$K = K_1(\widehat{\mathbb{F}}_2) = \begin{pmatrix} K_1(\widehat{\mathbb{Z}}_2[\pi]) \to K_1(\widehat{\mathbb{Z}}_2[\varrho]) \\ \downarrow \qquad\qquad\qquad \downarrow \\ K_1(\widehat{\mathbb{Z}}_2[G]) \to K_1(\widehat{\mathbb{Z}}_2[H]) \end{pmatrix}$$

for the square $\widehat{\mathbb{F}}_2$.

THEOREM 5.1. – *Suppose that the groups in the square F are finite 2-groups. Then we have the following isomorphisms (subscript indices mod 4)*

$$LS_n^K(\widehat{\mathbb{Z}}_2[F]) \cong \mathbb{Z}_2 \quad \textit{and} \quad LP_n^K(\widehat{\mathbb{Z}}_2[F]) \cong L_{n+1}^K(j_\gamma^!) \cong L_{n+1}^K(i_\gamma^!) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

PROOF. – For any morphism of geometric antistructures the induced map of groups L^K is an isomorphism, and hence the corresponding relative groups are trivial (see for example [47]). This implies the first isomorphism of the theorem. Finally, we obtain the other isomorphisms by diagram (4.18) ■

The square of decorations $Y \subset K = K_1(\widehat{\mathbb{F}}_2)$, for which conditions (4.6) are satisfied, is defined in a natural way. Hence the groups $LS_n^Y(\widehat{\mathbb{F}}_2)$ and $LP_n^Y(\widehat{\mathbb{F}}_2)$ are defined for all n. Applying Theorem 5.1 we can write exact sequences (4.26) and (4.27) in the following forms

$$(5.5) \qquad \cdots \to LS_n^Y(\widehat{\mathbb{F}}_2) \xrightarrow{\nu_s} \mathbb{Z}_2 \to H^n(LS(K/Y)) \to LS_{n-1}^Y(\widehat{\mathbb{F}}_2) \to \ldots$$

and

$$(5.6) \qquad \ldots \to LP_n^Y(\widehat{\mathbb{F}}_2) \xrightarrow{\nu_p} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to H^n(LP(K/Y)) \to LP_{n-1}^Y(\widehat{\mathbb{F}}_2) \to \ldots$$

We now describe how to reduce computations of the maps ν_s and ν_p to previous results proved by Wall in [47]. Assume that all groups in diagram (5.3) are finite 2-groups. Then for $T = \widehat{\mathbb{Z}}_2$ and the associated geometric diagram of antistructures in (5.4) Φ we have the following result.

THEOREM 5.2. – *If one of the maps*

$$L_n^Y(\widetilde{\widehat{\mathbb{Z}}_2[G]}) \to L_n^K(\widetilde{\widehat{\mathbb{Z}}_2[G]}) \cong \mathbb{Z}_2 \quad or \quad L_n^Y(\widetilde{\widehat{\mathbb{Z}}_2[\varrho]}) \to L_n^K(\widetilde{\widehat{\mathbb{Z}}_2[\varrho]}) \cong \mathbb{Z}_2$$

is trivial, then the map ν_s *in* (5.5) *is trivial.*

If one of the maps

$$L_n^Y(\widetilde{\mathbb{Z}_2[\pi]}) \to L_n^K(\widetilde{\widehat{\mathbb{Z}}_2[\pi]}) \cong \mathbb{Z}_2 \quad or \quad L_{n+2}^Y(\widehat{\mathbb{Z}}_2[H]) \to L_n^K(\widehat{\mathbb{Z}}_2[H]) \cong \mathbb{Z}_2$$

is an epimorphism, then the map ν_s *in* (5.5) *is an epimorphism.*

PROOF. – The results follow by considering natural maps from diagrams (4.12) and (4.19) for the square $\widehat{\mathbb{F}}_2$ with decorations Y to the similar diagrams with decorations K. ■

Diagram (4.18) gives a natural decomposition of the group $LP_n^K(\widehat{\mathbb{F}}_2)$ into a direct sum of two groups, each of them isomorphic to $\mathbb{Z}_2$, i.e.

$$LP_n^K(\widehat{\mathbb{F}}_2) \cong LS_n^K(\widehat{\mathbb{F}}_2) \oplus LS_n^K(\widehat{\mathbb{Z}}_2[H]) \cong LS_n^K(\widehat{\mathbb{Z}}_2[\varrho]) \oplus LS_n^K(\widetilde{\widehat{\mathbb{Z}}_2[G]}).$$

Now the results of [47], Theorems 4.1 and 5.2 give us the possibility of computing the map ν_p in the long exact sequence (5.6). To describe the Tate cohomology groups $H^n(LS(K/Y))$ and $H^n(LP(K/Y))$ in the exact sequences (5.5) and (5.6) we can use the diagrams of Theorem 4.6. In order to use the Wall methods for computing the groups $LS_n^Y(\mathbb{F})$ and $LP_n^Y(\mathbb{F})$ we must now determine the relative members $LS_n^Y(\mathbb{F} \to \widehat{\mathbb{F}}_2)$ and $LP_n^Y(\mathbb{F} \to \widehat{\mathbb{F}}_2)$ which fit in the following long exact sequences:

$$\cdots \to LS_n^Y(\mathbb{F}) \to LS_n^Y(\widehat{\mathbb{F}}_2) \to LS_n^Y(\mathbb{F} \to \widehat{\mathbb{F}}_2) \to \cdots \tag{5.7}$$

and

$$\cdots \to LP_n^Y(\mathbb{F}) \to LP_n^Y(\widehat{\mathbb{F}}_2) \to LP_n^Y(\mathbb{F} \to \widehat{\mathbb{F}}_2) \to \cdots \tag{5.8}$$

Let us consider, in addition, the square of decorations $X = SK_1(\mathbb{F})$. It is clear that the decorated groups $LS_n^X(\mathbb{F})$ and $LP_n^X(\mathbb{F})$ are well defined. Following [47], we denote similarly by X the square of decorations $SK_1(\widehat{\mathbb{F}}_2)$. Then the groups $LS_n^X(\widehat{\mathbb{F}}_2)$ and $LP_n^X(\widehat{\mathbb{F}}_2)$ are well defined. For the groups decorated by X there exist relative exact sequences similar to (5.7) and (5.8).

THEOREM 5.3. – *Under the hypotheses above, we have isomorphisms*

$$LS_n^X(\mathbb{F} \to \widehat{\mathbb{F}}_2) \cong LS_n^Y(\mathbb{F} \to \widehat{\mathbb{F}}_2) \quad and \quad LP_n^X(\mathbb{F} \to \widehat{\mathbb{F}}_2) \cong LP_n^Y(\mathbb{F} \to \widehat{\mathbb{F}}_2).$$

PROOF. – It follows from Theorem 4.6 and [47] that relative Tate cohomology groups $H^n_{\mathrm{rel}}(LS(Y/X))$ fitting in the following commutative diagram

$$\to LS^X_n(\mathbb{F}\to\widehat{\mathbb{F}}_2)\to LS^Y_n(\mathbb{F}\to\widehat{\mathbb{F}}_2)\to H^n_{\mathrm{rel}}(LS(Y/X))\to$$

are trivial. The case of the groups LP can be treated similarly. ■

Now we describe how to reduce the computation of relative members $LS^X_n(\mathbb{F}\to\widehat{\mathbb{F}}_2)$ and $LP^X_n(\mathbb{F}\to\widehat{\mathbb{F}}_2)$ to the corresponding results of Wall [47]. The group ring $\mathbb{Q}[\pi]$ splits into a direct sum of simple algebras

$$\mathbb{Q}[\pi]=\prod A_\phi,$$

where $A_\phi=a_\phi(\mathbb{Q}[\pi])$ for a central idempotent a_ϕ, $1=\sum a_\phi$, and each a_ϕ is not expressed as a sum of nontrivial central idempotents (see [22], [23], and [47]).

This decomposition induces a decomposition of the associated antistructure

$$(\mathbb{Q}[\pi],\,\alpha,\,u)=\prod_{\phi=\alpha(\phi)}(A_\phi,\,\alpha_\phi,\,u_\phi)\times\prod_{\phi\cong\alpha(\phi)}(A_\phi\times A_{\alpha(\phi)},\,\alpha_{\phi\times\alpha(\phi)},\,u_\phi\times u_{\alpha(\phi)}),$$

where $a_{\alpha(\psi)}$ denotes $\alpha(a_\psi)$. We have similar decompositions

$$(\mathbb{Z}[1/2][\pi],\,\alpha,\,u)=\prod_{\phi=\alpha(\phi)}(\Lambda_\phi,\,\alpha_\phi,\,u_\phi)\times\prod_{\phi\cong\alpha(\phi)}(\Lambda_\phi\times\Lambda_{\alpha(\phi)},\,\alpha_{\phi\times\alpha(\phi)},\,u_\phi\times u_{\alpha(\phi)}),$$

$$(\widehat{\mathbb{Q}}_2[\pi],\,\alpha,\,u)=\prod_{\phi=\alpha(\phi)}(\widehat{\Lambda_{\phi_2}},\,\alpha_\phi,\,u_\phi)\times\prod_{\phi\cong\alpha(\phi)}(\widehat{\Lambda_{\phi_2}}\times\widehat{\Lambda_{\alpha(\phi)_2}},\,\alpha_{\phi\times\alpha(\phi)},\,u_\phi\times u_{\alpha(\phi)}).$$

THEOREM 5.4. – [22] *For any geometric antistructure $(\pi,\,w,\,\theta,\,b,\,\varepsilon)$ we have the following decomposition of the relative $L^Y\cong L^X$-groups with the associated antistructure*

$$L^X_n(\mathbb{Z}[\pi]\to\widehat{\mathbb{Z}}_2[\pi],\,\alpha,\,u)\xrightarrow{\cong}\prod_{\phi=\alpha(\phi)}L^S_n(\Lambda_\phi\to\widehat{\Lambda_{\phi_2}},\,\alpha_\phi,\,u_\phi),$$

where the decorations S denote the trivial subgroups of the corresponding groups K_1.

Let $g:(G,\,w,\,\theta,\,b,\,\varepsilon)\to(H,\,w',\,\theta',\,b',\,\varepsilon')$ be the epimorphism of geometric antistructures, where the finite 2-groups G and H arise from square (5.3).

THEOREM 5.5. – *The epimorphism g of geometric antistructures induces the splitting projections*

$$g_*:\,L^X_n(\mathbb{Z}[G]\to\widehat{\mathbb{Z}}_2[G])\to L^X_n(\mathbb{Z}[H]\to\widehat{\mathbb{Z}}_2[H])$$

for every $n=0,\,1,\,2,\,3\ \mathrm{mod}\,4$.

PROOF. – The statement follows from the natural decompositions given by Theorem 5.4, and by using results proved in [20] and [21]. ■

The results of [47] give a complete description of the relative groups $L_n^X(\mathbb{Z}[H] \to \widehat{\mathbb{Z}}_2[H])$ and $L_n^X(\mathbb{Z}[G] \to \widehat{\mathbb{Z}}_2[G])$ in the considered case. So Theorem 5.5 gives the groups $L_{n+1}^{X(\mathrm{rel})}(g)$ for any epimorphism $g\colon G \to H$ of geometric antistructures in the case of finite 2-groups G and H.

THEOREM 5.6. – *Let F be a square of geometric antistructures with finite 2-groups. Then there are isomorphisms*

$$LS_n^X(\mathbb{F} \to \widehat{\mathbb{F}}_2) \cong L_{n+1}^{X(\mathrm{rel})}(g^-) \oplus L_n^X(\widetilde{\mathbb{Z}[\varrho]} \to \widetilde{\widehat{\mathbb{Z}}_2[\varrho]})$$

for every $n = 0, 1, 2, 3 \mod 4$.

PROOF. – The result follows from diagram (4.14) for the relative groups and from Theorem 5.5. ■

THEOREM 5.7. – *Under the hypotheses of Theorem 5.6, we have isomorphisms*

$$LP_n^X(\mathbb{F} \to \widehat{\mathbb{F}}_2) \cong L_{n+1}^{X(\mathrm{rel})}(g^-) \oplus L_{n+1}^{X(\mathrm{rel})}(j_-^!)$$

for every $n = 0, 1, 2, 3 \mod 4$, *where* $L_{n+1}^{X(\mathrm{rel})}(j_-^!)$ *are the relative groups of the transfer map* $j_-^!$ *fitting in the following exact sequence*

$$\cdots \to L_n^X(\mathbb{Z}[H^-] \to \widehat{\mathbb{Z}}_2[H^-]) \to L_n^X(\mathbb{Z}[\varrho] \to \widehat{\mathbb{Z}}_2[\varrho]) \to L_n^{X(\mathrm{rel})}(j_-^!) \to \dots$$

PROOF. – The result follows from diagram (4.16) for the relative groups and from Theorem 5.5. ■

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